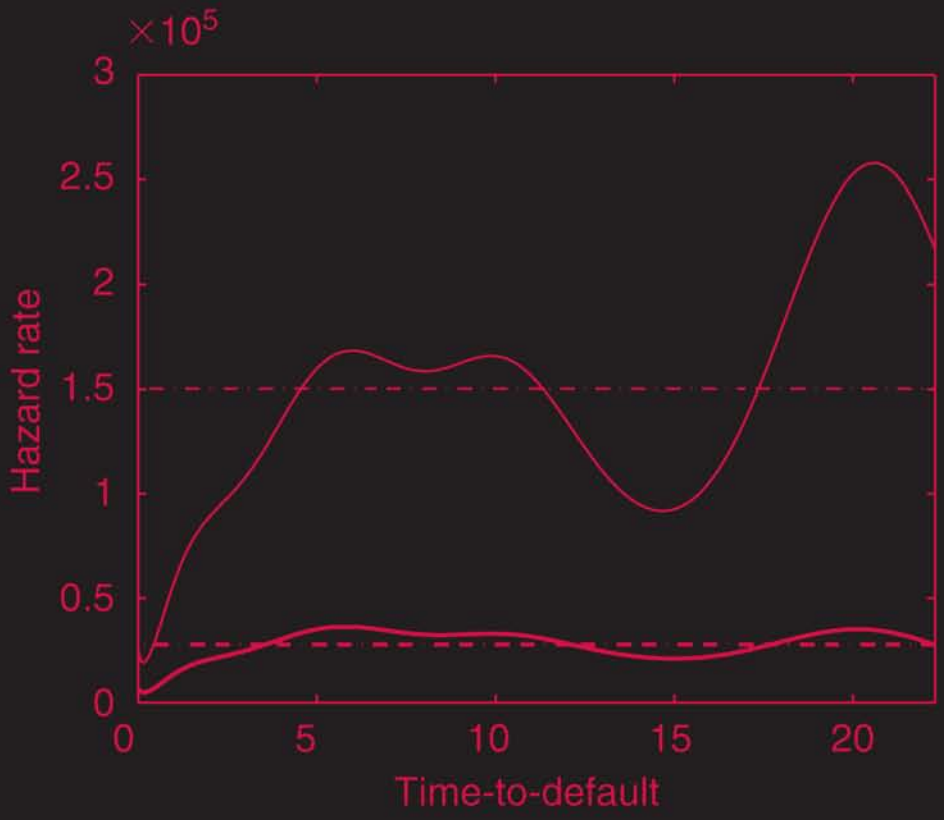
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Models, Derivatives, and Management



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Preface

During the last decade, international financial market regulation has had a notable impact on the business of financial intermediaries worldwide. Today, this process is ongoing as regulation as well as research impacts continuously find their way to trading and financial risk management departments. For financial institutions the *Basel Accord*, BIS (2001), is one of the major driving forces behind regulation and credit risk intermediation. While the accord sets the fundamental framework, each institution will need a sound understanding of its portfolio of credit risks before devising an implemented risk management system. Also, product innovations in credit risk transfer need a basis for sound applications in practice. This requires the choice of a set of suitable models together with the derivation of reliable empirical results on the validity of these models, before a system is implemented.

The present volume provides insights with respect to these questions. It provides 26 chapters in the areas of applied credit risk modeling, implementation, and management written by experts in the field with unique academic and industry backgrounds. The volume is structured along six main parts (Parts I–VI) covering aspects of credit risk management, pricing, and the uses of derivative products. The remainder of this preface first gives a current view on the credit risk topic and then points out the particular contributions of the volume, whose chapters address various challenging fields within the area.

A VIEW ON CREDIT RISK—RESEARCH AND ANALYSIS

While relevant to academics in finance for decades, the development of new methods, the appearance of new products, and their increased application in the financial services industry have heavily fostered research in the area of credit risk. Hence, for academics and practitioners alike, credit risk has grown to become an enormous field during the last decade, while it is still in a state of rapid expansion.

In fact, credit risk has long been of academic interest in the business and finance communities. For example, rigid statistical analysis of corporate default was carried out by Edward Altman starting in the 1960s. The structural credit risk modeling approach follows the celebrated option-pricing framework of Black, Scholes, and Merton. Also, among others, we may trace back intensity-based credit risk models to results from statistical renewal theory, about which Cox (1962) wrote an early monograph. Given these roots, the 1990s were a period that brought about intensified research in the area

of credit risk modeling and pricing.* In the present volume, Parts II–IV cover the traditional credit risk perspective, which includes credit risk models, pricing issues, default prediction, and empirical findings.

It took some time for comprehensive treatments of credit risk appeared, making the field available to a broader audience. Some of today's state-of-the-art treatments with a focus on risk models include, for example, Bielecki and Rutkowski (2002), Bluhm et al. (2003), Cossin and Pirotte (2001), Duffie and Singleton (2003), Giesecke (2004), Lando (2004), McNeil et al. (2005, Chapters 8 and 9), and Schönbucher (2003), among many others that appeared between the year 2000 and today.† Saunders and Allen (2002) offer a discussion of the regulatory context and explain industry benchmark models. Fong (2005) is an edited volume with a focus on advanced modeling issues. Altman and Hotchkiss (2006) is a state-of-the-art volume that covers corporate analysis and bankruptcy prediction.

A VIEW ON CREDIT RISK—DERIVATIVES MARKETS AND APPLICATIONS

Increased industry interest in credit risk emerged parallel to the development of academic research. As such, starting in the mid-1990s, credit risk management using credit derivatives gained much attention in practice. Advances in bank regulation and banks' capital requirements, and also the—then largely unmet—trading and speculation needs of various other market participants (including, e.g., hedge funds) created enormous growth in the market for credit derivatives.

Credit default swaps (CDSs) whose payoff is contingent on the default of a single obligor have now become the most important and liquid class of credit derivatives. These instruments provide insurance against obligor default and hence allow for elimination of individual risks in a lender's credit portfolio. With the emergence of the credit derivatives market, growth in outstanding credit derivatives notional was outstanding. Global notional in CDS reached $4.5 trillion at the end of June 2004 (see BIS 2005, p. 116), which is amazing for a market that had just emerged with the end of the 1990s. According to surveys of the International Swaps and Derivatives Association (ISDA),‡ during the period June 2001 to June 2006, growth in CDSs was in a range between 50% and 128% annually. Comprehensive details on the derivatives markets and a perspective on risk management can be found, for example, in Anson et al. (2003), Batten and Hogan (2002), and Chaplin (2005), among several others. In the present volume, Parts I–III address the topic of CDSs.

More recently, portfolio products such as baskets and collateralized debt obligations (CDOs) have gained much practical attention. CDOs, whose payoff is contingent on the loss of a predefined portfolio of obligors, can help banks and other lenders to structure their credit risk profiles. In detail, the tranching of portfolio loss distributions enables market

* See Altman and Saunders (1997) for a detailed review and a discussion on how the academic assessment of what was perceived as credit risk has changed.

† For an in-depth review of three of the books mentioned above, namely Duffie and Singleton (2003), Lando (2004), and Schönbucher (2003), see Schuermann (2005).

‡ See the ISDA summaries of midyear market survey results at www.isda.org.

participants to buy or sell protection for a given range of percentage portfolio losses, called "tranches." Hence, as an example, a bank may keep "regular" credit risks in its credit portfolio, while it may buy protection for tail risks, i.e., any percentage portfolio losses exceeding a given threshold, via trading CDO tranches with a reference portfolio identical (or similar) to the bank's credit portfolio.* Other typical applications include the release of regulatory capital, funding improvements, rating arbitrage, and finally speculation.

To ease trading and improve liquidity, standardized portfolio products are frequently based on established indices of liquid single-name CDSs. Important index examples are two credit risk benchmark indices, namely the Dow Jones CDX North America Investment Grade (DJ CDX.NA.IG) index for the U.S. index and the Dow Jones iTraxx Europe (DJ iTraxx Europe) index. Both market indices comprise 125 equally weighted large-cap investment-grade obligors, which have their individual default risks traded in a liquid CDS market. Several subindices cover, for example, sectors (e.g., DJ iTraxx Europe energy) or high-volatility index members (e.g., DJ CDX.NA.IG.HVol). Further indices cover regions (e.g., DJ iTraxx Asia) or subinvestment-grade obligors (e.g., DJ CDX.NA.HY).† Given such CDS indices, a special class of CDOs is index tranches, where the underlying reference portfolio is given by a standardized index portfolio.

Bluhm and Overbeck (2007) provide a comprehensive overview on credit portfolio products and models. A crucial concept for portfolio products is dependence modeling, which is outlined in the finance context, for example, in Cherubini et al. (2004). Elizalde (2006) reviews CDOs and their pricing. In the present volume, Parts V and VI address the emerging field of credit portfolio risk.

RECENT TOPICS IN CREDIT RISK—MODELS, DERIVATIVES, AND MANAGEMENT

The recent development in the markets for corporate credit makes it obvious that credit risk modeling and management are being confronted with a variety of challenges. The present volume addresses the gap between theory and new theoretical developments on one side and empirical findings and model implementation on the other.

Part I of the volume offers a view on the quickly developing area of credit derivative products. In Chapter 1, Anouk Claes and Marc De Ceuster review the mechanics of the single-name CDS market, the essential building block for any type of credit derivative product. By definition, the CDS is a contract in which a protection seller insures a protection buyer for a well-specified credit event on a notional amount for a specified period and for a specified portfolio of reference assets. As the risk of counterparty default is inherent in nearly all financial transactions, including credit derivatives, Volker Läger, Andreas Oehler, Marco Rummer, and Dirk Schiefer point out methods of treating

* While such a transaction takes place in an established derivatives market, it much resembles the situation of an insurance company that buys protection from a reinsurance company.

† For index product details refer to www.djindexes.com, www.indexco.com, and www.markit.com. In general, www.defaultrisk.com proves to be a useful Web resource for an overview on recent working papers as well as publications in the credit risk area.

counterparty risk in the pricing of CDSs. In Chapter 2, they consider valuation models based on observable market data, a structural model approach, an intensity model, as well as a rating-based approach. Jochen Felsenheimer and Philip Gisdakis take a view on future developments in integrated credit portfolio management in Chapter 3. They assess a broad range of structured credit instruments and their optimal uses, including new instruments such as iTraxx Futures, iTraxx default swaptions, and constant proportion debt obligations (CPDOs). Development in the CDS markets offers opportunities for additional products in banking and risk insurance. In Chapter 4, Rachel Campbell and Christian Wiehenkamp propose a new CDS instrument for commercial banks that use alternative assets such as art works as collateral, yielding art-backed loans. A CDS then transfers undesirable art price risk from banks to third parties who are willing to buy the art held as collateral at a guaranteed price in the case of default.

Part II of the volume addresses credit risk and credit spreads. CDSs represent traded credit spreads, which depend on a variety of economic state variables. In Chapter 5, Hans Byström analyzes the DJ iTraxx CDS index and its relation with the stock market. He finds that index spreads narrow when stock prices rise; the stock market leads the CDS market and index spreads tend to increase with increasing stock market volatility. In Chapter 6, Danielle Sougné, Cédric Heuchenne, and Georges Hübner study the relationship between CDS spreads and macroeconomic factors constructing synthetic time series of sector spreads for the automobile, consumer, energy, and industrial sectors. Confirming the results of Byström, they find that the spread exposure to a stock market index is negative. They also document that the exposure to the consumer price index and to a bond index is positive, while the main sources of systematic risk exposures generally tend to be temporary and unstable. Kannan Thuraisamy, Gerry Gannon, and Jonathan Batten examine the recent behavior of credit spreads on sovereign issuers from Latin America in Chapter 7. They find that credit spread changes are driven by interest rate and asset factors with both factors negatively related to changes in yield spreads. In Chapter 8, Umberto Cherubini discusses recent famous cases of corporate bankruptcy, namely Tyco (bankruptcy filed in 2002), WorldCom (bankruptcy filed in 2002), Enron (bankruptcy filed in 2001), and Parmalat (bankruptcy filed in 2003). These crises were typically announced by an inversion of the term structure of CDS spreads, while the values of equity and debt tend to co-move, while evidence of causality between both categories appears weak. He concludes that the financial scandals surveyed share many common features, including management's philosophy that transparency is a cost to evade rather than an opportunity to reduce the firm's cost of capital. As an example of a sovereign crisis, Chapter 9 examines the 2002 debt crisis in Argentina. Jorge Chan-Lau proposes to take a closer look at the maximum recovery rate that is compatible with observed CDS prices. Using such measures, he suggests constructing early warning indicators of financial distress.

Single-name credit risk models are focused on in Part III. While the structural modeling approach following Merton is very intuitive and elegant, there is wide consensus that structural models are difficult to implement. Gurdip Bakshi, Dilip Madan, and Frank Zhang (Chapter 10) empirically investigate systematic and firm-specific default risk factors. They analyze a set of three-factor credit risk models, where two factors cover interest

rate risk and a third factor covers firm-specific distress (proxied for, e.g., by firm leverage, book-to-market, profitability, equity-volatility, or distance-to-default). Among other findings, they point out that the mis-pricing of low-grade bonds indeed significantly tends to decline when taking leverage into account. Starting also from the structural model intuition, Christian Stewart and Niklas Wagner consider the pricing of CDSs in Chapter 11. They use CDSs from the DJ CDX.NA.IG index universe and compare the pricing results from the CreditGrades industry benchmark to a trinomial tree approach. Their results indicate that the models' pricing performances can be relatively poor, while the trinomial model outperforms CreditGrades. Company-specific variables hardly explain cross-sectional differences in model performance, while applying the trinomial model to companies with above average R^2 characteristics yields improved pricing performance. In Chapter 12, the Hull–White intensity-based model is applied to the pricing of names again from the CDX index universe. Bastian Hofberger and Niklas Wagner obtained satisfying pricing results subject to the base condition that liquid bonds with appropriate maturities are available for the respective issuers. Testing for co-integration of model and market spreads confirms a generally stable pricing relationship, while in several cases the model fails. While the cross-sections and time series of the CDS quotes in Chapters 11 and 12 overlap only partly (given constrained respective data availability), the results also provide some tentative indication that, on average, Hull–White pricing results are not dominated by those obtained in Chapter 11.

Part IV of the volume covers default and recovery risks, credit ratings, and applications within the Basel II framework. In Chapter 13, Edward Altman, Brooks Brady, Andrea Resti, and Andrea Sironi analyze aggregate default and recovery rates on corporate bond defaults over a 20-year period. In principle, default rates should increase during periods of economic stress, which also causes the value of assets of distressed companies to decrease. However, the statistical relationship is less significant than one might expect. Hence, the authors argue that recovery rates are a function of supply and demand for defaulted securities. Supply is driven predominantly by default rates and demand by alternative investment managers who act as purchasers of defaulted securities. Their empirical results have important implications for credit risk models and for Basel II implementations. As default rates increase during periods of economic stress and vice versa, risk models need to consider determinants of default probability changes. In Chapter 14, Fabien Couderc, Olivier Renault, and Olivier Scaillet investigate common determinants of default probability changes of individual firms by analyzing the responses of hazard rates to changes in a set of economic variables which should describe the state of the default cycle. They find that, in comparison to market factors (stock and bond market indicators), business cycle and credit risk factors become dominant as the issuer quality decreases. Past economic conditions are important in explaining probability changes, where lagged stock volatility is an overall significant determinant. In Chapter 15, Gabriele Sabato considers low-default portfolios that face the statistical problem that the number of defaults is low or equal to zero. He considers several methodologies and argues that generic scoring models are neither the only way nor the most accurate to solve the problem. The proposed logit model predicts the 1-year probability of default through several credit customer attributes.

It is shown to be a suitable model that may facilitate risk assessment in the absence of sufficient historical default data in the Basel II context. Chapter 16 outlines tests on the accuracy of the Basel II framework, considering two of its crucial assumptions, namely that the risk of individual credit exposures is driven by one systematic factor only and that the bank's credit portfolio is fully diversified. Using a ratings-based credit risk model, which accounts for multiple risk factors and portfolio concentration, Simone Varotto shows that these Basel assumptions may yield substantial biases and can well have an economically significant impact on risk assessments.

The topic of credit risk and default dependence is addressed in Part V of the volume. Today's market benchmark models of implied credit correlation risk surely have drawbacks. In Chapter 17, Vineer Bhansali analyzes the market situation saying, "What the Market Is Telling Us and Does It Make Sense?" He points out that—based on a very probably mis-specified Gaussian copula model, which is frequently used in practice—compound correlations (base correlations) implied by tranche prices typically appear to have a "correlation smile" (correlation skew). Additionally, equity tranches typically have positive correlation sensitivity, i.e., as correlation rises the equity tranche rises in value, whereas more senior tranches exhibit negative correlation sensitivity. Chapter 18 deals with copula-based default dependence concepts. Elisa Luciano reviews the state-of-the-art and points out open modeling issues in structural and intensity models. Starting from the well-known Li model, she points out dynamic inconsistencies, in that structural models with diffusive asset values do not have an intensity-based representation. This in turn allows for a calibration under the historical measure only, since historical equity returns are usually taken as asset return proxies. To calibrate dependence under a risk-neutral measure, the so-called factor copula is then presented. An empirical investigation of the fit of various copula models is provided in Chapter 19. Here, Sanjiv Das and Gary Geng model, simulate, and assess the joint default process of a broad cross-section of U.S. corporate issuers during the period 1987–2000. Default dependence has several characteristics that can be accounted for when using copula functions, namely level, asymmetry, and extreme behavior. They find that the skewed double-exponential distribution is a suitable choice for the marginal distribution of each issuer's hazard rate process. In dependence modeling, the Gumbel, Clayton, and Student-t copula functions in fact allow for improvements over the standard Gaussian copula. In Chapter 20, Sofiane Aboura and Niklas Wagner discuss a common factor model of systematic credit risk and analyze credit risk factor sensitivities and extreme dependence. They do this under the risk-neutral measure by relating single-name CDS spread changes to those of an observable factor, namely the DJ CDX index. The empirical results suggest pronounced time variation and asymmetry in factor dependence, where upward factor jumps matter. The latter finding is in line with models of infectious default and supports the result of Chapter 19 that portfolio credit models should consider asymmetric dependence.

As pointed out above, the market for credit derivatives has been innovative recently. Options and futures on index spreads and index tranches (e.g., with the DJ CDX or DJ iTraxx index serving as underlying) are at the edge of this development. Part VI of the volume considers options, credit portfolios, and the pricing of loss distribution tranches.

In Chapter 21, Damiano Brigo presents models for the pricing of options on single-name CDSs, the so-called swaptions, where he takes the perspective of the LIBOR (London interbank offered rate) market model into account. He considers standard CDS payoffs as well as alternatives and approximates CDS options via defaultable callable notes. He derives a Black-type pricing formula as well as an analytical formula for options under a CIR (Cox–Ingersoll–Ross)-type stochastic-intensity model. In Chapter 22, Siu Lam Ho and Lixin Wu discuss the pricing of credit derivatives from a general perspective. Like Damiano Brigo in Chapter 21, they suggest using a unified, LIBOR market model-type framework. They price single-name CDS options as well as portfolio credit derivatives such as CDOs. Pricing is based on forward credit spreads, which are implied by observable CDS rates, as well as on CDS rate correlations, and on implied swaption volatilities. Single-name swaptions are priced via a Black-type model, while prices of portfolio credit derivatives are derived via Monte Carlo simulation. In Chapter 23, Vincent Leijdekker, Martijn van der Voort, and Ton Vorst conduct an empirical analysis of CDO price data, using the misspecified base correlation benchmark method as a starting point. The authors perform an empirical analysis of market prices for CDO tranches on the iTraxx and CDX indices by filtering out the effect of the portfolio credit risk by means of a simple regression analysis, which shows that correlation matters. Also, the evolution of the base correlation over time and three different mapping methodologies are investigated. The latter can be used to determine correlations for nonstandard tranches, given base correlations derived from the index market.

Manuel Moreno, Juan Peña, and Pedro Serrano devote their contribution (Chapter 24) to the pricing of CDO tranches with generalized multifactor models, where the Vasicek one-factor structural model forms the starting point. Their model extensions include (1) the relaxation of the asset homogeneity assumption, (2) the introduction of a second, industry-specific systematic risk factor, which adds to the business cycle factor, and (3) the relaxation of the factor normality assumption via considering Student-t distributions. Their empirical factor model specifications use six explanatory variables to explain a set of historical default rates, namely the real gross domestic product (GDP), the consumer price index, the annual return on the S&P500 index, its annualized volatility, the 10-year treasury yield, and the index of industrial production. In Chapter 25, Jean-Michel Bourdoux, Georges Hübner and, Jean-Roch Sibille study applications of the Gaussian and the Student-t copula functions in an intensity-based pricing approach for iTraxx index tranches. They attest that correlation has a major impact on the prices of CDOs. They also verify Vineer Bhansali's statement from Chapter 17 that the correlation smile mirrors the fact that the Gaussian copula is not an exact distribution yielding the well-known correlation smile and that correlation effects are opposite between the equity tranche and the rest of the tranches. The authors further emphasize that the introduction of a random recovery rate should be seriously considered. Their tests on the Gaussian and the Student-t copulas document an equally poor job in describing the CDO correlation structure, although the Student-t copula can in principle accommodate tail effects, which the Gaussian cannot. In the volume's final contribution, in Chapter 26, Manuel Moreno and Pedro Serrano study the pricing of CDOs via Monte Carlo simulation. Their approach

is also motivated by intensity-based modeling, in which they consider different multifactor specifications, jumps in the default intensity process, and random default losses. Their simulation analysis shows that a three-factor model with constant loss impact is flexible enough to reproduce the spreads given by the market, where modeling random losses can be helpful when dealing with one- or two-factor models.

As a final point to this preface, I would like to mention that credit risk will prevail as a challenging topic after this book is published. Reports on subprime mortgage loan and CDO-related credit problems accumulated recently. The current U.S. subprime mortgage loan crisis, CDO-related credit problems, several bankruptcies of low-grade lending institutions, and the related substantial financial problems of major financial institutions worldwide are an indication of the ongoing relevancy of thoughtful credit risk modeling and management. Also at this point, I would like to thank all contributors, Darrell Duffie and Greg Gregoriou as well as Sunil Nair and Jessica Vakili, both of Taylor & Francis group, for their kind respective support of the book.

Niklas Wagner
Munich and Passau

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Part I

A View on Credit Derivatives

CHAPTER 1

Single Name Credit Default Swap Valuation: A Review

Anouk G.P. Claes and Marc J.K. De Ceuster

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> The potential of credit derivatives is immense. There are hundreds of possible applications: for commercial banks which want to change the risk profile of their loan books; for investment banks managing huge bond and derivatives portfolios; for manufacturing companies over-exposed to a single customer; for equity investors in project finance deals with unacceptable sovereign risk; for institutional investors that have unusual risk appetites (or just want to speculate); even for employees worried about the safety of their deferred remuneration. The potential uses are so wide spread that some market participants argue that credit derivatives could eventually outstrip all other derivative products in size and importance.
>
> (*EUROMONEY*, MARCH 1996)

1.1 PRODUCT

A credit default swap (CDS) is a bilateral and privately negotiated contract in which a protection seller (the insurer, the risk taker) insures a protection buyer (the risk shedder) for a well-specified credit event on a notional amount for a specified period and for a specified (portfolio of) reference asset(s). The protection buyer periodically pays (in arrears) a fixed fee (a CDS premium) to the protection seller as long as no credit event occurs. In case the credit event occurs, the buyer or the seller can send a credit event notice to the other party, after which the seller has to pay compensation to the protection buyer. The buyer often still has to pay a final accrual payment.

1.1.1 Settlement

Depending on the contract specifications, the default event allows the protection buyer:

- To *settle physically*, which means that the defaulted bond (i.e., the deliverable obligation) will be delivered at par. Physical delivery is still the most common way of settling, according to the 2006 BBA credit derivatives report, representing 75% of the settlements. Delivery normally takes place within 30 days after the credit event. The Delphi bankruptcy case in 2005, however, illustrated a disadvantage of physical

settlement. The demand from protection buyers looking to source bonds and speculators anticipating a short squeeze pushed the Delphi bond price severely upwards immediately after the bankruptcy. Pool and Mettler (2007) observed that this caused a disorder in the cash and the derivatives market. Very large—relative to the deliverable bonds outstanding—notional (single name) CDSs' exposures put protection buyers at risk of not receiving their contingent payment due to the difficulties they experienced in sourcing deliverable bonds.

- To *settle in cash*, i.e., to receive the difference between par and the bond's recovery value. Since obtaining quotes for a distressed reference asset is not trivial, cash settlement is less common. If chosen, settlement normally takes place within 5 business days after the credit event. A calculation agent then polls dealers to determine a mid-market price of the reference obligation.
- To *receive a preagreed fixed payoff*, irrespective of the recovery rate (fixed amount settlement or digital CDS structure) (Figure 1.1).

1.1.2 Positions

Market terminology relies on the fact that the traded asset in a CDS is the default protection. Consequently, the protection buyer is considered to be long in the CDS whereas the protection seller is considered to be short. Alternatively, one can also refer to the position taken with respect to credit risk. The CDS buyer (seller), who is long (short) in the CDS, will be short (long) in the credit.

1.1.3 CDS Rate

Since a CDS is designed as a swap, the parties do not exchange money when entering the contract, but determine the periodic payments in such a way to enter the CDS at zero value. The premium paid is often called the CDS spread. Note that contrary to what the name might suggest, the CDS spread is not quoted vis-à-vis a risk free benchmark such as yield spreads. The term "CDS rate" would therefore be more appropriate. This CDS rate, s, is quoted in basis points per annum of the contract's notional value. The premiums are usually paid quarterly.

Since the protection buyer can deliver the defaulted bond at a prespecified price when the credit event occurred, the CDS rate can be interpreted as a put option premium, paid over the term of the contract. If you have to pay 160 bps for a 5 year CDS on a notional of

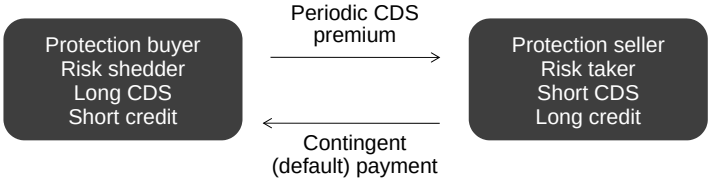


FIGURE 1.1 Basic credit default swap structure.

$10 million on Ford, you pay 40 bps (i.e., $40.000) every quarter to insure yourself against the default of Ford.

A bid of 150 bps on a CDS means that the bidder is willing to enter into a long CDS at a rate of 150 bps. An offer of 180 bps represents a seller willing to enter into a short CDS at 180 bps. As always, the offer quotes will exceed the bid quotes.

1.1.4 Reference Credit and the Reference Asset

The reference credit (credit entity, reference entity) is one (or several) issuer(s) whose default triggers the credit event. This can be one or several (basket structure) defaultable issuers. In case of multiple defaultable issuers, the credit event is triggered by the default of m-of-n credit entities.

The reference asset is one (or a set of) asset(s) issued by the reference credit (loans, bonds), needed to determine the credit event and the recovery rate.

Using this terminology, Daimler Chrysler could be the reference credit. All senior unsecured bonds issued by Daimler Chrysler with an issue size of minimum €10 million would be an example of the reference asset.

1.1.5 Credit Event

The ISDA's Master Agreement launched in 1999, and revised in 2002 and 2003, provides six potential trigger events:

1. Bankruptcy
2. Failure to pay
3. Repudiations/moratorium
4. Obligation acceleration
5. Obligation default
6. Restructuring

In practice, (1), (2), and (6) are most common. Bankruptcy occurs if the reference entity becomes insolvent or when it is unable to repay debt. The precise bankruptcy event needs to be specified in the contract. Failure to pay occurs when the reference entity, after a certain grace period, fails to pay any interest claim or the principal. Restructuring refers to a change in terms of the debt contracts of the reference entity that are adverse to the creditors. This credit event also needs to be specified further in the contract.

1.1.6 Example

Consider a 3 year CDS with annual payments represented in Table 1.1. The CDS rate is set at s per annum on a notional of €1. The time grid in the table is biannual. In case no default occurs during the maturity of the CDS, the protection buyer has to pay s at the end of every year. In case default would take place at $t = 3$, i.e., after 1.5 years, a terminal cash flow would be paid to the protection buyer.

TABLE 1.1 Credit Default Swap with an Annual Premium on a Notional of €1

Time Index (t)	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Time (in years)	0	0.5	1	1.5	2	2.5	3
Cash flows in case of no credit event		0	$-s$	0	$-s$	0	$-s$
Cash flows in case of default at time 1.5		0	$-s$	$(F+AI)\ (1-R)-AF$	0	0	0

- This terminal cash flow equals the $F(1-R)$ where F denotes the notional and R the recovery rate.
- Alternatively, the recovery rate can be expressed as a percentage of the market value. In doing so, we take into account the accrued interest earned, AI, on the reference asset and the terminal cash flow becomes $(F+AI)(1-R)$.
- Finally, CDS contracts often state that the last premium will be due pro rata temporis (i.e., an accrued fee, AF). The accrued fee is netted with the recovery value. Hence the terminal cash flow would become $(F+AI)(1-R)-AF$.

1.2 MARKET

Credit derivatives were first introduced at the annual meeting of the International Swaps and Derivatives Association in 1992 (Houweling and Vorst, 2005). Figure 1.2 shows that the growth of the global market has been overwhelming. From 1996 to 2004, the market size doubled biannually. Over the (last) 2004–2006 period the market even quadrupled to over US $20 trillion.

The market is organized as an OTC market. On the basis of Fitch 2005 data, Pool and Mettler (2007) indicate that Morgan Stanley, Deutsche Bank, Goldman Sachs, and JP Morgan have consistently been the top four OTC market makers. The interdealer market has been facilitated with the development of electronic trading systems such as the

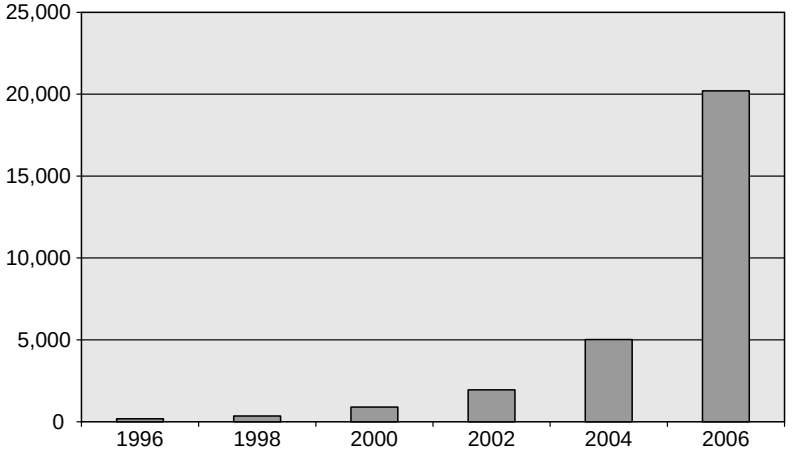


FIGURE 1.2 Global credit derivatives market (in billions of U.S. dollars). (From Barrett, R. and J. Ewan, 2006, *BBA Credit Derivatives Report 2006*, British Bankers Association, London, p. 39.)

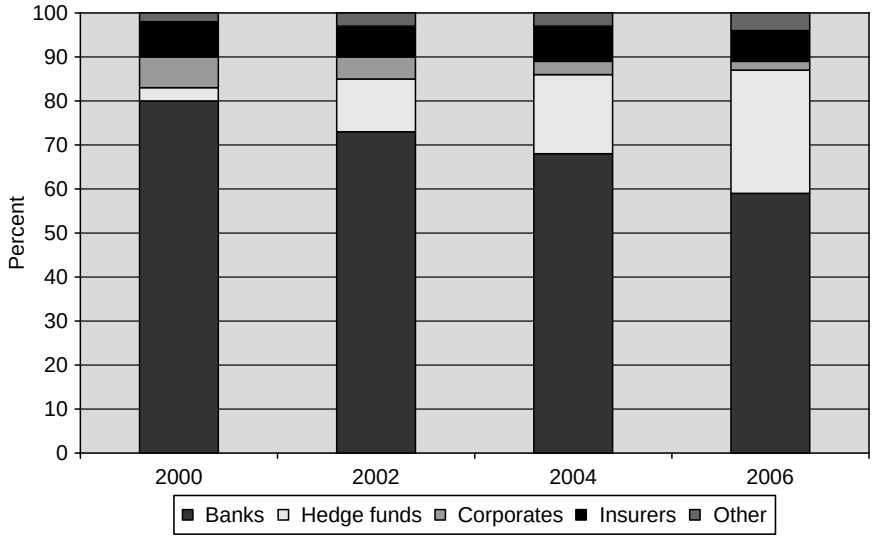


FIGURE 1.3 Distribution of the buyers of credit protection. (From Barrett, R. and J. Ewan, 2006, *BBA Credit Derivatives Report 2006*, British Bankers Association, London, p. 39.)

Creditex RealTime Platform. These platforms provide price and trade transparency as well as operational efficiencies. Over 35% of all European Credit Derivatives are now being traded electronically, with North America in hot pursuit (http://www.creditex.com/web/electronic-trading.html).

Figures 1.3 and 1.4 summarize the results of the 2006 BBA survey (Barrett and Ewan, 2006). Banks still constitute the major market players both on the buy and the sell side of

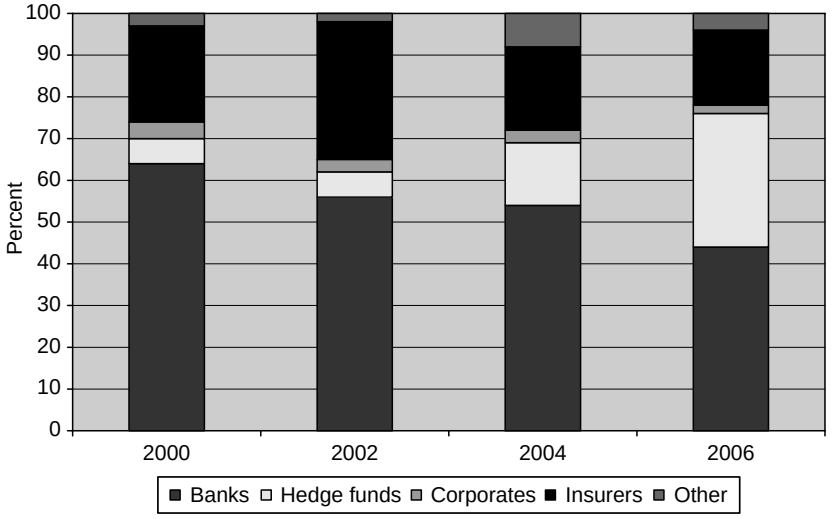


FIGURE 1.4 Distribution of the sellers of credit protection. (From Barrett, R. and J. Ewan, 2006, *BBA Credit Derivatives Report 2006*, British Bankers Association, London, p. 39.)

the market, although their market share has been decreasing on both sides over the past years. They use credit derivatives for trading purposes (2/3) and for managing their own loan books (1/3). During the last years, hedge funds have mainly become a driving source of market expansion.

Insurers, however, typically sell credit protection. The figures clearly show that corporates still have to discover the CDS market. "Other" includes among others, mutual funds and pension funds.

Although the market growth is fierce, the evolution of the market is still hampered by legal issues (e.g., the precise credit event definition) and by the backlog of unconfirmed trades (Pool and Mettler, 2007). The 2006 BBA survey, for example, reports that 10% of the trades had more than 2 month delays in confirmation. Demand for standardized exchange-traded alternatives is clearly growing and is giving rise to the development of products such as credit futures and credit index options.

Although the variety of credit derivatives has proliferated quickly, in 2006 the BBA estimated single name credit default swaps still constitute one third of the market (Barrett and Ewan, 2006).

1.3 ITS USE

CDSs can be used both from an investment angle and from a risk management perspective. Moreover, they have been recognized as leading indicators of bond and equity markets.

- By definition the CDS can be used to "insure" a long corporate bond (portfolio) against credit risk. Adding long CDSs transforms the corporate bond portfolio into a credit risk free portfolio.
- Alternatively, the CDS allows all parties involved to take bidirectional positions in pure credit risk. They can go long and short in credit risk without an initial funding requirement. A CDS can be entered into even if the cash bond of the reference entity of a specified maturity is illiquid or even unavailable.
- The CDS market has been recognized as a leading indicator for equity and bond markets (Hull, Predescu, and White, 2004).
- Backshall (2004) argues that CDSs can also be used to estimate (or speculate) on the timing of the default. If a 5 year single name CDS is traded at 2000 bps, we would have to pay US $2 million per year for a US $10 million position. Hence the market expects the single name to default within roughly 2.5 years.

1.4 CDS REPLICATION

Whenever a product is complex, one can try to decompose it into known building blocks in order to replicate its cash flows. Several "quasi" replication strategies have been presented (Cossin and Pirotte, 2001; Schonbucher, 2003). We will consider two strategies, a fixed

coupon bond strategy and an asset swap strategy, that both approximately replicate the CDS payoffs.

1.4.1 Fixed Coupon Bond Strategy

Consider on the one hand a long CDS on the defaultable bond with CDS rate s and notional amount F.

Consider on the other hand a portfolio consisting out of

- Short position in a T year defaultable coupon bond, B, with coupon c and face value F
- Long position in a T year default free coupon bond $B^\star$ with coupon $c-s$ and face value F

The coupon rate of the default free bond is adjusted in such a way that the initial bond prices equal.

- Assume for example, two 3 year coupon bonds both with a face value of 100. The risk free coupon bond pays a coupon, $c-s$, of 5%. If the yield to maturity of this bond ($y^\star$) is 6%, the risk free bond prices at 97.32699. The yield to maturity of the risky bond (y) is assumed to be 7%. In order to price the risky bond at 97.32699 as well, a coupon (c) of 5.9814% needs to be paid. This implicitly defines $s=0.9775$.

In case of default at time t_j, the portfolio is unwound and a recovery rate, R, is recovered of the face value.

The value of a bond at time t is denoted $B(t)$.

The cash flows of these two portfolios are presented in Table 1.2.

In case no default takes place, the cash flows of the portfolios are identical. Consequently, if (and only if) also the cash flows in the default state coincide, the initial prices of the two bonds should be exactly the same. A perfect replication therefore requires that the default free bond $B^\star(j)$ should be priced at par at the time of default (t_j). This makes the replication based on fixed coupon bonds, in general, only an approximate one.

TABLE 1.2 Replicating the Credit Default Swap as a Defaultable Coupon Bond

Time	**Portfolio 1**		**Portfolio 2**
No Default	Defaultable Bond	Default Free Bond	CDS
$t=0$	$B(0)$	$-B^\star(0)$	0
$t=t_i$	$-c$	$c-s$	$-s$
$t=T$	$-(F+c)$	$F+c-s$	$-s$
Default			
$t=t_j$	$-RF$	$B^\star(j)$	$(1-R)F$

If the replication would be perfect, the CDS could be replicated by:

$$\begin{aligned} B(0) &= \sum PV_{(y^\star)}(c-s) + PV_{(y^\star)}(F) \\ &= \sum PV_{(y^\star)}(c) - \sum PV_{(y^\star)}(s) + PV_{(y^\star)}(F) \\ &= cA_{(y^\star)} - sA_{(y^\star)} + PV_{(y^\star)}(F) \\ s &= \frac{PV_{(y^\star)}(F) - B(0) + cA_{(y^\star)}}{A_{(y^\star)}} \end{aligned}$$

with $PV_{(y^\star)}(F)$ the present value of the face value at the default free bond's yield, $y^\star$, $B(0)$ the price of the defaultable bond, c the coupon of the defaultable bond, and $A_{(y^\star)}$ the annuity factor for the default free bond.

$$\text{In our example, } s = \frac{83.9619 - 97.3270 + 15.9885}{2.6730} = 0.9814$$

$$\text{In the real world, } s \approx \frac{PV_{(y^\star)}(F) - B(0) + cA_{(y^\star)}}{A_{(y^\star)}}$$

For small probabilities of default, this approximation should work well. It becomes an empirical question: what the impact of imperfect replication in the default state is on the CDS rate.

If both bonds would be priced at par, the yields would equal the coupon rates so that the CDS rate can easily be approximated by the differences in the yields (Houweling and Vorst, 2005). Duffie (1999) shows that this relationship also holds exactly for par floating rate notes.

The rough approximation of CDS rates by the difference of two yields crucially depends on the chosen $y^\star$ and hence on the matching of a risk free bond to the underlying risky bond. Houweling and Vorst (2005) show that the treasury yield curve is no longer used as the risk free benchmark. Market practitioners clearly preferred to use swap rates as proxy for the risk free rate, for liquidity reasons.

1.4.2 Asset Swap Strategy

An alternative to approximately replicate the CDS is a portfolio of risky bonds and an interest rate swap. The key idea is to hedge the interest rate risk of the risky bond and to remain with a credit risk exposure. Suppose we want to replicate a short position in the CDS. In order to create a synthetic exposure to credit risk, the protection seller first would need to borrow money in the repo market and invest in a defaultable par bond, with the defaultable bond as collateral. The periodic cost of borrowing in the repo market would equal Libor, L, minus a spread on the repo, r_s. On the other hand, he would receive the interest on the bond which is made up by the "risk free rate," R_f, and a credit spread, c_s.

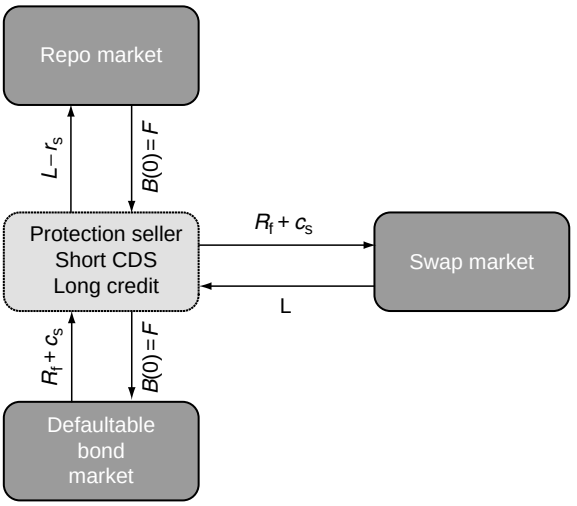


FIGURE 1.5 Asset swap replication of a credit default swap.

The interest rate risk in the bond can be hedged by entering in an interest rate swap receiving floating for fixed. The construction is visualized in Figure 1.5. The package of a defaultable bond (the asset) and an interest rate swap that swaps the (fixed) coupon ($c = R_f + c_s$) of the bond into Libor, L, is called an asset swap.

Table 1.3 shows that the CDS rate, s, can be computed as $c_s - s_s + r_s$.

Unfortunately, there is a gap between the theory and the practical implementation of this asset swap based strategy.

1. First, the risky bond that we have to buy has to be a par bond. The maturity of the CDS and the bond coincide so that the repayment of the principal of the loan can be financed by the received terminal value of the bond. This value should be the face value due to the pull to par phenomenon. The existence of par bonds in the market is obviously far from guaranteed.
2. Secondly, a repo contract on risky collateral has to be found providing money. If such a contract is available, it is unlikely that the full value of the bond will be received. Typically a haircut will be applied. Of course the credit risk, inherent in the repo, is ignored.

TABLE 1.3 Replicating the Credit Default Swap Based on an Asset Swap

Protection Seller	**No Default**			**Default**
	$t = 0$	$t = t_i$	$t = T$	$t = t_j$
Repo	F	$-(L - r_s)$	$-F$	$-F$
Swap	0	$+L - (R_f + s_s)$	0	Mtm loss/gain on swap
Bond	$-F$	$R_f + c_s$	F	$F\,R$
Total	0	$c_s - s_s + r_s$	0	$-F(1 - R)$ + loss/gain on swap
CDS	0	s	0	$-F(1 - R)$

Note: Mtm = marked to market.

3. Swap market might be very liquid, but we assume in this strategy that it efficiently prices credit risk. For investment grade fixed coupons this might be acceptable, for speculative grade bonds, however, it will be rather questionable.
4. Contract specifications of the CDS must be exactly as pointed out. Other schemes such as a final premium payment in case of default only make the replication less perfect.

1.5 RISK NEUTRAL PRICING

Nowadays, financial products are valued as the discounted value the expected cash flows under a risk neutral probability measure. We can discount the payments at the risk free rate using risk neutral valuation in the lines of Jarrow and Turnbull (1995). This implies that the default probabilities in the risk neutral world will be relevant.

1.5.1 Default Status Tree

We discretize the remaining time to maturity of the CDS, i.e., the time interval [0 T], into n intervals with equal distance Δt between the grid points. For $n = 3$, Figure 1.6 pictures the evolution of the default status of the reference bond.

At t_0 the company is alive (nondefault, *ND*) with probability 1. At t_1, the company can default (*D*) or survive (*ND*). If it defaults, this state remains the same for the future. If the firm did not default, the tree will further branch out with two possible states (*D* and *ND* at t_2). Default at any t_i acts as an "absorbing state" for the rest of the tree.

1.5.2 CDS Cash Flows

Let us refer back to our example in Table 1.1, where the cash flows for the buyer of a 3 year CDS with annual payments were summarized. Time was discretized in $n = 6$ steps of 0.5 years. Bankruptcy can occur on every node (column). In case of *ND* at time T, the last payment occurs at t_4 or at T depending on the contract specification.

In case of *D*, the payment of the premium payments stops but a last accrued premium may contractually be due at the time of default. This accrued fee is netted with the payment the protection seller has to pay in case of *D*. For the buyer, the t_3-value of the settlement of the CDS is $F(1 - R)$. Depending on the contract specifications the par value can be augmented with the accrued interest from t_2 to t_3. In most general terms, the payment becomes $(F + AI)(1 - R) - AF$.

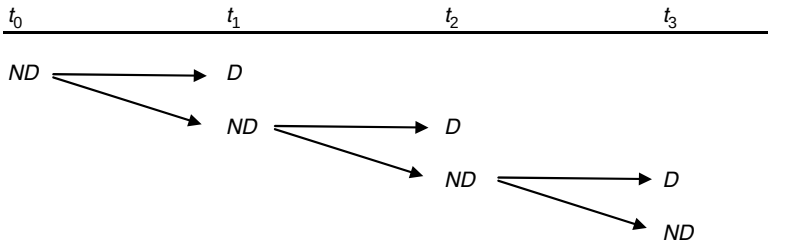


FIGURE 1.6 Default status tree.

1.5.3 Default Probabilities

1.5.3.1 Discrete Time Default Process

The default status tree (Figure 1.6) neglected the probabilities. The definition of the various default and survival probabilities makes the literature not very transparent. Deciphering notation is most of the time the hardest part of understanding the model.

First let us define the probability of default at each node in the tree, $p_{(i)}$, as the probability to default during period i. The firm still was alive at t_{i-1}, it defaults at t_i. Whenever the default probabilities are kept constant over time, such as in Figure 1.7, we will drop the subscript. In Figure 1.7 $p = 5\%$.

These default probabilities define a number of conditional probabilities characterizing the various states of the default process.

For the first period the marginal probabilities apply:

$$P(D_1) = p_{(1)} = 5\%$$
$$P(ND_1) = (1 - p_{(1)}) = 95\%$$

For the second period,

$$P(D_2|ND_1) = (1 - p_{(1)})\ p_{(2)} = 4.75\%$$
$$P(ND_2|ND_1) = (1 - p_{(1)})\ (1 - p_{(2)}) = 90.25\%$$

For the third period, we get

$$P(D_3|ND_2) = (1 - p_{(1)})\ (1 - p_{(2)})\ p_{(3)} = 4.51\%$$
$$P(ND_3|ND_2) = (1 - p_{(1)})\ (1 - p_{(2)})\ (1 - p_{(3)}) = 85.74\%$$

Two series of probabilities emerge:

- On the one hand, we find the probabilities of survival until time t_i, which we will denote π_i: 95%, 90.25%, and 85.74%.
- On the other hand, we obtain the probability of default at time t_i, conditional upon survival until the previous period: 5%, 4.75%, and 4.51%. We will denote these probabilities by p_i.

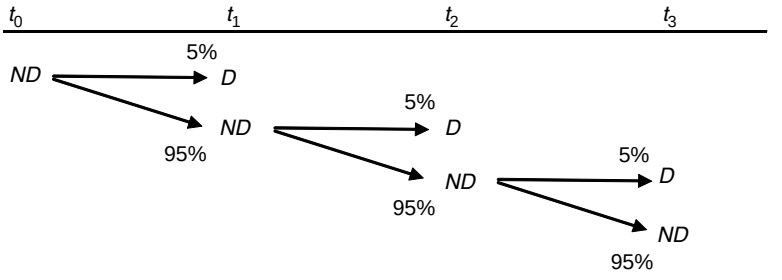


FIGURE 1.7 Default probabilities.

In general, we can express these conditional probabilities for t_i as

$$\begin{cases} p_1 = p_{(1)} \\ p_i = P(D_i|ND_{i-1}) = p_{(i)} \prod_{j=2}^{i} (1 - p_{(j-1)}) \quad \text{for } i = 2, 3, \ldots, n \end{cases}$$

$$\pi_i = P(ND_i|ND_{i-1}) = \prod_{j=1}^{i} (1 - p_{(j)}) \quad \text{for } i = 1, 2, \ldots, n$$

If the default probability is constant over the tree, these formulas for $i = 1, 2, \ldots, n$ collapse to

$$p_i = P(D_i|ND_{i-1}) = p(1-p)^{i-1}$$
$$\pi_i = P(ND_i|ND_{i-1}) = (1-p)^i$$

Note that the following relationships hold:

- Probability of survival until time t_i is equal to 1 minus the sum of the conditional probabilities of default in the previous periods: $\pi_i = 1 - \sum_{j=1}^{i} p_j$.
- Probability of survival until time t_i is equal to the probability of survival until the previous time minus the conditional probability to default in period i: $\pi_i = \pi_{i-1} - p_i$.
- Probability to survival until time t_i is equal to the probability of survival until the previous period multiplied by the factor 1 minus the default probability (given that the default probability is constant): $\pi_i = \pi_{i-1}(1-p)$ or even more general $\pi_i = \pi_{i-j}(1-p)^j$ with $j < i$ and $j \in N$.

1.5.3.2 Continuous Time Default Process: Hazard Rate

In continuous time we relate the survival probability until period 2 to the survival probability until period 1 by expressing the probability to default as $\lambda \, dt$:

$$\pi_2 = \pi_1(1 - \lambda \, dt)$$

where λ represents the probability of default during period dt. λ is called the hazard rate. $(1 - \lambda \, dt)$ is the probability of survival in the next time interval dt.

$$\begin{aligned} \text{In general, } \pi_{i+1} &= \pi_i(1 - \lambda \, dt) \\ \text{or } \pi_{i+1} &= \pi_i - \pi_i \, \lambda \, dt \\ \text{or } d\pi &= -\lambda \pi \, dt \\ \text{or } \frac{d\pi}{dt} &= -\lambda \pi \end{aligned}$$

The solution of this differential equation is $\pi_t = e^{-\lambda t}$.

Hence the survival probability can be modeled conveniently as an exponentially declining function of λ.

In order to choose the continuous hazard rate consistent with the constant discrete default probability p, the following relationships hold:

$$\lambda = \frac{-\ln(1-p)}{t} \quad \text{or} \quad p = 1 - e^{-\lambda t}$$

We notice that a hazard rate of 5.13% corresponds to a constant default probability of 5%.

1.5.4 Basic Assumptions

- We assume that default rates are constant throughout the tree, and that interest rates and recovery rates are nonstochastic or at least mutually independent.
- Claim in the event of default is the face value augmented with the accrued interest.
- No premium or accrued premium is due in case of default.
- Default can occur on every node of the grid: $t_1, t_2, t_3, \ldots, t_n$.
- $\Delta t =$ the time step in the grid.
- Let g be number of nodes between two payment dates $+1$. In our example $g = 2$. Consequently, there are n/g payment dates which occur on $t_g, t_{2g}, t_{3g}, \ldots, t_n$.

1.5.5 Risk Neutral Pricing of the CDS

The periodic premium payments are generally labeled in the literature as the floating leg. The contingent terminal payment is known as the fixed leg of the CDS.

1.5.5.1 Under the Discrete Default Process

1.5.5.1.1 Floating Leg At the payment dates $t_g, t_{2g}, \ldots, t_n$ the protection buyer pays the premium $P = sF(g\Delta t)$, until (and not including) default.

The expected payment for time t_i, seen today, is $P(ND_i|ND_{i-1})P + P(D_i|ND_{i-1})0 = \pi_i P$. Consequently, the present value of the expected payments is $\sum_{i=1}^{n/g} df_{ig}\, \pi_{ig}\, P$ where df_i denotes the discount factor for the time t_{ig} cash flow.

1. Note that the *discount function* can be expressed in various ways:
   - Cheng (2001) uses the general zero bond price formulation: $B(t_0,t_n)$ i.e., the time t_0 value of a default free zero bond maturing at t_n.
   - Brooks and Yan (1998) use simple one period forward rates: $df_i = \frac{df_{i-1}}{1 + r_{fw}\frac{NAD_i}{NTD}}$. NAD is the number of accrued days and NTD the total number of days, defining the applicable day count convention.
   - Continuous interest rates would yield: $e^{-r_i t_i}$.
   - Scott (1998), Aonuma and Nakagawa (1998), Houweling and Vorst (2005), Longstaff, Mithal and Neis (2005) use the stochastic interest formulation: $e^{-\int_0^t r(u)du}$.

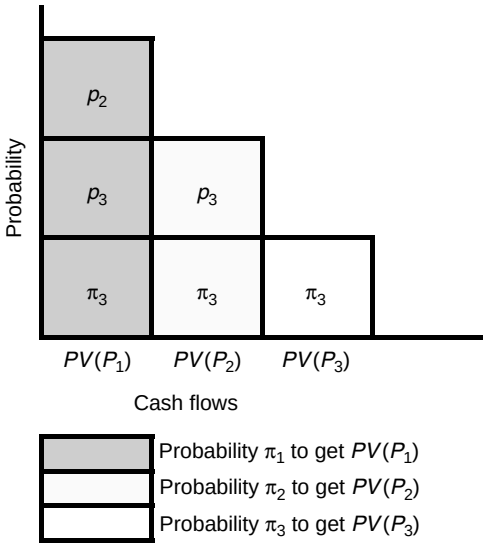


FIGURE 1.8 Redistribution of the default probabilities.

2. Also with the probabilities we can juggle a bit. Note that, as Figure 1.8 illustrates, the survival probability, π_i, $i < n$ equals the sum of the survival probability at the end of the contract, π_n, plus the sum of the probabilities of defaulting between t_i and t_n.

$$\pi_i = \pi_n + \sum_{j=i}^{n} p_j$$

$$\sum_{i=1}^{n/g} df_{ig}\,\pi_{ig}\,P \text{ then becomes } \sum_{i=1}^{n/g} df_{ig}\left(\pi_{n/g} + \sum_{j=i+1}^{n/g} p_j\right)P$$

which is a formulation Brooks and Yan (1998) use.

1.5.5.1.2 Fixed Leg Upon default, the protection buyer receives $(1-R)F$ where R is the constant recovery rate on the reference bond.

The present value of these cash flows is $(1-R)F\sum_{i=1}^{n}(p_i\,df_i)$.

1.5.5.1.3 CDS Rate In order to initiate the swap with zero value, the CDS rate, s, is set in such a way that the present value of the expected cash flows received by the buyer equals the present value of the expected cash flows paid by the buyer.

We get $\sum_{i=1}^{n/g} df_{ig}\,\pi_{ig}\,P = (1-R)F\sum_{i=1}^{n}(p_i\,df_i)$. Consequently, $P = \dfrac{(1-R)F\sum_{i=1}^{n}(p_i\,df_i)}{\sum_{i=1}^{n/g} df_{ig}\,\pi_{ig}}$

Recall $P = s\,F(g\,\Delta t)$ so that $s = \dfrac{(1-R)\sum_{i=1}^{n}(p_i\,df_i)}{g\Delta t\sum_{i=1}^{n/g} df_{ig}\pi_{ig}}$

1.5.5.2 Under the Continuous Default Process Formulation

1.5.5.2.1 Floating Leg In the continuous world we defined $\pi_t = e^{-\lambda t}$, so we can rewrite the present value of the expected payments the buyer has to make as $\sum_{i=1}^{n/g} e^{-r_{t_{ig}} t_{ig}} e^{-\lambda t_{ig}} P$ or $\sum_{i=1}^{n/g} e^{-(r_{t_{ig}}+\lambda)t_{ig}} P$.

1.5.5.2.2 Fixed Leg In the discrete case, the present value of these cash flows is $\sum_{i=1}^{n} p_i\, e^{-r_i t_i}(1-R)F$.

Note that $p_i = \pi_{i-1} - \pi_i$ can use the definition of the hazard rate to obtain

$$\left[\sum_{i=1}^{n}(e^{-\lambda t_{i-1}} - e^{-\lambda t_i})e^{-r_i t_i}\right](1-R)F$$

Note that $(e^{-\lambda t_{i-1}} - e^{-\lambda t_i})$ can be written as $(e^{-\lambda(t_i - dt)} - e^{-\lambda t_i})$ since $t_{i-1} = t_i - dt$.

Rearranging gives $e^{-\lambda t_i}(e^{\lambda dt} - 1)$.

We know that $\ln(1 \pm \varepsilon) \approx \varepsilon$ for small values of ε. This implies that $(1 \pm \varepsilon) = e^{\varepsilon}$ and that $e^{\varepsilon} \pm 1 = \varepsilon$.

Hence $e^{-\lambda t(i)}(e^{\lambda dt} - 1) = \lambda\, dt\, e^{-\lambda t_i}$.

The present value of the cash flows becomes

$$\left[\sum_{i=1}^{n}\lambda\, dt\, e^{-\lambda t_i} e^{-r_i t_i}\right](1-R)F \quad \text{or} \quad \left[\sum_{i=1}^{n} e^{-(r_i t_i + \lambda t_i)}\right](1-R)F\lambda\, dt$$

1.5.5.3 CDS Rate

In the swap we equate

$$\left[\sum_{i=1}^{n} e^{-\lambda t(i)} e^{-r_i t_i}\right](1-R)F\lambda\, dt \quad \text{to} \quad \sum_{i=1}^{n/g} e^{-(r(t_{ig})+\lambda)t_{ig}} P$$

Substitution $P = s\,F\,g\,\Delta t$ and solving for s gives

$$s = \frac{\left[\sum_{i=1}^{n} e^{-(r_i t_i + \lambda t_i)}\right](1-R)\lambda\, dt}{\sum_{i=1}^{n/g} e^{-(r_{t_{ig}}+\lambda)t_{ig}}\, g\, dt}$$

Note that if all default dates are payment dates as well, $g = 1$, the formula for the CDS rate shrinks to $s = (1 - R)\lambda$.

1.6 CONCLUSION

In this note we introduce the reader to credit default swaps and their pricing. The key drivers of the CDS rate are identified as the (risk neutral) default probability or the (risk neutral) hazard rate and the recovery rate. The basic model presented can be refined by making the default probabilities, the recovery rate, and the interest rates stochastic.

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CHAPTER 2

Valuation of Credit Derivatives with Counterparty Risk

Volker Läger, Andreas Oehler, Marco Rummer, and Dirk Schiefer

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2.1 INTRODUCTION

The valuation of credit derivatives has for a long time been based on default-free counterparties (i.e., contractual partners), as this allows a risk-free valuation of the payments made under credit derivatives. Even though financial institutions own subsidiaries, which could act as counterparties in OTC derivatives (over-the-counter derivatives), and reach strong ratings of "AAA", Ammann (2001) shows that less than half of the market participants have a rating of "A" or above. Moreover, no exchange traded credit derivatives exist up to now. Following these arguments, the consideration of counterparty risk is essential for a correct and consistent valuation of credit derivatives.

Given the stated findings and reasoning, this chapter discusses the valuation of credit derivatives with defaultable counterparties.* Especially, we focus on the modeling of joint default risks of risk buyers and reference parties. The following section concentrates on the valuation of standard credit derivatives, i.e., credit default swaps (CDSs). As described in Hull and White (2000), a CDS is a contract that provides protection against the default of a particular company. This company is referred to as reference entity and the default of the reference entity is the credit event. The buyer of protection has the right to sell to the protection seller at par a specified bond issued by the reference entity when a credit event occurs. That bond is commonly referred to as the reference obligation and the total volume of the bond that can be sold is the notional of the CDS.

The most important aspect of modeling counterparty risk is the treatment of correlations between the credit risk of the underlying asset and the credit risk of the counterparty. If, for example, perfect correlation is assumed, one can easily see the importance of correlation for valuation purposes. If a financial institution with an AAA-rating sells CDSs (risk buyer, protection seller) and the reference asset is its own bond, then the swap with an assumed recovery rate of zero is worthless. This is due to the fact that in the event of default of the reference entity, which triggers the credit event under the CDS, the risk buyer—being identical to the reference entity—defaults as well. If, in contrast, the risk buyer is solvent, i.e., no credit event occurs, the risk buyer will not be drawn on. Tavakoli (2001) summarizes these results as follows: "Counterintuitive as it may seem, it is better to buy credit protection from an uncorrelated lower-rated protection seller than from a protection seller that is highly correlated with the reference asset one is trying to hedge" (Tavakoli 2001, p. 25).

This chapter is organized as follows: Section 2.2 presents the approach of Hull and White (2001). The authors define variables that are directly or indirectly observable from market data as endogenous and all other variables as exogenous. In Section 2.3, we review how defaultable CDSs can be evaluated based on Merton (1974). Sections 2.4 and 2.5 present the models of Jarrow and Yu (2001) and Lando (2000), respectively. Both approaches focus on a joint default process, which explicitly incorporates the reciprocal action between both obligors. Therefore, the latter concepts abandon the conditional independencies of default events. Section 2.6 concludes the chapter.

* For the valuation of credit derivatives with risk-less counterparties, see Läger (2002), Chapter 7.

2.2 VALUATION BASED ON OBSERVABLE MARKET DATA

The approach of Hull and White (2001) defines both directly and indirectly observable market data as exogenous. Because of this approach, the model is commonly considered as rather practical and applicable. The valuation of CDSs follows a three-stage procedure:

1. Calculation of risk-neutral default probabilities of the relevant contractual agents (i.e., protection seller and reference entity). To accomplish this, one can either draw on listed bonds issued by the reference entity or bonds from companies with the same risk of default as the reference entity.
2. Calculation of the default correlation between the protection seller and the reference entity. Hull and White (2001) suggest the usage of market data.
3. Calculation of the expected future stream of payments, which are linked with the default swap (i.e., premium payments and loss compensations at default).

It should be noted that the model of Hull and White (2001) cannot be solved analytically. Instead, one has to use a simulation-based approach, e.g., Monte Carlo simulation.

2.2.1 Assumptions

Hull and White (2001) base the modeling of the default on a first-passage-time approach. Given this methodology, the default occurs, if the (obligor-specific) default barrier variable $\tilde{G}_t^j$ (where j represents the index of the analysed obligor) reaches the default barrier $\tilde{g}_t^j$. $\tilde{G}_t^j$ is referred to as the credit index for company j at time t. Hull and White (2001) propose two different interpretations of this variable. First, they define this variable as a function of the value of the total firm assets and second as a discrete credit rating. A more specific valuation is not important as the default probability will not be calculated within this model. Instead, it is an exogenous input parameter taken from observable market data. The risk-neutral default probability is used to calibrate the default barrier $\tilde{g}_t^j$.

To determine the correct default barrier $\tilde{g}_t^j$, the following assumptions have to be considered. First, it has to be assumed that the risk-neutral process for $\tilde{G}_t^j$ follows a geometric Brownian motion with zero drift and variance of one per year. Second, the default occurs at discrete points of time referred to as t_i (where $1 \leq i \leq n$). Therefore, the cumulative default probability of company j till time t_i could be expressed as follows, if no default occurs before t_{i-1}:

$$\mathrm{P}^*\{\tau_j = t_i | \tau_j > t_{i-1}\} = 1 - \int_{\tilde{g}_i^j}^{\infty} f_i^j(x)\,\mathrm{d}x$$

where

τ_j is the default time of the company j

f_i^j is the (conditional) density function of the credit index $\tilde{G}_t^j$ with $\tau_j > t_{i-1}$

If the (unconditional) default probability $p_i^j = \mathrm{P}^*\{\tau_j = t_i\}$ is known, the conditional density function, $f_1^j(x)$, and default barrier, p_i^j, can be calculated as follows:

For $i = 1$

$$f_1^j(x) = \frac{1}{\sqrt{2\pi\Delta_1}} e^{-x^2/2\Delta_1}$$
$$p_1^j = \Phi\left(\frac{\tilde{g}_1^j}{\sqrt{\Delta_1}}\right) \tag{2.1}$$

with $\Delta_i = t_i - t_{i-1}$, $1 \le i \le n$ and $t_0 = 0$. Φ is the standard normal distribution function.

From Equation 2.1 and $\tilde{g}_1^j = \sqrt{\Delta_1}\Phi^{-1}(p_1^j)$, the default barrier for $i = 1$ can be determined. On the basis of this result, the default barrier for $2 \le i \le n$ can be calculated as follows:

$$p_i^j = \int_{\tilde{g}_{i-1}^j}^{\infty} f_{i-1}^j(x)\Phi\left(\frac{\tilde{g}_i^j - x}{\sqrt{\Delta_1}}\right)dx$$

The conditional density function $f_i^j(x)$, $x > \tilde{g}_i^j$ can be calculated by solving the following equation:

$$f_i^j(x) = \int_{\tilde{g}_{i-1}^j}^{\infty} f_{i-1}^j(u)\frac{1}{\sqrt{2\pi\Delta_i}} e^{-(x-u)^2/2\Delta_i}du$$

The inductive calculated default barriers $\tilde{g}_i^j$ are not constant over time, but are time dependent. This is due to the usage of observable (risk-neutral) default probabilities for the calibration process.

As a proxy for the correlation coefficient ρ_{jk} between company j and k, Hull and White (2001) use the correlation between the respective equities.* The required default correlation coefficient $\tilde{\rho}_{jk}(T)$ can be derived from the following equation, if the correlation coefficient ρ_{jk} is known:

$$\tilde{\rho}_{jk}(T) = \frac{\tilde{F}^{jk}(T) - F^j(T)F^k(T)}{\sqrt{[F^j(T) - (F^j(T))^2]\,[F^k(T) - (F^k(T))^2]}}$$

with $\tilde{F}^{jk}(T) = P^*\{\tau^j \le T; \tau^k \le T\}$ and $F^l(T) = P^*\{\tau^l \le T\}$, $l = j, k$.

The cumulative default probability $F^l(T)$ is based on the risk-neutral default observable from market data. On the contrary, the joint default probability $\tilde{F}^{jk}(T)$ has to be estimated through simulations of default processes of company j and company k by using ρ_{jk}.

The above outlined steps allow investigation of the data. The valuation based on discounted expected payment streams is presented in the next section. Moreover, additional assumptions have to be made such as the independence of the recovery rate

* For a description of the methodology applied for private companies or debtors, who are not publicly traded see Hull and White (2001). Hull and White (2001) note that the commonly used methodology results in rather imprecise estimates.

from risk-less interest rates and from default events. Additionally, the independence of default events from risk-less interest rates has to be assumed.

2.2.2 Valuation of Credit Default Swaps

Hull and White (2001) focus on CDSs with periodical risk-free premium payments in arrear. Moreover, the protection buyer is regarded as a risk-free agent and therefore the payments of the buyer are risk-free as well.

The following swap payment streams are considered for our analysis:

1. Risk seller A pays the swap rate till maturity T. If the first default occurs before T then A terminates his regular payments and
   a. pays a final accrual payment* of e, if the defaulted party is the reference obligator R, otherwise
   b. pays no final accrual payment, if the risk buyer B defaults.
2. If R defaults during the duration of the swap and before B, then B pays the difference between the face value of the debt claim of A against R and the recovery rate. The recovery rate is calculated as the face value of the bond including accrued interests.
3. If B defaults before R during the duration, then no payment will be made and the swap is going to be terminated.

At $t=0$ the present value of the expected premium paid by A is calculated as follows:

$$\ddot{\kappa}'_{\rho v} = \ddot{\kappa}' \left(\int_0^T \left\{ f_t^{R\bar{B}} [u_t + m_t(ew^{-1})] + f_t^{B\bar{R}} u_t \right\} \mathrm{d}t + G_T^{BR} u_T \right) \tag{2.2}$$

where, w represents the annual payments of the premium calculated as a proportion of the face value of the swaps, $f_t^{R\bar{B}}$ is the default density of R under the condition that B did not default until $t=0$,† $f_t^{B\bar{R}}$ stands for the default density function of B under the assumption that R do not default until $t=0$,‡ u_t represents the present value of the annuity factor for annuity payments of 1 within the period $[0; t]$, $m_t(ew^{-1})$ describes the present value factor of an one-off final payment of ew^{-1} in t, and G_T^{BR} is the probability that the counterparty B and the reference entity R do not default until T.

The value of the swap for A is calculated as the present value of the expected payments generated by the swap as follows:

$$\ddot{\varpi}'_0 = \int_0^T \left[1 - E\left(\delta_t^R\right)(1 + c_t)\right] f_t^{R\bar{B}} m_t(1)\, \mathrm{d}t \tag{2.3}$$

---

* The final accrued payment represents a fraction of the premium that has been paid between the last payment and the default event.

† Under the assumption that no default of B has occurred so far, $f_t^{R\bar{B}}\Delta t$ represents the risk-neutral default probability of R within the time horizon Δt.

‡ Under the assumption that no default of R has occurred so far $f_t^{B\bar{R}}\Delta t$ represents a risk-neutral default probability of B within a time horizon Δt.

where δ_t^R is the expected recovery rate at the default of R and c_t represents the interest accruals until t is calculated as a percentage of the face value on the debt claims.

Hull and White (2001) calculate the fair swap-rate $\ddot{\kappa}'_{\rho\nu}$ drawing on Equations 2.2 and 2.3 through using a Monte Carlo simulation. They use different sets of parameters first for the quality of the risk buyer and second for the correlation between the risk buyer and the reference entity. This fair swap rate equals w, from which it follows that $\ddot{\varpi}'_0 = \ddot{\kappa}'_{\rho\nu}$. High correlations between the risk buyer and the reference entity as well as low ratings of the swap seller show a significant impact on the swap rate.

2.3 VALUATION BASED ON MERTON'S APPROACH

The value of a CDS equals the value of a put option on the assets of the company. The strike price is equivalent to the representative receivables. The writer of this put option is the risk buyer (i.e., swap seller). In an event as described in the last section in which the credit risk of the protection buyer is neglectable (i.e., because the lump-sum swap rate is paid up front) the valuation of a CDS with a defaultable counterparty equalizes the valuation of a vulnerable put.*

2.3.1 Assumptions

To evaluate vulnerable puts, different methods (e.g., Johnson and Stulz 1987, Klein 1996, Klein and Inglis 1999) based on Merton's (1974) credit risk model have been proposed.† Looking at defaultable CDSs, both credit risks have to be considered explicitly: First, the credit risk of the reference asset and second the credit risk of the risk buyer have to be evaluated. It should be noted that the valuation of these credit risks does not have to be based on an identical methodology. For the purpose of this review chapter, we assume a standard Merton (1974) firm value model:

- Firm value of the risk buyer V_t^{B} follows a lognormal distribution. The process of V_t^{B} can be expressed as follows: $\mathrm{d}V_t^R = V_t^{\mathrm{B}}\mu_{V^{\mathrm{B}}}\mathrm{d}t + V_t^{\mathrm{B}}\sigma_{V^{\mathrm{B}}}\,\mathrm{d}W_t^{V^{\mathrm{B}}}$.
- Firm value of the reference party V_t^R follows a lognormal distribution. We are therefore able to use the following expression:

$$\mathrm{d}V_t^R = V_t^R\mu_{V^R}\,\mathrm{d}t + V_t^R\sigma_{V^R}\,\mathrm{d}W_t^{V^R}$$

  The correlation coefficient between the firm values B and R, both of which follow a Brownian motion, is denoted as $\rho_{\mathrm{B}R}$.

* Ammann (2001) suggests an alternative model for the valuation of defaultable credit derivatives. This approach is based on Merton's firm value model. Within this framework, credit derivatives are modeled as exchange options. It should be noted that the equation for the valuation of exchange options was developed by Margrabe, who describes an exchange option as an option "... to exchange one risky asset for another." (Margrabe 1978).

† For the presentation of the models see Johnson and Stulz (1987), Klein (1996), Klein and Inglis (1999), and Klein and Inglis (2001). A summary can be found in Läger (2002).

- Risk buyer B defaults only at T. The default occurs, if the firm's value V_t^{B} drops below an assessed fixed default barrier. This threshold level differs, as shown below, between the various models used to evaluate vulnerable puts:
  a. Option (a): The claim of the swap can be used as the default barrier. This is equivalent to the assumption that the default swap is the sole liability of the option writer and therefore this follows Johnson and Stulz (1987).
  b. Option (b): Alternatively, the total amount of all liabilities of the risk buyer can be drawn on as the default barrier. If the liabilities are assumed to be constant over time, then this is equivalent to the assumption that the claim of the swap is negligibly small. This is based on the model of Klein (1996) and Klein and Inglis (1999).*
  c. Option (c): Moreover, the threshold level can be set equivalent to the sum of all other liabilities of the option writer additional to the claim on the swap. This is based on the approach of Klein and Inglis (2001).
- Reference party R can only default at T. The default occurs, if the firm value V_T^R drops below the value of the liabilities.
- At default, the recovery rate $\delta_T^{\mathrm{B}}\left(V_T^{\mathrm{B}}\right)$ of the risk taker B is calculated as the ratio of the firm value V_T^{B} and the total sum of all liabilities multiplied with the factor $(1-\alpha)$. The latter factor represents the dead-weighted costs associated with the default.
- At default, the recovery rate of R corresponds to the ratio of V_T^R to the overall liabilities.

The default swap is constructed as follows:

- Reference bond of the swap is a zero-coupon-bond of the reference party R with maturity T and face value F. This bond is the only liability of R.
- Maturity of the swap is T and this maturity equals the maturity of the reference bond.
- At the default of the reference party, the creditor A receives—given the above-stated assumptions as well as the fact that the zero-coupon bond is the only liability—a payment equivalent to the total firm value V_T^R. Owing to this, the creditor has a claim of a deficiency payment against the protection seller, which is equivalent to $F-V_T^R$.
  a. If the risk buyer B has not (yet) defaulted at the time of the default of R, the total deficiency payment has to be paid.
  b. If B defaults at the same time as R, the risk seller or the creditor receives only a fraction of his claim, which is equivalent to $\delta_T^{\mathrm{B}}\left(F-V_T^R\right)$.

* To be more precise, both references differ from each other in the threshold level D^* and the (higher) value of the liabilities D. Both D^* and D are assumed to be constant over time from which it follows that the overall statement about the assumed delectability of the writer's option does not have to be changed.

2.3.2 Valuation of Credit Default Swaps

The value of the CDS, $\ddot{\varpi}_0$, can be derived from the discounted expected stream of payments. The expected values each depending on the default barrier of B can be expressed as follows:

$$\begin{aligned}
\ddot{\varpi}_0' &= E_{P^\star}\left\{B_T^{-1}\left[\left(F - V_T^R\right)^+ \mathbf{1}_{\left\{V_T^{\mathrm{B}} \geq F - V_T^R\right\}} + V_T^{\mathrm{B}} \mathbf{1}_{\left\{V_T^{\mathrm{B}} < F - V_T^R\right\}}\right]\right\} \\
\ddot{\varpi}_0'' &= E_{P^\star}\left\{B_T^{-1}\left[\left(F - V_T^R\right)^+ \mathbf{1}_{\left\{V_T^{\mathrm{B}} \geq D^{\mathrm{B}}\right\}} + (1-\alpha)\frac{V_T^{\mathrm{B}}}{D^\star}\left(F - V_T^R\right) + \mathbf{1}_{\left\{V_T^{\mathrm{B}} < D^{\mathrm{B}}\right\}}\right]\right\} \\
\ddot{\varpi}_0''' &= E_{P^\star}\left[B_T^{-1}\left(\left(F - V_T^R\right)^+ \mathbf{1}_{\left\{V_T^{\mathrm{B}} \geq D^{\mathrm{B}} + F - V_T^R\right\}}\right.\right. \\
&\quad \left.\left. + (1-\alpha)\frac{V_T^{\mathrm{B}}}{D^\star + \left(F - V_T^R\right)}\left(F - V_T^R\right)^+ \mathbf{1}_{\left\{V_T^{\mathrm{B}} < D^{\mathrm{B}} + F - V_T^R\right\}}\right)\right]
\end{aligned} \tag{2.4}$$

The expressions $\ddot{\varpi}_0'$ to $\ddot{\varpi}_0'''$ represent the above described options (a) to (c), respectively.

The CDS equals considering its payout-structure a defaultable put option, which is expressed as $\tilde{P}_0(F, V_t^R)$. Owing to this, the equations for European put options derived by Johnson and Stulz (1987), Klein (1996), and Klein and Inglis (2001) can be used to determine the expected rate of return in Equation 2.4. We use $\ddot{\varpi}_0''$ below as illustrative example:

$$\begin{aligned}
\ddot{\varpi}_t'' = &- V_t^R \Phi_2(-d_1, b_1, -p) + F\mathrm{e}^{-r(T-t)}\Phi_2(-d_2, b_2, -p) \\
&- (1-\alpha)\frac{V_t^{\mathrm{B}}}{D^\star} V_t^R \mathrm{e}^{p\sigma\nu^{\mathrm{B}^\sigma}\nu^{R^{(T-t)}}}\mathrm{e}^{r(T-t)}\Phi_2(-\tilde{d}_1, \tilde{b}_1, p) \\
&+ (1-\alpha)\frac{V_t^{\mathrm{B}}}{D^\star} F\Phi_2(-\tilde{d}_2, \tilde{b}_2, p)
\end{aligned}$$

where $\Phi_2(\cdot)$ is the function with a standard bivariate normal distribution and d_1, d_2, b_1, b_2, $t^\star$, and p are give by:

$$\begin{aligned}
d_1 &= \frac{\ln\left(V_t^R/F\right) + \left(r + \frac{1}{2}\sigma_{V^R}^2\right)t^\star}{\sigma_{V^R}\sqrt{t^\star}} = d_1\left(t^\star, V_t^R\right), & \tilde{d}_1 &= d_1 + p\sigma_{V^{\mathrm{B}}}\sqrt{t^\star} \\
d_2 &= d_1 - \sigma_{V^R}\sqrt{t^\star}, & \tilde{d}_2 &= d_2 + p\sigma_{V^{\mathrm{B}}}\sqrt{t^\star} \\
b_1 &= \frac{\ln\left(V_t^{\mathrm{B}}/D^\star\right) + \left(r - \frac{1}{2}\sigma_{V^{\mathrm{B}}}^2 + p\sigma_{V^B}\sigma_{V^R}\right)t^\star}{\sigma_{V^{\mathrm{B}}}\sqrt{t^\star}} = b_1\left(t^\star, V_t^{\mathrm{B}}\right), & \tilde{b}_1 &= -b_1 - \sigma_{V^{\mathrm{B}}}\sqrt{t^\star} \\
b_2 &= b_1 - p\sigma_{V^R}\sqrt{t^\star}, & \tilde{b}_2 &= -b_2 - \sigma_{V^{\mathrm{B}}}\sqrt{t^\star} \\
t^\star &= T - t \\
\rho &= \rho_{\mathrm{B}R}
\end{aligned}$$

This equation follows directly from Klein's (1996) equation for European put options with a defaultable option writer and a constant interest rate. Within this framework, the firm

value V of the option writer, the value of the underlying U, and the strike price K are specified as follows:*

$$V_t = V_t^B$$
$$U_t = V_t^R$$
$$K = F$$

In the case of stochastic interest rates, the equation framework of Klein and Inglis (1999) has to be used instead of Klein's framework.†

2.4 VALUATION BASED ON TWO-STATE INTENSITY MODEL

The coherence on default risk between the protection buyer and the protection seller plays a significant role in the valuation of defaultable credit derivatives. The assumption of conditional independence leads to a simplification in mathematical terms and therefore eases the valuation. It should be noted that the dependence between two contractual parties is only created through similar reactions of both parties on conjoint state variables. A statement on the specific dependence of one contractual partner cannot be made as the underlying relationship remains concealed as a black box. Especially, when it comes to modeling of default probabilities of financial institutions, this assumption must be pointed out as being a major disadvantage. The drawback is based on the fact that financial institutions depend regarding the financial solvency on the non-default of their creditors.

Jarrow and Yu (2001) recommend a framework based on Lando (1998a) in which the default probability of one creditor is influenced by the default of a third party. The main aspect in terms of modeling the default process will be outlined in Section 2.4.1. On this basis, we discuss the usage of defaultable credit derivatives in the context of adjusted reduced-form models.

2.4.1 Assumptions

The main assumptions and specifications of the framework of Jarrow and Yu (2001) are as follows:

- Default is modeled using Cox-processes from which it follows that the default is triggered if the default process H_t^j jumps. The stochastic intensity of the Cox-processes is considered as being dependent on the d-dimensional state variable Y_t.
- Jarrow and Yu (2001) assume equivalent recovery after the default.‡ This recovery rate expressed as $\delta^j \in [0;1]$ depends on the obligor and is constant over time.
- Process of the short rate is not specified but every arbitrage free (riskless) term structure model can be drawn on.

* For a general specification of the equation used to evaluate vulnerable puts see Equation (12) by Klein 1996 at page 1221.

† See proposition 2, Equation (12) in Klein and Inglis (1996).

‡ An overview of different models can be found in Läger (2002).

Under the assumption that no default of the obligor has been occurred so far, the price of a default-free zero bond with maturity T can be expressed at time t as follows:

$$D^j(t,T) = \delta^j B(t,T) + (1-\delta^j) E_{Q^\star}\left(\mathrm{e}^{-\int_t^T (r_s+\lambda_s^j)ds}\right), \quad T \geq t \tag{2.5}$$

Equation 2.5 can only be used for debtors if their default probability is solely influenced by the macroeconomic state variables Y_t. In cases in which the likelihood of the default of a debtor depends on his counterparties or another third party, Equation 2.5 is not valid. To model firm-specific dependencies, two distinct methodologies are being discussed in the literature: unilateral or bilateral. In the latter case, A influences the default probability of B, which itself influences the default probability of A (i.e., looping default). One explanation for this would be if an interlocking participation exists between two counterparties. This kind of dependency in terms of modeling is rather difficult to handle but for the primary objective namely the modeling of a default of financial institutions not necessary.* Given this, Jarrow and Yu (2001) focus in their analysis on a one-sided dependency, which they label primary–secondary framework.

For the above-described purpose, the set of firms $\mathbb{J}$ is being separated into two different subsets S_1 and S_2:

- $\mathcal{S}_1 \subset \mathbb{J}$ contains all primary firms. The default probabilities of those companies depend solely on state variables Y_t.
- $\mathcal{S}_2 \subset \mathbb{J}$ contains all secondary firms. The default probabilities of these companies are affected by Y_t as well as the default process of the above-described primary firms.

For the intensity of secondary firms, the following structure is assumed:

$$\lambda_t^j = a_{0,t}^j + \sum_{k=1}^{m^j} a_{k,t}^j \mathbf{1}_{\{\tau^{i_k} \leq t\}}, \quad \forall j \in \mathcal{S}_2 \tag{2.6}$$

where

i represents the index of primary firms
j stands for the index of secondary firms
m^j is the number of primary firms that influence the default probability of j

Thereby, it is assumed that the secondary firm j influences the primary firms $i_1, i_2, \ldots, i_{m^j}$, and therefore, for example, the secondary firm holds equity of debt stakes of the primary firm. The impact of the defaults of primary firms depends on the sign of $a_{k,t}^j$. In cases in which $a_{k,t}^j > 0$, the default of the primary firm will increase the default probability of the secondary firms. On the other side in cases in which $a_{k,t}^j < 0$, the probability of default is going to decrease given the dependency between both groups.

* Even if it seems plausible to assume that financial institution will face financial difficulties after the default of one of its major customers, it seems also reasonable to assume that the default of a financial institution will not show such an impact on its major customer.

The equation for the valuation of zero-coupon bonds is used below to state an example. For simplicity we assume that intensities do not depend on state variables. Moreover, we assume that primary firms R exhibit a constant intensity of λ^R. From Equation 2.5, it follows that the price of a zero bond can be expressed as

$$D^R(t,T) = \delta^R B(t,T) + (1-\delta^R)B(t,T)\mathrm{e}^{-\lambda^R(T-t)}$$

Furthermore, we consider a secondary firm B with the intensity λ_t^{B}. The intensity λ_t^{B} is specified as follows while facilitating Equation 2.6:

$$\lambda_t^{\mathrm{B}} = b_0 + 1_{\{t\geq\tau^R\}}b_1$$

where τ^R represents the default time of the primary firm R.

Following Jarrow and Yu (2001), Equation 2.5 can be simplified for B as follows:

$$D^{\mathrm{B}}(t,T) = \delta^{\mathrm{B}}B(t,T) + (1-\delta^{\mathrm{B}})B(t,T)\frac{b_1\mathrm{e}^{-(\lambda^R+b_0)(T-t)} - \lambda^R\mathrm{e}^{-(b_0+b_1)(T-t)}}{b_1-\lambda^R}$$

with $b_1 \neq \lambda^R$.

2.4.2 Valuation of Credit Default Swaps

Following Jarrow and Yu (2001), we consider for our analysis a CDS inheriting the following properties:

- Zero bond of the reference party R with maturity T' and face value 1 is considered as reference bond of the swap. From $R \in S_1$ follows that the reference party is a primary firm.
- Maturity date of the swap is expressed as T, with $T < T'$.
- Risk seller A pays a periodical fixed swap-rate $\dot{\kappa}''$ until the maturity of the swap. This payment has to be paid even if a default of the reference party or the risk buyer B has occurred.
- If the reference party defaults till T, B sets off the loss A. This payment is due at the maturity of the swap. Furthermore, zero recovery is assumed for R as recovery rate. From this it follows that R faces a loss of 100% of his claims in the event of default.
- If during the duration, the risk buyer defaults before or at the same time as R, B does not pay in the event of the default of R, which is equivalent to a zero recovery rate of B. From $\mathrm{B} \in S_2$ follows that the risk buyer is a secondary firm.
- If the risk seller defaults during the duration, the payment of the swap rates is going to be suspended (i.e., zero recovery rate). For simplicity it is assumed that at the default of A, the swap is not going to be terminated. From this it follows that in cases in which R defaults during the duration of the swap at a later stage, B has to pay the agreed compensation.

Each participating party (A, B, and R) is assumed to be inheriting a likelihood of default. The expected value of the premium payments can be calculated as follows:

$$E_{Q^\star}\left(\int_0^T \dot{\kappa}''\mathbf{1}_{\{\tau^A>s\}}\mathrm{e}^{-\int_0^s r_u\,\mathrm{d}u}\mathrm{d}s\right) = \dot{\kappa}''\int_0^T D^{\mathrm{A}}(0,s)\,\mathrm{d}s$$

From the risk seller's perspective (i.e., A), the price of the swap is equivalent to the expected rate of return of the discounted payment stream received by A at $t = 0$:

$$\ddot{\varpi}_0'' = E_{Q^\star}\left(\mathbf{1}_{\{\tau^R\le T\}}\mathbf{1}_{\{\tau^{\mathrm{B}}>T\}}\mathrm{e}^{-\int_0^T r_u\,\mathrm{d}u}\right) \tag{2.7}$$

This leads to the following solution for the expected rate of return:*

$$\ddot{\varpi}_0'' = D^{\mathrm{B}}(0,T) - \mathrm{e}^{-\int_0^T (r_s+\lambda_s^{\mathrm{B}}+\lambda_s^R)\,\mathrm{d}s}$$

2.5 VALUATION DRAWN FROM A RATING-BASED MODEL

To describe the joint loss distribution of several loans, a concentration on the above-described correlations is not sufficient.† The limitations of a correlation analysis with regard to expressiveness also matters for modeling the joint distribution of defaults of several loans. Lando (2000) shows, using different examples, that the possible distribution of defaults can vary considerably despite identical default correlations. Given this finding, it is not sufficient to obtain information on the correlation between two agents. Moreover, information about the structure and the mechanism that cause the dependence between the issuer and the seller on defaultable bonds is essential for a sophisticated analysis. To examine the impact of different ways of modeling dependencies, Lando uses defaultable swaps. Especially, he models the joint default of reference seller and risk buyer and its impact on the swap in various ways.

Section 2.5.1 states briefly the underlying rating-based model. On the basis of this section, Section 2.5.2 presents selective illustrative results while drawing on Lando (2000).

2.5.1 Assumptions and Valuation

The rating-based model, which is described further in this section, is based on Huge and Lando (1999) as well as Lando (1998a,b).‡ For the joint development of the rating of two obligors, a two-dimensional stochastic process described by $A_t = \left(A_t^{\mathrm{B}}, A_t^R\right)$ is assumed.§ The state space with a continuous-time process is given by

$$\mathbb{K}^K = \{1,\ldots,K\}\times\{1,\ldots,K\}$$

* See Equation (42) in Jarrow and Yu (2001) for details.

† A discussion about the limitations of correlations can be found in Embrechts et al. (1999), Frey et al. (2001) as well as Huschens (2000).

‡ See Huge and Lando (1999) and Lando (1998a,b). A summary can be found in Läger (2002).

§ For details see Lando (2000).

Denote K as default state. The intensity of the joint rating-transition from pair $(i^{\mathcal{B}}, i^{\mathcal{R}}) \in \mathbb{K}$ to pair $(j^{\mathcal{B}}, j^{\mathcal{R}}) \in \mathbb{K}$ is denoted by $\lambda[(i^{\mathcal{B}}, i^{\mathcal{R}}), (j^{\mathcal{B}}, j^{\mathcal{R}})]$. These intensities are combined in the $K^2 \times K^2$-intensity matrix Λ. Furthermore, the subset of the state space is denoted as $\mathbb{D} = \mathbb{K}^K / \mathbb{K}^{K-1}$, in which at least one of the two firms is subject to default.

The price of a financial security is denoted by $S(t, r_t, A_t)$. This financial security consists of a stream of payments, which are due to at fixed dates T_n or at those points of times where transitions in the Markov-chain take place. The maturity of the security is $T_N = T$. It is imperative that $A_t \in \mathbb{D}$: $S(t, r_t, A_t) = 0$ because in the case of default, a final payment ensues and the contract will be ceased prematurely.

Assume a point of time just before maturity with $t \in [T_{N-1}; T)$ but after the last fixed payment. The K^2-dimensional vector $\underline{S}(t, r_t)$ consists of all $S(t, r_t, i)$, $i \in \mathbb{K}^K$ and can be given by the solution of the subsequent stochastic differential equation:

$$\frac{\partial \underline{\mathbf{S}}(t,r_t)}{\partial t} + \mu_r \frac{\partial \underline{\mathbf{S}}(t,r_t)}{\partial r} + \frac{1}{2}\sigma_r \frac{\partial^2 \underline{\mathbf{S}}(t,r_t)}{\partial r^2} + \mathbf{\Lambda}\underline{\mathbf{S}}(t,r_t) + \underline{\mathbf{diag}}(\mathbf{\Lambda}\Xi_t^{\mathrm{T}}) - r_t \underline{\mathbf{S}}(t,r_t) = 0 \qquad (2.8)$$

where

$\underline{diag}(\bullet)$ represents the vector of the diagonal elements
Λ is an intensity matrix
Ξ stands for a $K^2 \times K^2$ matrix

The side condition of the stochastic differential equation (Equation 2.8) is $S(T, r_t, i) = d(T, r_T, i)$, $i \in \mathbb{K}^K$. The elements of the matrix Ξ correspond to the payments $\hat{\varpi}(t, r_t, \mathrm{A}_{t-}, \mathrm{A}_t)$ which are going to be paid at the time of rating-transitions from A_{t-} to A_t. The valuation at time $t < T_{N-1}$ is based on Equation 2.8 and was done by Lando (2000) in a recursive way. Equation 2.8 is the basis for a numerical valuation of defaultable CDSs.

2.5.2 Credit Default Swap with Different Correlations Structures

Following Lando (2000), we now focus on CDSs with the following contract specifications:

- Initiation at date $t = 0$, maturity at $T = 5$ (years).
- Risk seller A pays a semiannual swap rate of $\frac{1}{2}c$. If the first default occurs between two dates of payment, which is equivalent to min $(\tau^{\mathrm{B}};\tau^{R}) \in (T_{n-1};T_n)$ then A has to pay either
  a. the fraction of the swap rate for the period between the last coupon payment and default (hence $(\tau^R - T_{n-1})c$), if the reference party R defaults or
  b. nothing, if the risk buyer B defaults.

The swap premium payments will be ceased irrespective of the identity of the defaultable party. Therefore, the protection buyer A and his coupon payments will be regarded as being risk-free.

- If R defaults during the duration, A receives a payment from B to offset his losses. As equivalent recovery is assumed for R, the payment received by A is equivalent to $(1 - \delta^R)\, B(\tau^R, T)$. This is generally referred to as default swap.

- If B defaults before R during the duration, the two-way payment rule for the settlement of this financial contract is applied. Under this rule, the protection buyer A has to pay the entire value of the swap to B, if the predefault swap value of the contract is positive for B. On the other side, A gets only partly compensated with δ^B of the predefault swap value from B, if the value is positive for A.*

Lando analyzes three possibilities of modeling correlations between B and R, which are as follows:

1. Correlation of rating migrations between B and R is generated through the dependency on joint state variables, i.e., the short rate. This is the weakest form of dependency between B and R. The stream of payments to A is given by:

$$\hat{\boldsymbol{\varpi}}(\tau, r_\tau, \mathrm{A}_{\tau-}) = \mathbf{1}_{\{\tau=\tau^{\mathrm{B}}\}}\left[\delta^{\mathrm{B}}\mathbf{1}_{\left\{\ddot{\boldsymbol{\varpi}}'_{\tau-}>0\right\}}\ddot{\boldsymbol{\varpi}}'_{\tau-} + \mathbf{1}_{\left\{\ddot{\boldsymbol{\varpi}}'_{\tau-}<0\right\}}\ddot{\boldsymbol{\varpi}}'_{\tau-}\right] + \mathbf{1}_{\{\tau=\tau^R\}} \times \left[(1-\delta^R)\mathrm{B}(\tau,T) - (\tau - T_{n-1})c\right]$$

   where $\tau = \min(\tau^{\mathrm{B}};\tau^R)$ represents the date of first default and $\ddot{\boldsymbol{\varpi}}'_{\tau-}$ is the value of a default swap for A just before default.

2. Default of R increases the default probability of B without triggering the default. After the default of the reference party R, the protection buyer B is not going to receive the compensation for these losses immediately. The payment will be made at a later point of time. This implies that A will probably not receive his promised (entire) compensation as B can also default during this period between the event of default of the reference party and the settlement of the compensation. The longer the period between the default of R and the payment of B, the higher the default probability will be. The default window with a constant default intensity of B ultimately equals an increase of the default intensity without such a window. Therefore, this is equivalent to the modeling of default correlations (correlation-like effect). The payment stream to A is as follows:

$$\hat{\boldsymbol{\varpi}}(\tau, r_\tau, \mathrm{A}_{\tau-}) = \mathbf{1}_{\{\tau=\tau^{\mathrm{B}}\}}\left[\delta^{\mathrm{B}}\mathbf{1}_{\left\{\ddot{\boldsymbol{\varpi}}'_{\tau-}>0\right\}}\ddot{\boldsymbol{\varpi}}'_{\tau-} + \mathbf{1}_{\left\{\ddot{\boldsymbol{\varpi}}'_{\tau-}<0\right\}}\ddot{\boldsymbol{\varpi}}'_{\tau-}\right] + \mathbf{1}_{\{\tau=\tau^R\}}\left[(1-\delta^R)\mathrm{B}(\tau,T) - (\tau - T_{n-1})c\right]D^{\mathrm{B}}(\tau,\tau+\Delta T)$$

   where $D^{\mathrm{B}}(\tau,\tau+\Delta T)$ is the value of a zero bond of issuer B with maturity $t+\Delta T$ and face value 1.

3. Modeling simultaneously the defaults of R and B. If R defaults, it seems rather likely that B defaults at the same time and vice versa. The maximum probability depends on the initiated rating of both parties.† This is considered as being the strongest form of default correlation. The payment stream for A equals

* For a description of the full two-way payment rule and the alternative limited two-way payment rule see Läger (2002).

† For example, the default of the reference party with a rating of BBB will not trigger a default of the risk buyer rated "AAA" at a probability of one as this will simply not be compatible with its rating.

$$\begin{aligned}\hat{\ddot{\boldsymbol{\varpi}}}(\tau,r_\tau,\mathrm{A}_{\tau-}) = {} & \mathbf{1}_{\{\tau=\tau^{\mathrm{B}}<\tau^{R}\}}[\delta^{\mathrm{B}}\mathbf{1}_{\{\ddot{\boldsymbol{\varpi}}'_{\tau-}>0\}}\ddot{\boldsymbol{\varpi}}'_{\tau-} + \mathbf{1}_{\{\ddot{\boldsymbol{\varpi}}'_{\tau-}<0\}}\ddot{\boldsymbol{\varpi}}'_{\tau-}] \\ & + \mathbf{1}_{\{\tau=\tau^{\mathrm{R}}<\tau^{\mathrm{B}}\}}[(1-\delta^{R})\mathrm{B}(\tau,T) - (\tau - T_{n-1})c] \\ & + \mathbf{1}_{\{\tau=\tau^{\mathrm{B}}<\tau^{R}\}}\delta^{\mathrm{B}}[(1-\delta^{R})\mathrm{B}(\tau,T) - (\tau - T_{n-1})c]\end{aligned}$$

The impact of the first two variations on the swap value is negligible (Lando 2000). This statement is still valid for cases in which a longer default window is implemented, i.e., 1 month as stated in the second alternative. Only the assumption implemented in the third approach (i.e., simultaneous defaults) reduces the value of the swap tremendously—especially for lower-rated risk buyer.

2.6 CONCLUSION

The modeling of default correlations has a decisive meaning in the above outlined valuation of credit derivatives as at least two different credit risks are involved, i.e., the credit risk of the underlying asset and the credit risk of the counterparty. We focus our review on CDSs with uni- and bilateral default risks. The simple structure of default swaps enables us to focus on questions about the best-suitable modeling methodology without the difficulties of complex product structures. Besides the presentation of an equation for the valuation of defaultable CDSs, which is based on Merton (1974), other results of alternative approaches are being reviewed. First, the pure numerical valuation process by Hull and White (2000, 2001), the intensity approach by Jarrow and Yu (2001), and the rating-based model by Lando (2000). The latter examines the impact of default correlation between the risk buyer and the risk seller on the value of swaps.

Despite the widespread and active research, no generally accepted approach for the valuation of credit risks and credit derivatives has been presented so far. Given the current debate, it seems rather unlikely that this will happen in the near future for the following reasons.

Modeling credit risk is more complex than modeling market risk. For example, if one compares interest rate models with intensity models, it can be seen that the intensity models are expanded interest rate models. Therefore, questions concerning the modeling of credit risk have to be added on top of the questions about the term structure.

The most commonly stated hazard with regard to the valuation of credit risk models is the limited availability of data. Especially, credit events are rather rare and therefore information is only available at lower frequencies than market data (Huschens and Locarek-Junge 2002). This leads to the usage of Poisson processes to model default events.

As systematic data gathering and storage were started only a few years ago, only limited databases in terms of period are available up to now. Nevertheless, such time series are essential for the calibration and backtesting of credit risk models (see Stahl et al. 2002 as well as Bühler et al. 2002).

Both problems regarding the data should be solvable in the long run, as more and more data are going to be gathered and stored. Moreover, the growing usage of credit derivatives enhances the data collection as these instruments (for the first time) are subject to a regular marking-to-market and due to this (relatively), high frequent data are available.

Consequently, further research should integrate counterparty risk in the valuation of credit derivatives as the isolated modeling of credit risk as discussed above is inadequate in the long run. All these developments should enable us to achieve a more complete approximation of the reality. Finally, it should be noted that a separate analysis of credit and market risk is not sufficient. Therefore, from both a theoretical as well as practical point of view, an integrated approach for the valuation of credit derivatives is urgently needed (Oehler and Unser 2002).

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CHAPTER 3

Integrated Credit Portfolio Management: A Preview

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3.1 INTRODUCTION

The recent development in credit markets is characterized by a flood of innovative credit risky structures. State-of-the-art portfolios contain derivative instruments ranging from simple, nearly commoditized contracts such as credit default swap (CDS), to first-generation portfolio derivatives such as first-to-default (FTD) baskets and collateralized debt obligation (CDO) tranches, up to complex structures involving spread options and different asset classes (hybrids). These new structures allow portfolio managers to implement multidimensional investment strategies, which seamlessly conform to their market view. Moreover, the exploding liquidity in credit markets makes tactical (short-term) overlay management very cost efficient. While the outperformance potential of an active portfolio management will put old-school investment strategies (such as buy-and-hold) under enormous pressure, managing a highly complex credit portfolio requires the introduction of new optimization technologies (Bluhm and Overbeck, 2007; Felsenheimer et al., 2005).

New derivatives allow the decoupling of business processes in the risk management industry (in banking, as well as in asset management), since credit treasury units are now able to manage specific parts of credit risk actively and independently. The traditional feedback loop between risk management and sales, which was needed to structure the desired portfolio characteristics only by selective business acquisition, is now outdated. Strategic cross asset management will gain in importance, as a cost-efficient overlay management can now be implemented by combining liquid instruments from the credit universe, e.g., the iTraxx Future, with well-established instruments, like the DAX Future.

In any case, all these developments force portfolio managers to adopt an integrated approach. All involved risk factors (spread term structures including curve effects, spread correlations, implied default correlations, and implied spread volatilities) have to be captured and integrated into appropriate risk figures. In this chapter, we present an idea how such an integrated portfolio approach can be achieved. We have a look on the iTraxx Future as the core investment, constant proportion debt obligations (CPDOs) as a leveraged exposure on credit indices, constant proportion portfolio insurance (CPPI) as a capital guaranteed instrument, CDO tranches to tap the correlation market, and equity futures to include exposure to stock markets in the portfolio (Brommundt et al., 2006).

For an integrated credit portfolio management approach, it is of central importance to aggregate risks over various instruments with different payoff characteristics. In this chapter, we will see that a state-of-the-art credit portfolio contains not only linear risks (CDS and CDS index contracts, such as the iTraxx and the CDX) but also nonlinear risks (such as FTD baskets, CDO tranches, or credit default swaptions). From a practitioner's point of view there is a simple solution for this risk aggregation problem, namely delta-gamma management. In such a framework, one approximates the risks of all instruments in a portfolio by its first- and second-order sensitivities and aggregates these sensitivities to the portfolio level. Apparently, for a proper aggregation of risk factors, one has to take the correlation of these risk factors into account. However, for credit risky portfolios, a simplistic sensitivity approach will be inappropriate, as the following list of key characteristics of credit portfolio risks shows:

- Credit risky portfolios usually involve a larger number of reference entities. Hence, one has to take a large number of sensitivities into account. However, this is a phenomenon that is already well known from the management of stock portfolios. The solution is to split the risk for each constituent into a systematic risk (e.g., a beta with a portfolio hedging tool, such as the iTraxx) and an alpha component which reflects the idiosyncratic part of the risk.
- However, in contrast to equities, credit risk is not one dimensional (i.e., one risky security per issuer) but at least two dimensional (i.e., a set of instruments with different maturities). This is reflected in the fact that there is a whole term structure of credit spreads. Moreover, taking also different subordination levels (with different average recovery rates) into account, credit risk becomes a multidimensional object for each reference entity.
- While most market risks can be satisfactorily approximated by diffusion processes, for credit risk the consideration of events (i.e., jumps) is imperative. The most apparent reason for this is that the dominating element of credit risk is event risk. However, in a market perspective, there are more events than the ultimate default event that have to be captured. Since one of the main drivers of credit spreads is the structure of the underlying balance sheet, a change (or the risk of a change) in this structure usually triggers a large movement in credit spreads. The best-known example for such an event is a leveraged buyout (LBO).
- For credit market players, correlation is a very special topic, as a central pricing parameter is named implied correlation. However, before we jump into implied correlation, we should mention that there are two kinds of correlation parameters that impact a credit portfolio: price correlation and event correlation. While the former simply deals with the dependency between two price (i.e., spread) time series under normal market conditions, the latter aims at describing the dependency between two price time series in case of an event. In its simplest form, event correlation can be seen as default correlation: what is the risk that company B defaults given that company A has defaulted? While it is already very difficult to model this default correlation (note that there are numerous models that deal with this topic), for practitioners event correlation is even more complex, since there are other events than just the default event, as already mentioned above. Hence, we can modify the question above: what is the risk that spreads of company B blow out given that spreads of company A have blown out? In addition, the notion of event correlation can also be used to capture the risk in capital structure arbitrage trades (i.e., trading stock versus bonds of one company). In this example, the question might be: what is the risk that the stock price of company A jumps given that its bond spreads have blown out? The complicated task in this respect is that we do not only have to model the joint event probability but also the direction of the jumps. A brief example highlights why this is important. In case of a default event, spreads will blow out accompanied by a significant drop in the stock price. This means that there is a

negative correlation between spreads and stock prices. However, in case of an LBO event, spreads will blow out (reflecting the deteriorated credit quality because of the higher leverage), while stock prices rally (because of the fact that the acquirer usually pays a premium to buy a majority of outstanding shares). In this example, there is a positive correlation between spreads and stocks.

This list shows that a simple sensitivity approach—e.g., calculate and tabulate all deltas and gammas and let a portfolio manager play with this list—is not appropriate. Further risk aggregation (e.g., beta management) and risk factors that capture the event risk are needed. For the latter, a quick solution is the so-called instantaneous default loss (IDL). The IDL expresses the loss incurred in a credit risk instrument in case of a credit event. For single-name CDS, this is simply the loss given default (LGD). However, for a portfolio derivative such as a mezzanine tranche, this figure does not directly refer to the LGD of the defaulted item, but to the changed subordination of the tranche because of the default. Hence, this figure allows one to aggregate various instruments with respect to credit events. Moreover, taking also the time horizon into account—i.e., the term structure of the IDL—we can obtain very valuable information about the timing problem of a credit event.

A simple example will shed some light. Let us assume that we have an FTD basket in our portfolio that, among others, refers to company A and has a remaining time to maturity of 5 years. In addition, we also have a forward CDS contract in our portfolio, which refers to buying protection in company A. The forward CDS contract has a forward start date in 1 year and an expiry date in 6 years. By aggregating both IDLs under the assumption that both contracts refer to the same notional amount, we can show that our portfolio is only prone to default risk of company A for one more year. Thereafter, we hedged against a default event for the following 4 years, and even over-hedged between 5 and 6 years.

3.2 INSTRUMENTS

3.2.1 Single-Name CDS and Corporate Bonds

Single-name CDSs can be considered as highly commoditized credit risk instruments which enjoy an even higher liquidity than the underlying bonds. In an integrated credit portfolio management approach, these instruments are the nuclei of the risk portfolio. They serve as underlyings of structured portfolio derivatives, such as CDS index contracts (iTraxx and CDX), FTD baskets, and CDO tranches, and as hedging tools for managing the idiosyncratic risks within credit portfolio derivatives.

A CDS is a bilateral over-the-counter (OTC) contract which transfers the credit risk from the protection buyer to the protection seller. As compensation for taking the credit risk, the protection seller usually receives quarterly premium payments. It is important to note that in liquid markets, CDS contracts can easily be terminated before maturity which involves a termination fee, reflecting the fair value at the closing date, comparable to interest rate swaps. Hence, a CDS contract is not only a buy-and-hold contract, but one which allows active trading, even with shorter time horizons. Current markets are so liquid—which means that bid/ask spreads are so narrow—that a trading time horizon of a

few days may already offer attractive opportunities, especially in more volatile markets, such as the crossovers. As a consequence, the CDS instruments can be used to implement an active management style which does not only focus on buy-and-hold investments, but also on more active strategies in which credit risky positions are managed on a shorter time horizon. This also means that the underlying risk perspective has to be adapted. While in a pure buy-and-hold portfolio default risk is the major focus, active credit portfolio management concentrates more on the mark-to-market risk.

3.2.2 iTraxx Future

In March 2007, the Eurex introduced credit future contracts referring to the iTraxx EUR indices. Also, the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME) are currently thinking about the introduction of exchange traded credit derivatives contracts. The first and most important aspect about the iTraxx Futures contract is that it is not a futures contract in the usual sense, as it does not have a forward payoff profile. It can be viewed as a standardized exchange traded total return index on an unfunded underlying. In contrast to a forward CDS contract, the iTraxx Futures involves during the holding period premium payments (however, not as real cash flows but as accruals), and default risk (forward CDS is usually knock-out-on-default contracts).

The iTraxx Futures contract is quoted in (dirty) price terms, which reflects the following:

- Accrued premium (referring to the deal spread of the underlying swap contract)
- Value change in the underlying iTraxx swap contract (because of spread changes)
- Losses owing to a credit event (including the recovery rate)

Regarding the CDS premium, it is important to note that there will be no real payments booked during the lifetime of the contract. Hence, in contrast to the iTraxx swap contract, in which premiums are paid quarterly, the future contract accounts for premiums only by accruing them over the entire lifetime. In technical terms, the underlying of the future can be viewed as a swap contract with a long first coupon where the pay-date of the premium is the expiry date of the future.

The future price consists of four components.

- Basis: The basis of the futures contract is given by the sum of the weightings of the non-defaulted underlyings. In case of no defaults, the basis is 100. In case of a default event, it will be reduced.
- Accrued premium: The premium of the futures contract starts to accrue from the effective date of the underlying iTraxx contract.
- (Clean) Value: The clean value component of the futures contract reflects the present value change in the contract owing to spread changes in the underlying iTraxx. It can be approximated by: Spreadchange $\times$ DV01.
- Recovery rate: In case of a default event, the recovery rate for the defaulted entity has to be reflected in the iTraxx Futures price.

The valuation formula used for the iTraxx Futures can be summarized as follows:

$$\underbrace{\underbrace{(\text{Basis} + \text{Recovery}) + \text{CleanPV}}_{\text{clean}} + \text{Accrued}}_{\text{dirty}} \tag{3.1}$$

Credit futures contracts are ideal instruments to buy systematic credit risk in a cost-efficient way. We view it as a core investment in our portfolio construction process.

3.2.3 CPDOs

The CPDO hype in 2006 raised some questions about the basic construction principles, risk factors, and potential improvements. In respect of the construction principle, the leverage mechanism was a major concern regarding first-generation CPDOs. At a first glance, these structures will come under pressure in case the positive carry and roll-down effects will be offset by wider spreads and hence the net asset value of the structure declines. However, back-testing showed that CPDOs even survived 2001/2002 scenarios as a dramatic widening was offset by the following strong tightening which more than offset the widening as leverage increased. Consequently, cash-in events of potential CPDO transactions issued before 2001/2002 took place within the maturity of the structures.

In addition, the assumption of a normally shaped credit curve has a strong impact on the expected performance of CPDOs. Assuming a roll-down of 2.5 bp for half a year, the performance impact, e.g., based on an initial average spread level of 25 bp (CDX and iTraxx), amounts to around 11 bp. However, in case of curve inversion, this positive performance contribution could move into negative terrain. The assumption of rolling down the curve is closely coupled with the assumption of a positive roll spread. A positive roll spread triggers increasing spread income at every roll-date, just by extending the maturity of the underlying contract. A positive roll spread is obviously based on the idea that no distressed debt emerges in the on-the-run period of a series. Assuming some names drop from investment grade to the sub-investment grade area, the roll spread might be negative. This means that leverage has to be higher in the new series to generate enough carry.

The leverage mechanism in a CPDO is constructed in a way that the relation between leverage (as a multiple of the underlying credit indices) and the net asset value of the portfolio remains constant.

One of the major concerns regarding the CPDO structures remains static, purely formula-based character with respect to the exposure to the risky underlying. As a result, the structure may underperform when rolling the underlying CDS index positions from an old series to a new one, as markets might adapt to this forced-to-roll behavior. Another negative effect of the purely rules-based trading strategy is that the CPDO might suffer in a falling angel scenario. In this case, the underlying risky asset incurs mark-to-market losses (because of the spread widening in the old series driven by the falling angels), which cannot be offset by a higher carry income in the new series, as the downgraded high

spread names were excluded from the new index. The next generation of CPDO structures is trying to cope with the potential weaknesses of the transactions we have seen in 2006. The second generation of CPDOs can be divided into adjusted rules-based and managed deals.

A CPDO is designed to take a leveraged long exposure on credit markets, with mark-to-market risk clearly dominating default risk. That said, we can use CPDOs as a satellite investment which allows us to take a leveraged view on directional moves in the credit market.

3.2.4 CDO Tranches

The iTraxx Europe universe comprises a series of tradable and standardized tranches offering leveraged exposure to the European high-grade credit market. Besides long positions, shorting tranches is also possible.

The tranches are split up to 0%–3% (equity), 3%–6% (BBB), 6%–9% (AAA), 9%–12% (junior super senior low), 12%–22% (junior super senior high), and 3%–100% (investment grade), with the percentage related to total loss. This means that in case 10% of all index constituents have a credit event and a recovery value of 80%, the first-loss-piece is still intact as total loss is only 2%. A new series will be launched every 6 months, while future attachment points could change in line with the risk profile of the underlying index. From a risk and return profile, iTraxx tranches equal other investment grade CDOs. The most important difference is high liquidity in iTraxx tranche products, while cash CDOs and bespoke transactions are rather a buy-and-hold investment. While all tranches above the equity piece are quoted in standard spread terms, the price quotation for the 0%–3% tranche is quite different. An iTraxx equity tranche always pays a constant spread of 500 bp on the (remaining) notional amount, while market makers quote a percentage of the notional amount that has to be paid as an up-front payment when entering into such a contract as a protection buyer. As tranche pricing is highly complex and directly linked to several risk factors (directional market moves, implied correlation, recovery assumptions, spread volatility, etc.), iTraxx tranches offer an attractive opportunity for pure credit derivative players rather than for credit portfolio managers.

The main risk factor in the tranched iTraxx universe—besides the underlying credit spreads—is default correlation among the reference entities. A high default correlation indicates a higher risk of joint defaults, which may erode higher protected tranches of the capital structure. The value of the tranches and, accordingly, the spread paid to investors are highly sensitive to default correlation. Therefore, trading CDO tranches is often referred to as correlation trading. An unhedged position in a CDO tranche indicates a market view on individual credit spreads and default risk in the underlying credits and a view on joint default risk in the pool. Usually, default correlation is an input parameter for a complex model, which results in a price. For iTraxx tranches, it is the other way around. Because of high liquidity in the market, prices (i.e., tranche spreads) are driven by supply and demand. Consequently, the level of default correlation can be extracted from these prices. Thus, the name of the game is price discovery (implied correlation) rather than model-based pricing.

By eliminating spread risk (delta hedging), a tranche position is primarily sensitive to changes in the implied correlation. Delta-hedging means that we enter the opposite position in the underlying index to immunize the tranche position (ignoring gamma) against spread moves in the underlying market. Hence, an investment in iTraxx tranches allows us to take exposure to default correlation.

3.2.5 Credit Default Swaptions

Liquid iTraxx default swaptions allow investors to implement simple hedging strategies (portfolio insurance/protective put), and also more complex strategies (straddles, strangles, butterflies, etc.). Standardized spread options are available for the iTraxx Europe and the iTraxx Crossover. There are payer options (calls) and receiver options (puts), with different strike prices, as well as straddles (combining receiver and payer options). The expiration dates are scheduled for 3, 6, and 9 months at the introduction date of a new series. Spread options are quoted in basis point, with implied at-the-money volatility being quoted, as well.

Credit default swaptions can be traded on a single-name and on an index basis. A single-name credit default swaption is an option to enter into a standard single-name CDS contract, while its index counterpart is an option to enter into a CDS index contract, such as the iTraxx. Both instruments are quite similar, but there is an important difference. While standard single-name swaptions do not offer default protection until the option's expiration (knock-out feature), options on credit indices are not knocked out by a default event in the underlying basket, and hence provide default protection also during the lifetime of the option.

The designation of the option type (call or put) is a bit confusing. Market participants use the same notation as for interest rate swaptions, by referring to the premium leg of the CDS. A payer credit default swaption refers to an option to buy protection at a fixed CDS spread at maturity, i.e., it is a put option on the value of the credit risk. It is named payer default swaption because the investor has to pay the premium in the underlying CDS. A receiver swaption refers to selling protection. The analogy to stock options can be somehow misleading when we use the common perspective of an investor who sells protection (and receives the premium payments). The value of such a CDS position declines if the CDS spread increases. Hence, a receiver option offers a positive payoff in case the spread increases. However, for increasing spreads, the value of the CDS decreases. A receiver swaption can therefore be considered a put option on the credit risk.

Because of the knock-out feature, credit default swaptions have to be European-style rather than American or Bermudan. Think of an American-style spread option, one which can be exercised at any time. The owner of a payer option, which refers to buying protection, can—and surely will—exercise his option shortly before default when financial distress of the reference credit has already become apparent. Hence, an American-style option is one that includes default protection owing to technical reasons, unless there are strict rules that circumvent the aforementioned exercise strategy. For receiver default swaptions (selling protection), the knock-out-on-default feature is not required. The holder of the option will not exercise it in case of a default event, as this would mean

selling protection on an already defaulted name. A receiver default option will only be exercised if the reference spread is tighter than the strike spread.

Credit default swaption is sensitive to spread volatility. By using the concept of delta-hedging (entering the delta-adjusted opposite position in the underlying index), we can extract spread volatility from credit default swaption. Therefore, we can use this instrument to take exposure to spread volatility without being exposed to directional spread risk.

3.2.6 First-to-Trigger Baskets

An equity default swap (EDS) is similar to a CDS contract from the construction principle, except for a major difference. The trigger event is not a classical credit event (default, failure-to-pay, and restructuring) but a certain drop of the share price. Most of the contracts are standardized with the threshold levels being fixed at 30% (a drop of the share price by 70%, respectively). The standardized recovery rate is 50%.

The payoff structure of an EDS contract is closely related to that of a CDS contract. The protection seller receives the EDS premium payments until a trigger event occurs, which triggers the conditional payment—the so-called default payment—by the protection seller to the protection buyer. However, there are two differences between EDS and CDS contracts: the trigger event and the recovery payment. An EDS default is triggered if the share price hits a prespecified default barrier—usually 30% of the initial share price. This is in contrast to a plain-vanilla CDS where default is triggered on the occurrence of a credit event, i.e., bankruptcy, failure-to-pay, and restructuring. However, as the probability that the company's stock price drops more than 70% in case a credit event occurs is quite large, the EDS can be viewed as a kind of CDS-plus.

The EDS provides protection against a significant drop in the share price and against credit risk (through the correlation between the risk of a jump in stock prices and default events). Hence, the EDS payoff resembles an American-style far out-of-the-money put option with swap-styled premium payments. Nevertheless, from a contractual point of view, a credit event may occur without an accompanied dip in the share price, triggering the CDS contract, but leaving the EDS contract intact, which is however unlikely. On the other hand, the EDS barrier can be triggered without the occurrence of a credit event. Another difference between EDS and CDS contracts is that the conditional payment in an EDS contract is fixed—usually 50% of the contract's notional amount—while the loss-given-default for a standard CDS is unknown in advance, as it is determined by the recovery value of the reference asset.

All the well-known instruments of the credit derivatives world can be translated into the EDS framework. The most basic correlation product in the credit world is an FTD basket, while its pendant in the equity EDS methodology is a first-to-trigger basket (FTT). The underlying mechanism of an FTT is similar to an FTD, except for the trigger event and the fixed recovery assumption. An FTT offers access to correlation within the equity universe.

By imposing the nth-to-default concept on EDS, we can implement equity-linked risks in a credit portfolio, which allows us to take a view on the capital structure of specific companies, which brings us back to Merton's (1974) initial idea of how to link equity and debt markets. An FTT is sensitive not only to a dramatic slump of share prices but also to

the correlation (comovement) of stock prices. As mentioned above, a more pronounced drop of share prices should be directly linked to default probability.

3.3 IMPLICATIONS FOR CREDIT PORTFOLIO MANAGEMENT

3.3.1 Sensitivity Analysis

The major problem is that we do not have only one risk parameter (e.g., volatility), but several sensitivities which, additionally, are correlated. We assume a portfolio consisting of iTraxx Futures as a core investment. Our satellites in these portfolios are CPDOs (highly spread-sensitive), correlation sensitive single-tranche CDOs (STCDOs), credit default swaptions (spread volatility), and FTTs (indirectly default sensitive). To optimize a portfolio including structured instruments, basic risk-return optimization will fail to generate satisfying results. The major problem is that we do not have only one risk parameter (e.g., volatility), but also several sensitivities which, additionally, are correlated. The first step of analyzing a structured portfolio is to extract sensitivities, shown in Table 3.1.

3.3.2 Optimization Process

A pure risk-return approach as introduced by Markowitz does not fit with such a multidimensional optimization problem. To cope with such complex portfolios, we have to take into account not only that the portfolio as a whole is sensitive to specific risk factors, but also that sensitivities are correlated. Rising default rates, e.g., do not only have an impact on default-risk instruments as such a scenario will be accompanied by wider credit spreads, but also affect primarily spread-sensitive instruments. That said, the portfolio of optimization is directly linked to the instruments included in the portfolio. The instruments introduced above argue against the efficiency of the implementation of

TABLE 3.1 Sensitivities of Different Credit Risky Instruments

Risk Type	Single-Name	Portfolio
Linear spread risk	• Credit default swap (CDS) • Digital default swap (DDS) • Forward CDS • Constant maturity CDS	• CDS indices (iTraxx, CDX) and Futures • Bespoke linear baskets (e.g., U.S. banks) • Leverage note (e.g., CPDO) • Dynamic CPPI structures
Default correlation		• First-to-default (FTD) baskets • Standard tranches (iTraxx, CDX) • Bespoke collateralized debt obligation (CDO) • CDO^2 • Equity tranches (POETs), tranchelets, etc.
Spread volatility	• Credit default swaptions	• Credit default swaptions, variance swaps • Options on Futures and total return locks
Hybrid risk	• Hybrid capital (financial/nonfinancials) • Subordinated CDS • Capital structure arbitrage	• First-to-trigger baskets (FTTs) on equity default swaps (EDS) • Collateralized fund obligations (CFO)

a traditional CAPM (capital asset pricing model) approach to generate individually optimized portfolios.

3.3.3 Replacement of the CAPM Regime by the CFO Regime

Recent developments in the credit derivatives universe not only have a limited impact on the credit arena itself, but will also affect investment behavior in other market segments. In the long term, we think that the well-established CAPM regime will be succeeded by the CFO (collateralized fund obligation) regime. The replacement of the CAPM regime by the CFO regime has a significant impact on portfolio management, on product development, and on the mainstream view on correlation.

A major trend with respect to product development is to impose credit derivatives technology on other asset classes. CDOs on EDS, collateralized commodity obligation (CCO, commodity swap CDOs), or foreign exchange CDO (CFXO) have already been introduced, reflecting the trend toward pricing established asset classes in a new and innovative format. We think that this will be a lasting trend rather than a short-lived one. In the long term, this so-called CFO regime is expected to replace old-fashioned risk-return optimization.

In the CAPM world, the risk-return optimization is already implemented in the market portfolio "M" which is defined as being efficient, while correlation between the portfolio constituents is simply the comovement of asset prices. The individual risk perception is simple, reflected by the investment share of the market portfolio and the risk-free asset (RF). In a CFO, the underlying portfolio is selectable without μ/σ—constraints, and there is no investment in RF. Correlation is rather joint default probability, while the specific tranche investment is done in line with the individual risk perception (Figure 3.1).

CFOs are smoothing the volatility of assets, while correlation as defined as comovement of assets (volatility-driven CAPM regime) will be replaced by a correlation of tail events (event-risk-driven CFO regime). How are tail events correlated? Obviously, systematic shocks will be a crucial risk factor to a CFO, while market fluctuations and cyclical moves

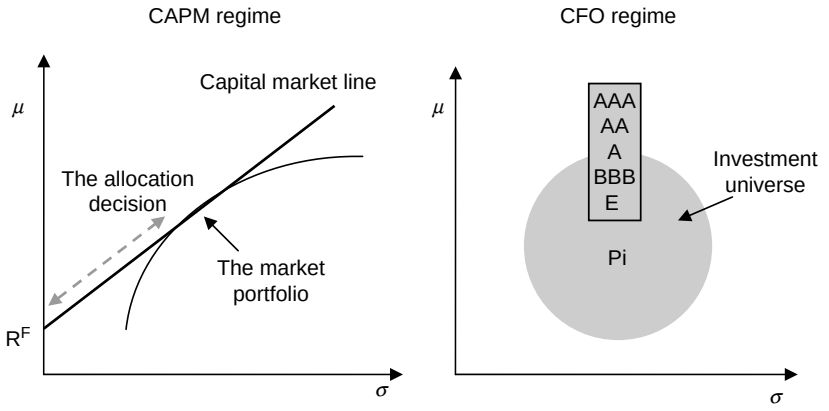


FIGURE 3.1 CAPM versus CFO regime.

are of minor importance for the performance of a CFO. Idiosyncratic risk factors are losing in importance versus systematic risk factors. The risk and return profile of CFO tranches differs significantly from the risk and return profile of the underlying assets, shifting risk away from pure volatility toward default. The effect on asset management is obvious: pure risk-return optimization will lose in importance versus structured investment portfolios, as, e.g., managed by Credit Derivatives Product Company (CDPC).

3.4 LATEST DEVELOPMENTS IN THE SYNTHETIC CREDIT MARKET

The structural change in the structured credit universe continues to accelerate. While the market for synthetic structures is already pretty well established, many real money accounts remain outsiders owing to regulatory hurdles and technical limitations, e.g., to participate in the correlation market. Therefore, banks are continuously establishing new products to provide real money accounts with access to the structured market, with CPDOs recently having been the most popular example. Against this background, three vehicles which offer easy access to structured products for these investors have gained in importance: CDPCs, PCVs (permanent capital vehicle), and SIVs (structured investment vehicles).

A CDPC is a rated company which buys credit risk via all types of credit derivative instruments, primarily super senior tranches, and sells this risk to investors via preferred shares (equity) or subordinated notes (debt). Hence, the vehicle uses super senior risk to create equity risk. The investment strategy is a buy-and-hold approach, while the aim is to offer high returns to investors and keep default risk limited. Investors are primarily exposed to rating migration risk, to mark-to-market risk, and, finally, to the capability of the external manager. The rating agencies assign, in general, an AAA-rating on the business model of the CDPC, which is a bankruptcy remote vehicle (special purpose vehicle [SPV]). The business models of specific CDPCs are different from each other in terms of investments and thresholds given to the manager. The preferred asset classes CDPC invested in are predominantly single-name CDS, bespoke synthetic tranches, ABS (asset-backed security), and all kinds of CDOs. So far, CDPCs main investments are allocated to corporate credits, but CDPCs are extending their universe to ABS and CDO products, which provide further opportunities in an overall tight spread environment. The implemented leverage is given through the vehicle and can be in the range of 15–60x. On average, the return target was typically around a 15% return on equity, paid in the form of dividends to the shareholders.

In contrast to CDPCs, PCVs do not invest in the top of the capital structure, but in equity pieces (mostly CDO equity pieces). The leverage is not implemented in the vehicle itself as it is directly related to the underlying instruments. PCVs are also set up as SPVs and listed on a stock exchange. They use the equity they receive from investors to purchase the assets, while the return on their investment is allocated to the shareholders via dividends. The target return amounts, in general, to around 10%. The portfolio is managed by an external manager and is marked-to-market. The share price of the

company depends on the net asset value (NAV) of the portfolio and on the expected dividend payments.

In general, an SIV invests in the top of the capital structure of structured credits and ABS in line with CDPCs. In addition, SIVs also buy subordinated debt of financial institutions, and the portfolio is marked-to-market. SIVs are leveraged credit investment companies and bankruptcy remote. The vehicle issues typically investment-grade rated commercial paper, MTNs (medium term notes), and capital notes to its investors. The leverage depends on the character of the issued note and the underlying assets, ranging from 3 to 5 (bank loans) up to 14 (structured credits).

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CHAPTER 4

Credit Default Swaps and an Application to the Art Market: A Proposal

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4.1 INTRODUCTION

Wealthy individuals are often asset rich and cash poor. With their capital tied up in assets they tend to want access to loans backed by assets. Asset-backed securities (ABS) were first

introduced in the United States in the 1970s in the form of Mortgage backed securities. More recent forms are Automobile backed securities and Credit-Card sacked securities. These loans all have the underlying asset as the collateral in the case of credit default. However, the move toward loans backed by artworks has been much slower in evolving. This is mainly due to the inability to correctly assess the risk from changes in the price movements of art. Since the collection of a number of price indices this is now becoming easier. Price movements may be more easily tracked for the art market in general. However, the art market is made up from heterogeneous goods, where the value of an individual artwork is still very difficult to evaluate.

In the arts sector a famous example of securitization was the issuance of loans using royalties as cash flows, consequently named Bowie Bonds after singer David Bowie who securitized 10 years worth of royalties. Another example is the recent move by some private banks to make loans against yachts as collateral.

The major problem for banks in the introduction of plain vanilla art-backed securities is the management of the art risk. At present the market is highly illiquid, which makes daily management of art risk almost impossible at market prices; a requirement of the Basel accord for banks' capital requirements.

However, through the introduction of art credit default swap contracts (ACDS) banks can transfer undesirable art risk from holding artworks on the banks' balance sheet to a third party that is willing to hold the art price risk. In return for doing so, a premium is received. The risk from incorrectly pricing the derivative contract is substantially smaller than the risk from holding the underlying art risk, providing an efficient solution to transferring risk for the banking sector. We explore this concept in this chapter.

The outline of the chapter is as follows. In the following section we introduce art as an alternative asset class, and the characteristic risk and return profile of art. In Section 4.3 we discuss art lending and the inefficient nature of the current practice. In Section 4.4 we show how risk management tools can be adopted to transfer undesirable art price risk from banks to third parties who are willing to protect the bank against the credit default risk on art-backed loans because they are willing to buy the art held as collateral at a guaranteed price in the case of default. This insures the bank against adverse movements on art markets, in times of high credit default probabilities. The introduction of this financing structure would result in a more efficient market for lending against art. The mechanism behind the trilateral financing structure is addressed in Section 4.5, with the introduction of ACDS contracts. Pricing equations for these ACDSs are introduced in Section 4.6 and an example is given. Conclusions are then drawn in the final section, Section 4.7.

4.2 ART AS AN ASSET CLASS

There have been a variety of studies devoted to the estimation of art indices to gauge average returns in the art market. Data goes back as far as the seventeenth century. "Art" is far too general a term to generate any meaningful analysis, therefore, much as traditional assets are categorized, we limit the discussion of art to visual fine art: paintings, drawings, and sculpture rather than the performing arts.

Art price indices have been created using three main methodologies: average prices, repeat sales regression, and hedonic regression. Chanel, Gerard-Varet, and Ginsburgh's study indicates that over long periods the respective methodologies are closely correlated (Chanel et al. 1996). Also see Ginsburgh, Mei, and Moses (2006) who look at the differences in the modeling processes of the various types of index construction. Goetzman (1993) points out the selection bias derived from looking only at repeat sales on auction house data. Repeat sales regressions require artworks to put on the block at least more than once to be included as a repeat sale. It is thought that artworks which fall drastically in value tend not to be resold at auction.

The information from databases that collect art sales information is problematic for a number of reasons (Ashenfelter and Graddy, 2003). Ashenfelter and Graddy's (2003) study contends that an empirical discrepancy in one year can materially alter the overall rate of return by up to 5%. Campbell and Pullan (2005) also find evidence of this phenomenon for the more recent Mei & Moses All Art Index compared to the General Art Index of Art Market Research for the period 1976–2004.

In Table 4.1 we have provided a summary of the approaches to data collection, extended from an excellent survey by Ashenfelter and Graddy (2003). Average real returns are

TABLE 4.1 Estimated Art Returns and Standard Deviation as Reported by Various Papers

Author	Sample	Period	Method	Nominal Return (%)	Real Return (%)	Standard Deviation (%)
Baumol (1986)	Paintings in general	1652–1961	RSR		0.60	
Frey and Pommerehne (1989)	Paintings in general	1635–1949	Hedonic		1.40	
		1653–1987	Hedonic		1.50	5.00
		1950–1987	Hedonic		1.70	
Buelens and Ginsburgh (1992)	Paintings in general	1700–1961	Hedonic		0.91	
	Paintings in general	1780–1970	RSR	3.70	3.00[a]	
Goetzmann (1993)	Paintings in general	1716–1986	RSR	3.20	2.00[a]	5.65
		1850–1986	RSR	6.20	3.80	6.50
		1900–1986	RSR	17.50	13.3	5.19
Anderson (1974)	Paintings in general	1780–1960	Hedonic	3.30	2.60[a]	
		1780–1970	RSR	3.70	3.00[a]	
Chanel, Gerard-Varet and Ginsburgh (1996)	Paintings in general	1855–1969	Hedonic		4.90	
		1855–1969	Repeat sales		5.00	
Mei and Moses (2002)	American, impressionist, and old masters	1875–1999	RSR		4.90	4.28
		1900–1986	RSR		5.20	3.72
		1900–1999	RSR		5.20	3.55
		1950–1999	RSR		8.20	2.13
		1977–1991	RSR		7.80	2.11

(*continued*)

TABLE 4.1 (continued) Estimated Art Returns and Standard Deviation as Reported by Various Papers

Author	Sample	Period	Method	Nominal Return (%)	Real Return (%)	Standard Deviation (%)
Goetzmann (1996)	Paintings in general	1907–1977	RSR		5.00	
Fase (1996)	Nineteenth century	1946–1966 1972–1992		11.00 10.60	7.50 1.10	
Stein (1977)	Paintings in general	1946–1968	Geometric mean	10.47		
Barre, Docclo and Ginsburgh (1996)	Great impressionist	1962–1991	Hedonic	12.0	5.00[a]	
	Other impressionist	1962–1991	Hedonic	8.00	1.00[a]	
Czujack (1997)	Picasso paintings	1966–1994	Hedonic		8.30	
Deutschman (1991)	Old masters	1971–1991		12.30	6.04	
Angnello and Pierce (1996)	Nineteenth century United States	1971–1992		9.30	3.25	
Campbell (2005)	Paintings in general	1976–2004	Average prices	5.73	1.44	8.27
	U.S. paintings	1976–2004	Average prices	7.94	3.66	8.73
Pesando (1993)	Modern prints	1977–1992	RSR		1.51	19.94
Pesando and Shum (1996)	Picasso prints	1977–1992	RSR	12.00	2.10	23.38
Frey and Serna (1990)	Old masters	1981–1988	Hedonic	10.59	3.20	
Candela and Scorcu (1997)	Modern contemporary paintings	1983–1994		3.89	0.21	

[a] Real returns estimated additionally by Ashenfelder & Graddy.

generally regarded to be significantly lower than for stocks, but higher than inflation and government bonds. Table 4.1 provides an overview of the average returns and standard deviations for the main studies cited in the literature for art price indices which have been constructed.

Behavioral aspects have implications for the cyclicality of the indices. In a boom, artworks sell quickly at prices above their reserve prices and often far above price estimates. In periods of downturn or in bust, artworks not reaching reserve prices are "bought in" without sale. These observations result in prices being less flexible downwards than buyer's offers. McAndrew and Thompson (2007) try and model the left hand tail of the distribution to produce a continuous distribution for modeling art risk for risk management purposes. The resulting illiquidity observed at times on the market has resulted in a lack of loans being provided using art as collateral. However, are the extremely high spreads observed reflective of this high liquidity risk observed in the market?

4.3 ART LENDING

At present the market for lending against art is relatively inefficient. This is partly due to the inefficient nature of the art market, and naturally the lack of liquidity in the market. This means that the use of art-backed loans is a highly risky form of loan securitization, with a low percentage of the artworks value being used as a guarantee for the loan, and also large spreads appointed to credit. In a bilateral agreement for the case of a loan using art as collateral, the bank would normally assign a higher interest rate on the loan as compensation for holding the art price risk stemming from holding art as collateral on its books. Given the highly volatile, illiquid, and transparency lacking global art market, the rate of interest on the loan would certainly occur at a premium. For an overview of art lending practices currently on offer, see Table 4.2.

TABLE 4.2 Art Lending Practices

Company	Rates	Terms	Caveat
Art Capital Group Inc.	8%–12% for bridge loans. Rates are in mid-teens where artwork is only collateral	Offers nonrecourse or pawn-shop deals, where art is used as sole collateral. Loan can be for 3 months or 3 years, depending on the situation. Loan is typically up to 50% of the low auction estimate	Art Capital Group must have verified ownership and reliable auction estimate
Sotheby's	Prime +300 bp with some flexibility	Loan is 40% of the low auction value of art or collection	Minimum loan is $1 million. With few exceptions, the company no longer gives loans to customers unless they plan to sell the art through Sotheby's
Christie's	Varies by client	Loan is up to 50% of the estimated auction value. Requires collateral of 2–2.5 times the amount of the loan	Company offers service to long-term clients and borrowers who intend to sell through Christie's
Citigroup Private Bank	Average is 225 bp above London Interbank offered rate, depending on the client	Loan is 50% of the value of the artwork. Prefers loans of more than $5 million	Bank typically requires the borrower to be current or potential client of the bank. Usually requires other assets as collateral
Bank of America Private Bank	150 to 300 bp over London Interbank offered rate	Value of loan is up to 50% of the value of artwork	Bank loans mainly to existing customers or prospects of the private bank

For example Art Capital Group appoints rates in the mid-teens for loans using art as collateral. The large spread covers the additional risk of holding art on the balance sheet, and the associated art price risk.* A risk has to be correctly accounted for in daily risk management purposes, as set out in the Basel accords. A further consequence of the greater risk inherent in art prices is that the price at which the art is valued for as collateral is likely to be significantly lower than its market value. Due to inefficient pricing in the art market, a sub-optimal amount of lending occurs in the market for asset-backed loans, when art is used as collateral. The inefficiencies from the art market are likely to be carried over to the financial sector resulting in an inefficient market for asset-backed loans incorporating art as the reference asset in the loan.

By entering into a trilateral agreement, banks are able to transfer the risk of fluctuations in the price of art—art price risk. Therefore, additional capital is not required for Basel requirements, and the only risk which the bank holds on its books is the default risk on the ACDS. The structure is conditional on the bank only entering into an agreement with a credit worthy party. Such a structure would then result in greater efficiency in the capital market for asset-backed loans.

4.4 RISK MANAGEMENT TO TRANSFER ART RISK

In managing the art risk for loans using art as collateral, there is a role for a third party to create greater efficiency in the market. As with all financial engineering, and the resulting products, there is a benefit of transferring risk away from the party not willing to hold the risk, to the party who is willing to hold this risk. One way of transferring the art risk off the banks' balance sheets is through the introduction of a credit option on the art loan.

At present the new regulations on risk management practices, from the Basel accord and Basel II mean that banks need to be able to adequately estimate all risks associated with changes in asset prices. The difficulty in being able to estimate and monitor risks associated with art price changes means that banks are unwilling to hold art price risk on their balance sheets. Since the introduction of the Basel accords, banks prefer to remove liabilities, which require a high allocation of capital. At present banks are required to hold 8% of their liabilities as capital. For more risky positions then a higher percentage is required.

4.5 ART CREDIT DEFAULT SWAPS

Credit default swaps (CDS) are contracts which provide protection on the par value of a specified reference asset, where a *protection buyer* pays a periodic fixed fee or a one off premium to a *protection seller*, in return for which the seller will make a payment on the occurrence of a specified credit event (Hull and White 2000; Chaudhry 2004). In this particular case the bank is the *protection buyer* and another third party, who has an interest

* Citigroup and Bank of America offer existing highly valued private clients a lower spread, often with other assets also required as collateral. Up to 50% of the value of the artwork is adopted as collateral.

in buying the reference asset, is the *protection seller*.* In this case the underlying asset is a specific asset, the artwork or the art collection held as collateral on the loan, commonly known as the reference asset. The third party is willing to accept the art price risk for a premium and enters into a contract with the bank which entitles him to also buy the reference asset at a set price if a credit event is triggered. Here this is the case if the reference entity, the art collector, defaults on the loan. On occurrence of the credit event, the swap contract is terminated and a settlement payment is made by the *protection seller* to the *protection buyer*. The settlement payment in this case is the exchange of the reference asset for a set termination price as specified in the contract from the third party to the bank.

Since the lender has covered his loan with his artwork as collateral, the credit event being triggered is also contingent on the price of the artwork falling below the outstanding loan payments. At the beginning of the loan agreement the value of the artwork to the bank, used as a guarantee, is the current value of the loan. This is likely to be set as a percentage of the current market valuation of the artwork.† Only if the artwork falls below this guaranteed price and the lender defaults on the loan is the credit event triggered and the protection seller has the obligation to buy the artwork at the set price. Since the value of the loan must be less than the market price for the artwork, in order for the credit event to be triggered, the bank with the long position in the ACDS is necessarily out-of-the-money. This is a necessary requirement for the third party to be obliged to fulfill his obligation on the contract and buy the artwork for the guaranteed price. The risk is transferred from the bank as long as the termination price for the artwork is set at least equal to the outstanding value of the loan.

An ACDS can be used to enable banks to transfer unwanted credit risk exposure to a third party. In this case the use of risk management techniques transfers the default risk from the loan backed by art off the balance sheet. Since the art is held as collateral the bank's assets are subject to changes in the value of the artwork, and the bank carries the art price risk. The benefit is that the bank can make a loan using the art as collateral, but could transfer the risk from holding the art as collateral. By writing a CDS on the loan, if default occurs, the bank removes the risk of carrying the art price change on its books. The bank is not required to monitor the underlying value of the art and importantly is not required to be tracked for risk management regulations stipulated in the Basel accords for capital requirements.

The basic premise is of course that the bank is able to find a counterparty for the ACDS. At present parties who would be willing to hold such a derivatives contract would be art funds, (or in the form in which they are more likely to emerge in future, art mutual funds‡), art museums, or from a more speculative disposition, by hedge funds, institutional investors, or wealthy individuals wanting to expose themselves to art price risk; willing

* In a credit default swap "going long" is to buy the swap, where the buyer purchases protection and pays the premium. The buyer of the CDS is frequently referred to in the market as "shorting" the reference asset.

† Such guarantees are commonly used by auction houses as percentages of the market value in auction sales, commonly referred to as the reserve price.

‡ See Campbell and Pullan (2005) for an overview of the potential for the mutual art fund industry.

to pay a low premium for the small probability of being able to buy a wealthy artwork or collection at a fraction of the cost.

The main steps for the financing structure introduced here are as follows:

1. Bank issues a loan to the art collector, with the artwork as collateral. The value of the artwork is estimated, and as is currently the practice, a price is guaranteed on the art as a percentage of the current value. The bank carries the art risk on its books in lieu for giving the loan. Commonly the uncertainty in the value of the art would be reflected in a high interest rate on the loan, manifesting itself in an inefficient market for asset-backed lending.
2. Bank, not wanting to hold the risk of art price fluctuations on its books, issues an ACDS to a third party. If the reference entity (art collector) defaults on the loan the bank has the right to deliver the reference entity to the seller in exchange for its face value. This is set at the face value of the loan.
3. Bank buys protection by taking a long position in the ACDS, but actually shorts the reference asset. The bank therefore transfers the risk of holding the art as collateral in exchange for the credit risk on the ACDS. If the third party is not sufficiently rated, the bank may want to enter into a funded credit agreement, and therefore mitigate all risk on the banks books; otherwise if the party is creditworthy it is likely that the bank will be willing to carry the credit default risk of the ACDS.*
4. On the credit event being triggered the ACDS is terminated and the protection seller (third party) makes a payment to the protection buyer. In this case the bank sells the artwork(s) at the termination value, calculated at the time the contract was specified, to the third party. See Figure 4.1 for an example of the art finance structure. The default occurring is contingent on the market price of the artwork falling below the outstanding loan payments. Otherwise the art is used by the bank in its primary role as collateral, its market price able to cover the value of the outstanding loan.

Another premise for the structure to work is that the ACDS is priced correctly. The risk to the bank is now the correct pricing of the ACDS, which is substantially lower than the risk from holding the artworks on the banks books. In the following section we shall discuss pricing the ACDS.

4.6 PRICING ART CREDIT DEFAULT SWAP DERIVATIVES

In order to correctly price the ACDS, we need to determine the credit spread so that the present value of the total fee payments made by the bank equals the expected credit loss in today's terms. As is common in the literature, default probabilities and interest rates are assumed to be independent. Following Hull and White (2000), the expected loss is a function of the default probability at time t and the amount by which the face value of the loan exceeds the value of the collateral at a particular instant in time, calculated in present terms.

* Many art museums have issued a variety of tax exempt bonds for financing solutions, requiring credit status to be acknowledged.

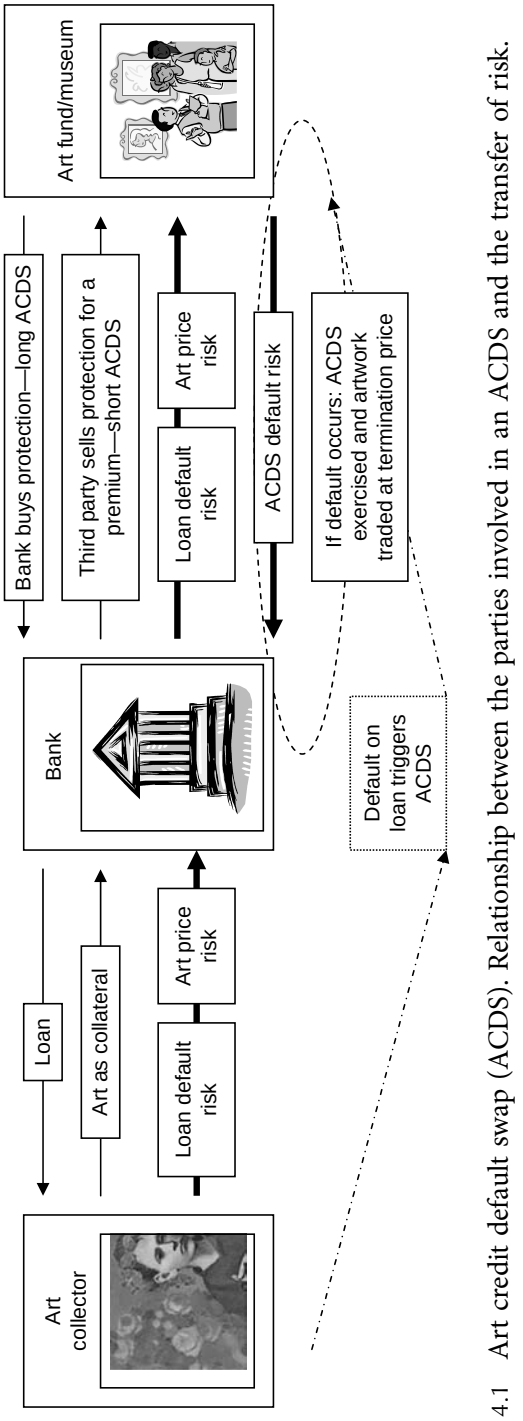


FIGURE 4.1 Art credit default swap (ACDS). Relationship between the parties involved in an ACDS and the transfer of risk.

As an example, consider a loan whose face value is set to x% of the value of the art piece or art collection at $t = 0$. At present the common practice is to use a maximum of 50% of the current market price of the artwork to back the loan. The idea of this financing structure using the ACDS is to mitigate the risk of the art price falling below the loan value. We could therefore use a much higher percentage, x, even up to the full value of the artwork if the collector is of high credit quality. Equation 4.1 shows this relation,

$$P_{t=0}^{\text{Loan}} = x\%\, P_{t=0}^{\text{Art}} \tag{4.1}$$

where $P_{t=0}^{\text{Loan}}$ and $P_{t=0}^{\text{Art}}$ are the prices of the loan and art work at $t = 0$, respectively.

Default is contingent on the full price of the artwork at time, t, falling below the outstanding value of the loan at time t as shown in Equation 4.2.

$$P_t^{\text{Art}} \leq P_t^{\text{Loan}} \tag{4.2}$$

From Equation 4.1 the price of the artwork is simply equal to the guaranteed percentage on the initial value of the artwork, assuming bullet repayment as shown in Equation 4.3.*

$$P_t^{\text{Art}} \leq x\%\, P_{t=0}^{\text{Art}} \tag{4.3}$$

The probability density function of default at time t required for the expected loss determination is the probability that the price of the artwork falls below the percentage of the artwork guaranteed at the initial market value of the artwork at time 0. Formally this distribution function is expressed in Equation 4.4 as q(t).

$$q(t) = \text{Prob}\left(P_t^{\text{Art}} \leq x\%\, P_{t=0}^{\text{Art}}\right) \tag{4.4}$$

In the case of default, the protection seller will be required to pay the difference between the face value of the loan and current market value of the collateral plus any accrued interest to the bank. Assuming (4) to represent a risk-neutral probability density function, this can be discounted at the risk-free rate. Setting

$$v(t) = \begin{cases} e^{-r_f t}\left(\left(x\%\, P_{t=0}^{\text{Art}} - P_t^{\text{Art}}\right) + y \cdot \delta_t \cdot P_{t=0}^{\text{Loan}}\right) & \text{if } P_t^{\text{Art}} \leq x\%\, P_{t=0}^{\text{Art}} \\ 0 & \text{if } P_t^{\text{Art}} > x\%\, P_{t=0}^{\text{Art}} \end{cases} \tag{4.5}$$

where δ_t is the year expressed as a fraction of time between 0 and 1, since the last interest payment and y is the interest rate charged on the loan to the art collector, the expected loss can be expressed by Equation 4.6.

$$\text{ECL} = \int_0^{\text{T}} q(t) \cdot v(t)\, \text{d}t \tag{4.6}$$

* If this happens the borrower would rationally default on the loan. Status in holding the art and reputation may result in him not actually defaulting.

As noted above, the expected credit loss (ECL) should equal the present value of the fee payments. With payments being made until default or maturity of the ACDS, whichever occurs earlier, the costs of default can be expressed as a weighted average of the two. For a maturity of T years and ω payments per year, it is straightforward to see that, in the case of no default, the present value of these payments is

$$s \cdot u(K) = \sum_{k=1}^{K} s \cdot \frac{1}{\varpi} \cdot \left(x\% \cdot P_{t=0}^{\text{Art}}\right) \cdot \mathrm{e}^{-r_f T_k} \tag{4.7}$$

where

s is the credit spread quoted on an annual basis

$T_1, \ldots, T_K$ are the fee payment dates

The probability weight corresponding to no credit event is one minus the probability of the art price falling below the face value of the loan during the duration of the ACDS and is given as π in Equation 4.8:

$$\pi = 1 - \int_0^{\mathrm{T}} q(t)\,\mathrm{d}t \tag{4.8}$$

If default occurs before maturity at time τ, the present value of the payments is the sum of the fees paid at payment dates before default and the accrued value since the last payment date T_n and the time of default:

$$s \cdot g(\tau) = \sum_{k=1}^{n(\tau)} s \cdot \frac{1}{\varpi} \cdot \left(x\% \cdot P_{t=0}^{\text{Art}}\right) \cdot \mathrm{e}^{-r_f T_k} + s \cdot \delta_t' \cdot \left(x\% \cdot P_{t=0}^{\text{Art}}\right) \cdot \mathrm{e}^{-r_f \tau} \tag{4.9}$$

with δ_t' being the year fraction since T_n.

Putting the equations together, we obtain the value of the fee payments:

$$s \cdot \left(\int_0^T q(t) g(t)\,\mathrm{d}t + \pi \cdot u(K) \right) \tag{4.10}$$

which gives, after equating fee payments in Equation 4.10 and expected credit loss in Equation 4.6, the credit spread of the ACDS, s:

$$s = \frac{\int_0^T q(t) \cdot v(t)\,\mathrm{d}t}{\int_0^T q(t) g(t)\,\mathrm{d}t + \pi \cdot u(K)}. \tag{4.11}$$

A numerical example will clarify the concept. The initial value of the art piece is assumed to be €125,000. The loan granted is set at 80% of the value of the art (P_t^{Art}) at time, $t = 0$.

The maturity of the loan is set to 5 years at a continuously compounded interest rate of 5%, which is 200 basis points above the risk-free rate. Interest is assumed to be paid at the end of each year. The ACDS is further assumed to match the loan with respect to maturity and payment dates. For the purpose of this example, suppose that annual art returns follow a normal distribution with mean 6% and a standard deviation of 15%.

In a Monte Carlo simulation 10,000 art price realizations using daily returns based on the distributional assumption are obtained. We compute the expected credit loss using Equation 4.6 and the value of the fee payments in Equation 4.10 by averaging over the credit losses and fee payments for all realizations, respectively. Dividing the former by the latter gives an ACDS spread of 20.92 basis points. This is the fair value for the protection that the bank will have to pay in order to transfer the credit risk stemming from the loan.

The spread is a positive function of the assumed variability of the art return. Figure 4.2 shows this relationship.

We assume a very simple example using an art price which is normally distributed. In this case, even with volatility at 25%, Figure 4.2 shows that simulated spreads are as low as 71 basis points. Consequently, the current spreads do not seem to be justified and point to a highly inefficient market for both banks and borrowers. To reconcile modeled credit spreads with market spreads at present, the liquidity premium must be substantial which is not accounted for in the presented model. By assuming the ratio of illiquid and liquid prices to be a fraction depending on a random number of available buyers, in line with Ericsson and Renault (2006), the model could, however, easily be extended to incorporate liquidity risk. This is an avenue for further research.

The sensitivity of the spread to changes in the maturity of the loan and ACDS contract is depicted in Table 4.3. The spread declines with increasing maturities. This can be attributed to credit events usually occurring at the beginning of the term of the loan when a negative return is more likely to cause a drop of the art price below the face value of the loan.

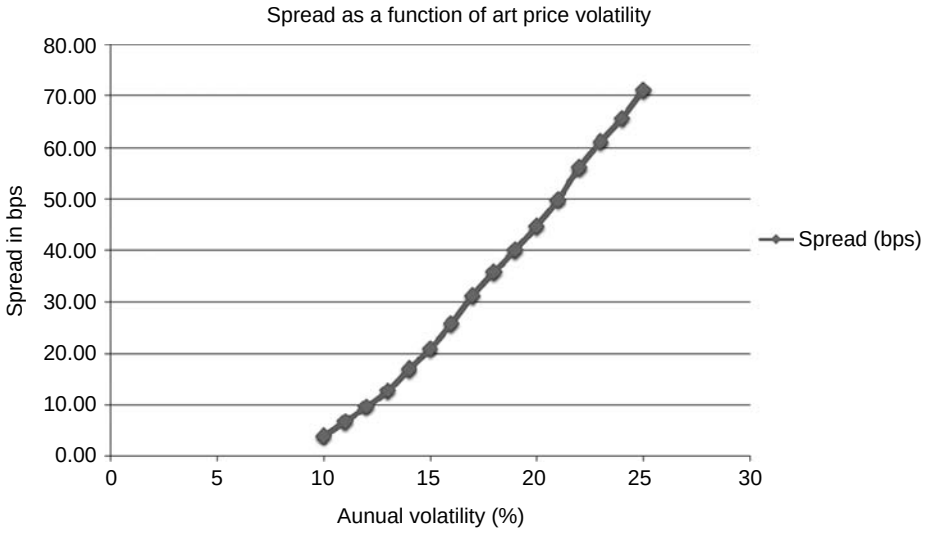


FIGURE 4.2 Credit spread as a function of art price volatility.

TABLE 4.3 Credit Spread as a Function of Maturity

Maturity (years)	Credit Spread (bps)
1	31.49
3	26.31
5	20.92
7	17.61
10	13.88[a]

[a] The simulation for 10 year maturities is based on 1000 realizations only.

4.7 CONCLUSION

Art lending and art finance are both in their infancy. However, with the demand for more asset-backed loans, involving art as collateral, and the willingness of a credit worthy third party to carry art price risk, such as through the emergence of art funds, and museums, banks are now able to transfer undesirable art price risk off its balance sheets through the use of ACDS derivative contracts. The structure presented in this chapter outlines the concept and highlights the necessary requirements for the introduction of lending against art in practice. Financial engineering improves efficiency in the banking sector, resulting in risk transfer and lower spreads paid for loans using art as collateral.

At present the market for pricing ACDSs is anywhere from mature. The simulation results have shown that the spread associated with credit risk is much lower than current practice which highlights the presence of a nonnegligible liquidity premium. This issue needs to be investigated in much greater detail. Other pricing problems which may occur due to dependent default functions are not yet relevant, however, are likely to become an issue once the market becomes more established.

Moreover, the derivatives market for art may bring the necessary liquidity to the art market which would deregulate the market and in turn fuel the move toward greater investment in art as an alternative asset class.

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Part II

Credit Risk, Spreads, and Spread Determinants

CHAPTER 5

Credit Default Swaps and Equity Prices: The iTraxx CDS Index Market

Hans Byström

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5.1 INTRODUCTION

Credit risk is the major source of risk for most commercial banks and it can be defined as the risk of loss resulting from failures of counterparties or borrowers to fulfill their

obligations. Credit risk appears in almost all financial activities, and it is therefore important to measure, price, and manage accurately.

Credit derivatives, a recent innovation in the credit risk market, can help investors with these issues. Credit derivatives are financial contracts that transfer the (credit) risk and return of an underlying asset from one counterparty to another without actually transferring the underlying asset. The credit derivatives market is growing rapidly and in June 2004 notional amounts for credit derivatives amounted to $4.5 trillion, compared to $0.7 trillion 3 years earlier (BIS 2004). Furthermore, the fact that global markets have much larger exposures to credit risk than to interest rate or currency risk indicates an almost unlimited growth potential for the credit derivatives market.*

Contracts similar to credit derivatives, such as letters of credit and credit guarantees, have been around for centuries, but credit derivatives are different in the sense that they are traded separately from the underlying assets; in contrast, the earlier arrangements were contracts between an issuer and a guarantor. Credit derivatives are therefore ideal risk reduction tools for any investor who wants to reduce the exposure to a particular counterparty but finds it costly to sell outright the claims on that counterparty. A related feature of credit derivatives is that credit risk is transferred without any funding actually changing hands. Only in case a credit event occurs does the buyer of credit risk provide funds *ex post* to the seller. This way of allowing financial institutions to manage credit risk separately from funding is an example of how modern financial markets divide financial claims into various building blocks (credit, interest rate, exchange rate, etc.) that each can be traded in a standardized wholesale market that better meets the needs of investors.

The use of credit derivatives is not limited to commercial banks' risk management departments; commercial banks also use them for regulatory arbitrage, hedge funds are active users to hedge other trades, nonfinancial firms use credit derivatives to buy protection against credit extended to customers or suppliers, and many investment banks bring together their various trading desks to encourage traders to identify arbitrage opportunities that arise between credit derivatives markets and their underlying bond and stock markets.

The value of any credit derivative is linked to the probability of the underlying reference entity being exposed to a credit event (delayed payment, restructuring, bankruptcy, etc.) at some point in the future, and for entities with traded equity the probability is often estimated using information from the stock market.† The (theoretical) link between stock prices and credit derivatives prices raises the question of whether there might be arbitrage possibilities available for investors who successfully exploit the link between the two markets. This is an example of a popular new line of business called capital structure arbitrage, or capital structure inbreeding (Currie and Morris 2002).

Currently, the most important credit derivative is the credit default swap (CDS). A CDS is essentially an insurance contract providing protection against losses arising from a

* The credit derivatives market is of course still small compared to the largest OTC (over-the-counter) derivatives markets in the world. Interest rate related OTC derivatives contracts constitute around $177 trillion in notional amounts and foreign exchange related OTC derivatives make up about $32 trillion (BIS 2004).

† The most well-known approach of calculating these probabilities using stock market information is the Merton (1974) model.

credit event, and with CDSs, investors can go both long and short in a particular credit without having to find the underlying asset. This makes them more accessible and easier to trade than their underlying reference entities. Recently, tradable CDS indexes have also been introduced that allow investors quick and easy ways to buy and sell market-wide or sectoral credit risk. In June 21, 2004, the two main CDS indexes, iBoxx and Trac-x, were merged into the Dow Jones iTraxx index that since has set a new standard when it comes to liquidity, transparency, and diversification. Large exposures (negative or positive) to a diversified pool of credit risks are now much easier to gain and the liquidity of the iTraxx market has attracted new participants such as hedge funds and capital structure arbitrageurs.

This chapter discusses the link between CDS spreads and equity prices as well as volatilities in the iTraxx market. The link between stock prices and CDS spreads has been studied before but as far as we know this is one of the first papers looking at the link between stock return volatilities and CDS spreads. Earlier literature has studied either the link between corporate bond yields and stock prices (Kwan 1996) or between single-name CDS spread changes and stock returns (Longstaff et al. 2003, Norden and Weber 2004, Blanco et al. 2005, Yu 2006). A few papers have discussed the important link between equity volatility and bond spreads (Collin-Dufresne et al. 2001, Campbell and Taksler 2003) and Blanco et al. (2005) investigate the link in the CDS market. Knowledge about the link between stock volatilities and CDS spreads is crucial to arbitrageurs in the CDS market and in this chapter, we provide early evidence of such a link in the quickly growing iTraxx CDS index market.

The rest of this chapter is organized as follows. Section 5.2 gives a brief introduction to CDSs and the iTraxx CDS indexes. Section 5.3 discusses the link between the equity markets and the CDS markets. Section 5.4 describes the data and Section 5.5 presents the results. Section 5.6 concludes the chapter.

5.2 CDS INDEX MARKET

A CDS is an insurance contract that protects against losses arising from some kind of predefined credit event involving a reference entity. In a CDS contract, the credit protection buyer pays the protection seller a periodic fee, the CDS spread, which is analogous to the spread between the yield on a typical defaultable bond and the risk-free interest rate. In case a credit event does strike the reference entity, the buyer typically delivers debt owed by the reference entity to the seller in return for a sum equal to the face value of the debt.

With the development of the CDS market, a new credit-linked instrument without many of the problems in the traditional credit market was created. CDSs allow investors to buy pure credit risk because the CDS isolates the credit risk component from other possible risks, such as interest rate risk and foreign exchange risk. Furthermore, liquidity in the CDS market is promoted through the use of standardized contracts concentrated around certain maturities, as well as through the ease with which short positions can be taken. Finally, investors in the CDS market can buy or sell arbitrarily large positions in a

particular credit for reasons of speculation, arbitrage, or hedging without having direct exposure to the underlying reference entity. Therefore, investors are less constrained by whether the underlying entity decides to issue debt or not and by the readiness of other debt holders to sell their debt.

For speculators, taking long positions in CDSs without exposures to the underlying reference entity offers good upside potential in the case of a deterioration of the underlying credit. Arbitrageurs can exploit unjustified spread differentials between the bond and the CDS market. And finally, for credit risk managers, one of the main contributions of the CDS market is to provide accurate credit quality assessments using readily available market data. In this context, some market participants even refer to the CDS market as an additional rating agency (CreditMagazine 2004) and a study by the Bank for International Settlements (BIS) shows that CDS spreads tend to widen well before rating actions (BIS 2004).

Similar to the way a stock index is created as a portfolio of individual stocks, a CDS index is a portfolio of single-name CDSs. CDS indexes are fairly new instruments that provide investors with market-wide credit risk exposure. With the announcement of a merger between the two CDS indexes, iBoxx and Trac-x, into the Dow Jones iTraxx index in June 2004, it has become even more straightforward to gain exposure to a diversified portfolio of credits. The iTraxx index family consists of various indexes of the most liquid CDS contracts in Europe and Asia (in the United States, since April 2004, a similar family of indexes is called Dow Jones CDX). In Europe, an index called iTraxx Europe, which is made up of 125 equally weighted European names selected by a dealer poll based on CDS volume traded over the previous 6 months, is used as a benchmark index. The European index is further split up into several sector indexes (autos, financials, etc.), a corporate index comprised of the largest nonfinancial names (from the 125 names), a crossover index comprising the 25 most liquid sub-investment grade nonfinancial names, and a HiVol index that consists of the 30 names with the widest CDS spreads. The iTraxx indexes typically trade with 5 as well as 10 year maturities and new series are issued every 6 months. The indexes are managed and administrated by a newly created company called International Index Company that is owned by a group of the largest global investment banks.

Thanks to the different iTraxx indexes it is now easy to exploit market beliefs by executing relative-value trades between sectors, buying single-names versus their sector, or perhaps construct tailored synthetic credit risky portfolios using risk-free covered bonds together with a position in a suitable iTraxx CDS index. Credit index trades can also be performed to reflect the slope of the credit spread term structure since 5 as well as 10 year maturities are traded. Various arbitrage strategies involving the CDS index, the constituent CDS, and corresponding stock and bonds are also possible. Overall, the introduction of liquid and easily tradable CDS indexes has opened the door for a new generation of credit derivatives products based on these indexes and it is quite likely that the iTraxx CDS index could outperform standard single-name CDSs in popularity in the near future.

5.3 CDS INDEX SPREADS AND EQUITY PRICES

A crucial parameter in CDS pricing is the amount of credit risk associated with the underlying reference entity and to quantify this amount an investor can follow different paths. One is to rely on rating agencies that rate individual firms' capability of servicing and repaying their obligations (Moody's and Standard & Poor's are two of the better known rating agencies). Another is to rely on traditional scoring models that typically attempt to measure the amount of credit risk using accounting information. A third alternative is to extract information about credit risk from the market; if credit risk is acknowledged by the market then there must be ways of filtering the information contained in market prices to get measures of credit risk.

The most well-known stock market based credit risk model is the Merton (1974) model. This model views a firm's liabilities (equity and debt) as contingent claims issued against the firm's underlying assets. By backing out asset values and volatilities from quoted stock prices and balance sheet information, the Merton (1974) model produces instantaneous updates of a firm's default probability. The default probability in the Merton (1974) model is a nonlinear function (where the default probability has to be solved for iteratively) of the firm's stock price, stock price volatility, and leverage ratio. Furthermore, in 2002, the risk management firm RiskMetrics presented CreditGrades, a commercial (but free) stock market based tool for default probability calculations that add simplifying assumptions to the standard Merton (1974) model. In the simplified model, the default probability is a simple function of the stock price volatility and the leverage ratio and Byström (2006) shows how the CreditGrades model can be deduced from the Merton (1974) model. A further simplification of the default probability expression in CreditGrades can actually be found in the earlier papers by Hall and Miles (1990) and Clare and Priestley (2002). In the Hall and Miles (1990) framework, the default probability is a simple function of the stock price volatility. For the link between these three stock market based default probability models we refer to Byström (2006).

Since the most important determinant of the CDS price is the likelihood that a credit event involving the underlying reference entity occurs, and since theory (Merton 1974) tells us that this probability should be linked to the stock market valuation as well as the stock return volatility of the reference entity, it is natural to investigate empirically the link between the stock market and the CDS market. As mentioned, one group of investors that are particularly interested in this link are those involved in capital structure arbitrage (Currie and Morris 2002). Basically, capital structure arbitrageurs try to detect inconsistencies between the stock market, the corporate bond market, and the credit derivatives market. Those who can price the credits and credit sensitive derivatives accurately, i.e., those who are able to calculate default probabilities accurately, can then earn substantial arbitrage profits by taking positions in the CDS market and hedging them in the stock market or vice versa. Such a strategy is only the simplest example of capital structure arbitrage where accurate modeling of the linkage between stock markets and CDS markets is essential.

There are several earlier studies dealing with related issues. Fama and French (1993), for instance, find some commonality in risk factors affecting the stock and the bond market. Kwan (1996) studies the relationship between the corporate bond market and the stock market and finds a negative correlation between bond yield changes and stock returns and indications of stocks leading bonds in reflecting firm-specific information. The U.S. CDS market and its relationship with the U.S. stock market are investigated in Longstaff et al. (2003) where both the CDS market and the stock market are found to lead the bond market. No clear lead–lag relationship is found between the stock market and the CDS market, however. Norden and Weber (2004) investigate the European CDS market and find CDS spread changes to be negatively correlated with stock returns. Furthermore, stock returns seem to lead CDS spread changes. Campbell and Taksler (2003) are one of the first papers to look at the relationship between stock return volatilities and bond yields and it shows that firm-level volatility can explain much of the variation in U.S. corporate bond yields.

The purpose of this chapter is to study iTraxx CDS indexes and their relationship with the stock price movements of the underlying entities making up the indexes. The size of the CDS index spread (the level) and its empirical relationship with the value and volatility of the underlying stock portfolio are interesting because of the Merton (1974) model predictions. One would expect a large CDS index spread when the stock market valuation is low and the volatility is high and vice versa. Any relationship between CDS index spread changes (first differences) and stock portfolio returns, on the other hand, is interesting since it is a signal of profit possibilities arising from trading strategies involving the various markets.

We look at iTraxx indexes covering the European market, and the seven sectoral indexes that we include in our study are as follows: industrials, autos, energy, technology-media-telecommunications (TMT), consumers, senior financials, and subordinated financials. Each index contains between 10 and 30 individual names. Corresponding portfolios of stocks are formed and both ordinary Pearson correlations and Spearman's rank correlations between the various CDS spreads, stock prices, and stock volatilities are computed. We also estimate the degree of contemporaneous and cross–serial correlations between the iTraxx market and the stock market by estimating the following empirical model:*

$$r\mathrm{CDS}_t = a_{0,t} + a_{1,t} r\mathrm{CDS}_{t-1} + a_{2,t} r_t + a_{3,t} r_{t-1} + \varepsilon_t \tag{5.1}$$

* As opposed to earlier studies on bond markets, we do not include the risk-free interest rate as an independent variable in the OLS (ordinary least squares) regressions. The reason is that credit default swaps (CDSs) are pure credit exposures without interest rate risk. This minimizes the effect of noncredit related components of the spread between the treasury market and the credit risky market on the regression. Essentially, macronews are expected to have limited impact on CDS prices. However, just like the stock market, we expect the CDS market to react firmly to firm-specific information.

where $r\text{CDS}_t$ is the change in iTraxx CDS index spread from $t-1$ to t (in %), r_t is the stock index return from $t-1$ to t (in %), $a_{i,t}$ is the regression coefficients, and ε_t is the normally distributed error term.

Stock indexes and CDS indexes are expected to be contemporaneously but not cross–serially correlated if information is simultaneously embedded into security prices in the two markets. This is probably true for public information, but with private information informed traders could systematically prefer to trade in either the stock or the CDS market. It the private information is not simultaneously embedded into the stock and CDS markets, a lead–lag relationship between the prices in the two markets can be observed. The contemporaneous correlations therefore reflect the degree of common firm-specific information driving the stock portfolios and corresponding iTraxx indexes, and the cross–serial correlations reflect which of the two markets is more likely to be used by informed traders and to what degree one of the markets might drive the other.

5.4 DATASET

The data used in this study consists of daily closing quotes (the midpoints between quoted bid and ask quotes) for seven sectoral iTraxx CDS Europe indexes.* Each index is traded with 5 as well as 10 year maturities and is denominated in Euro.† The period covered is June 21, 2004 to April 18, 2005, which is the very first 10 months in the history of the iTraxx market. The total number of midpoint CDS spread quotes in the panel is 1484. Compared to previous studies using CDS quotes our data series are very clean in the sense that all observations are market quotes, that all quotes are directly comparable to each other and that the quotes consistently are updated on a daily basis. Furthermore, no quote staleness is observed for any of the indexes.

Most of the 125 names in the indexes are large multinationals and all have traded equity.‡ This makes it possible to construct sectoral stock indexes comprising the same names as the reference portfolios behind the sectoral iTraxx indexes. Since the iTraxx indexes are equally weighted in its underlying single-name CDS contracts, the stock indexes are also constructed as equally weighted indexes. All stocks are converted into euros on a daily basis and are adjusted for stock splits. The first day in the sample we construct equally weighted indexes (we invest equal amounts in each stock) and thereafter we do no rebalancing of our index portfolios. The actual weights of the individual stocks in the portfolios therefore change slightly over time, but this investment strategy is more realistic than a daily rebalancing of the reference portfolios. Nevertheless, over our fairly short period, the weight drift was very limited. Finally, the way the iTraxx indexes are

* All index quotes have been made available by the International Index Company.

† We only present results for the 5 year maturity iTraxx indexes. However, an initial study of the 10 year maturities indicates very similar results to those presented for the 5 year maturities.

‡ An exception is Vattenfall AB that is a government owned group with a tightly held nontraded stock. Also, for some names (never more than 10% of the total number of names in each sector, respectively) there was no stock price data available in the EcoWin database.

defined means that slightly updated iTraxx reference portfolios are introduced every 6 months (IIC 2004).

5.5 EMPIRICAL INVESTIGATIONS

This section presents our empirical findings. Since this is one of the first studies dealing with the iTraxx market, we start with some descriptive statistics.

5.5.1 Descriptive Statistics

To investigate the link between the CDS market and the stock market, we look at levels as well as percentage changes of CDS index spreads and stock prices. We start by applying the Phillips Perron test to our data series to find out whether the data is stationary or not. In Table 5.1 we present the results for iTraxx CDS indexes, stock indexes, and stock index volatilities and most series are found to be stationary (with the exception of the stock index series that, not surprisingly, are stationary around a trend). Among the CDS index spreads, the "autos" sector stands out and the reason is the deterioration of the entire sector in the wake of the problems faced by General Motors (GM) toward the very end of the sample.* In fact, the autos CDS index series becomes stationary if only the last three observations (out of 212) were removed.

The size of the CDS spread, in basis points, varies somewhat over the time period as well as across the seven sectors. All the sectoral iTraxx indexes demonstrate a general narrowing of the spread over the first 9 months of the 10 month sample period, however, followed by a sharp widening of the spread over the last month or so. Among the sectors, the consumers sector (supermarkets, airlines, clothing, etc.) has the widest average spread and the senior financials sector (secured CDS contracts issued by financial firms) has the narrowest average spread as can be seen in Table 5.1. To the extent that the spread is a compensation for credit risk, this indicates that the European consumers products and services sector of 2004–2005 was considered riskier by the market than the other European sectors. The variability of the spread is also much larger for the consumers sector than for the tranquil senior financials sector. Finally, the largest quoted spread of any index (87.2 bp) is quoted for the autos sector the very last day of the sample.

Turning to the (unconditional) distribution of daily CDS index spread changes and stock price returns, we find the distribution of CDS index spread changes to be much more skewed and leptokurtic than the stock index return distribution. The iTraxx index is also at least two to three times as volatile (with a standard deviation equal to 30%–40% on an annual basis) as the corresponding stock portfolio (with a standard deviation equal to 10%–15% on an annual basis). The largest positive spread changes observed from 1 day to the next are as high as 20%–25% for some of the iTraxx indexes and a mere 2%–2.5% for the stock index returns. The same holds for the largest negative changes, but while extreme

* A downgrade of the giant automobile manufacturer General Motors' (GM's) credit rating was expected for many weeks in April 2005 and finally GM's debt, together with Ford's, was downgraded to junk status on May 5 (2 weeks after the last day of our sample). Much of this turbulence seems to have been spread across the Atlantic ocean to the European autos sector.

TABLE 5.1 Descriptive Statistics for 5 Year iTraxx CDS Index Spreads (Levels and Changes), Stock Indexes (Levels and Changes), and 3 Month Stock Index Return Volatilities

	Industrials	Autos	TMT	Energy	Consumers	Senior Financials	Subordinated Financials
CDS Index Spreads (Basis Points)							
Mean	38.1	46.0	39.3	26.6	51.0	18.2	30.9
Stdev	6.2	6.1	6.4	3.2	5.8	2.0	4.5
Max	56.8	87.2	52.4	34.3	64.9	22.6	39.5
Min	27.3	37.0	27.4	20.7	38.9	14.0	24.3
PP (no trend)	−1.6	9.5	−5.0***	−6.2***	−5.5***	−5.2***	−4.5***
CDS Index Spread Changes (Daily Log-Returns)							
Mean · 10^2	0.080	0.231	−0.036	−0.014	0.016	0.005	−0.005
Stdev · 10^2	2.63	2.99	2.52	2.22	1.97	1.91	2.11
Max · 10^2	26.1	19.0	19.0	13.5	11.5	11.5	11.6
Min · 10^2	−4.7	−15.8	−7.9	−5.9	−5.3	−6.8	−7.7
Skew	5.3	2.3	2.4	2.4	1.5	1.7	1.7
Kurt	46.9	18.9	17.6	11.7	6.6	9.5	9.0
PP (no trend)	−154.1***	−227.0***	−146.8***	−170.2***	−153.9***	−175.5***	−129.2***
Q(6)	26.7***	24.4***	20.7***	17.9***	33.8***	30.5***	39.9***
Q(12)	34.8***	39.7***	22.7**	30.9***	41.2***	36.3***	43.3***
Stock Index Levels (Normalized to Start at One)							
Mean	1.05	1.02	1.01	1.06	0.97	1.04	1.04
Stdev	0.06	0.05	0.05	0.05	0.04	0.07	0.07
Max	1.18	1.14	1.10	1.16	1.05	1.15	1.15
Min	0.96	0.94	0.90	0.98	0.90	0.92	0.92
PP (no trend)	−2.0	−3.8***	−2.2	−1.8	−3.0	−1.1	−1.1
PP (trend)	−15.3***	−9.7***	−9.8***	−19.0***	−7.7***	−14.0***	−14.0***
Stock Index Returns (Daily Log-Returns)							
Mean · 10^2	0.048	0.023	0.017	0.052	0.005	0.046	0.046
Stdev · 10^2	0.75	0.90	0.74	0.62	0.64	0.71	0.71
Max · 10^2	2.5	2.1	2.5	1.6	1.8	1.8	1.8
Min · 10^2	−3.1	−2.9	−2.6	−2.8	−2.5	−2.1	−2.1
Skew	−0.4	−0.2	0.1	−0.7	−0.4	−0.5	−0.5
Kurt	1.4	0.2	0.6	2.4	0.7	0.4	0.4
PP (no trend)	−206.9***	−189.3***	−197.8***	−187.6***	−188.4***	−198.0***	−198.0***
Q(6)	5.2	3.1	3.8	5.9	6.5	2.6	2.6
Q(12)	10.1	8.3	12.2	11.0	8.8	5.0	5.0
Stock Index Return Volatility (3 Months, on a Daily Basis)							
Mean · 10^2	0.79	0.92	0.78	0.59	0.66	0.73	0.73
Stdev · 10^2	0.15	0.13	0.12	0.09	0.08	0.13	0.13
Max · 10^2	1.2	1.2	1.1	0.8	0.9	1.0	1.0
Min · 10^2	0.58	0.69	0.60	0.45	0.53	0.53	0.53
PP (no trend)	−4.5***	−2.8	−4.2***	−2.6	−4.9***	−3.0	−3.0

Note: Period—June 21, 2004 to April 18, 2005. Skew indicates skewness and Kurt indicates excess kurtosis. PP indicates the Phillips Perron test for stationarity (with or without a trend and with four lags). Q(6) and Q(12) are Ljung-Box tests for autocorrelation. 1% and 5% significance levels are indicated by *** and **, respectively.

positive and negative stock returns in this particular sample are of the same magnitude, the most extreme positive CDS index spread changes are many times larger than the corresponding extreme negative changes. This observation, together with the much larger skewness for the iTraxx spread change distribution, is a possible indication of the CDS market reacting relatively more strongly to credit deteriorations than credit improvements compared to the stock market.

Last, but not least, we look at possible serial correlation in the CDS series. We find that while stock returns show no autocorrelation, as indicated by the small Ljung-Box test statistics in Table 5.1, the iTraxx CDS indexes all demonstrate significant (positive) autocorrelation. This is interesting since it indicates an inefficient CDS index market where predictable index changes could mean large profit possibilities for large investors. We investigate the magnitude of the autocorrelation further in the regression study below.

5.5.2 Correlations and Rank Correlations

In Table 5.2, we present correlations between stock index prices, CDS index spreads, and stock volatilities. Correlations between spread changes and stock returns are also presented. In the latter case, we also include cross–serial (lead–lag) correlations. In addition to ordinary (Pearson) correlations, we compute Spearman rank correlations. Rank correlations look at the similarity of rankings, from the smallest to the largest observation, in two data series, and we present rank correlation coefficients because various data comes from very different (non-normal) distributions.

Stock index volatilities are calculated using various windows of historical stock returns and in this chapter, we present results for 1 month (1M), 3 month (3M), and 1 year (1Y) windows.*

The large negative correlations, around −0.5 for both the ordinary correlations and the rank correlations, between levels in the upper part of Table 5.2 indicate a strong negative relationship between CDS spread levels and stock price valuations; the spread of the CDS index is large when the value of the stock portfolio is low and vice versa. This is what we would expect from theory (Merton 1974) and the results are robust across the various sectors.

Furthermore, a significant positive relationship between historical volatilities and CDS spread levels can also be found from inspection of the correlations in Table 5.2. The (positive) correlations between CDS spread levels and stock volatilities, particularly 3 month volatilities, are all highly significant. For some iTraxx sectors, the correlations are as high as 0.6–0.8. Although the link is strongest for volatility estimated using the 3 month window, the results are similar for the other two window sizes. For some reason, the link is weakest for the 1 month window and the only explanation we have is that window sizes as short as 1 month are too short for accurate volatility estimates, and that the majority of the investors therefore use at least 3 months of historical observations to estimate their volatilities. Overall, the positive relationship between CDS spreads and stock

* We have also redone all calculations using 1 week, 2 month, 6 month, and 3 year windows for a subsample covering the first 5 months of the sample. The results we get when we use these volatility estimates are overall similar to those presented in this chapter.

TABLE 5.2 Correlations between 5 Year iTraxx CDS Index Spreads (Levels and Changes), Stock Indexes (Levels and Changes), and Stock Index Return Volatilities

	Industrials	Autos	TMT	Energy	Consumers	Senior Financials	Subordinated Financials
Levels							
Correlation							
CDS-stock	−0.50***	−0.28***	−0.63***	−0.35***	−0.33***	−0.40***	−0.60***
CDS-1MVol	0.15**	0.15**	0.26***	−0.12**	0.24***	0.57***	0.69***
CDS-3MVol	0.68***	0.45***	0.78***	0.60***	0.74***	0.58***	0.76***
CDS-1YVol	0.52***	0.27***	0.59***	0.42***	0.46***	0.43***	0.64***
Rank Correlation							
CDS-stock	−0.56***	−0.38***	−0.59***	−0.39***	−0.26***	−0.42***	−0.59***
CDS-1MVol	0.20***	0.22***	0.26***	−0.01	0.24***	0.62***	0.74***
CDS-3MVol	0.65***	0.60***	0.71***	0.42***	0.69***	0.54***	0.71***
CDS-1YVol	0.53***	0.44***	0.61***	0.11**	0.51***	0.43***	0.60***
Changes							
Correlation							
CDS-stock	−0.29***	−0.27***	−0.23***	−0.17***	−0.22***	−0.24***	−0.29***
CDS-stock (lagged)	−0.19***	−0.23***	−0.27***	−0.09	−0.17***	−0.16**	−0.28***
CDS (lagged)-stock	0.02	−0.03	−0.09	−0.03	−0.01	−0.09	−0.05
Rank Correlation							
CDS-stock	−0.32***	−0.26***	−0.21***	−0.20***	−0.20***	−0.21***	−0.21***
CDS-stock (lagged)	−0.30***	−0.27***	−0.26***	−0.15**	−0.19***	−0.17***	−0.24***
CDS (lagged)-stock	−0.04	0.08	−0.17**	−0.05	−0.09	−0.06	−0.03

Note: Period—June 21, 2004 to April 18, 2005. Correlation indicates ordinary Pearson correlation and rank correlation indicates Spearman rank correlation. 1% and 5% significance levels are indicated by *** and **, respectively.

volatilities is in accordance with theory (Merton 1974) and it is consistent across all the various sectors. Finally, throughout the study the rank correlations and the ordinary correlations in Table 5.2 are very similar to each other.

In Figure 5.1, we present results for the data averaged across all seven sectors (normalized to start at one) and the strong negative relationship between CDS spread levels and stock price valuations seen in Table 5.2 can also be observed in Figure 5.1. Stock prices clearly have a tendency to increase when CDS spreads decrease and vice versa. The significant link between the CDS spread levels and the stock index volatility (the 3 month historical volatility) is also demonstrated in Figure 5.1. It is fairly clear that the spread widens with an increasing stock volatility and vice versa. The close relationships between the CDS spreads, on one hand, and the stock prices and the stock return volatilities, on the other hand, are partly broken toward the last month of the sample where we can observe a significant widening of all the CDS spreads. The widening is most significant in the autos sector, but all sectors suffer to various degrees. Both the stock price and the stock return volatility react in the expected way, compared to the CDS spread, but the size of the reaction is slightly delayed and much more modest. The lagged reaction in the case of the stock volatility is of course partly caused by the use of up to 3 month old observations and it is

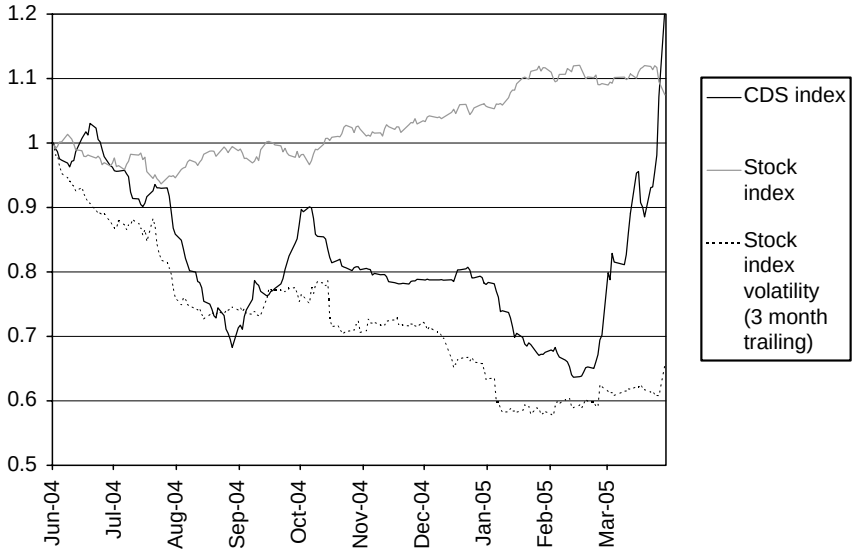


FIGURE 5.1 CDS (credit default swap) index level, stock index level, and stock index volatility.

possible that implied volatilities from stock options markets would react much faster and much more distinctly to the apparent credit deterioration in the European CDS market.

If we turn to CDS spread changes, they are found in the lower part of Table 5.2 to be negatively correlated, at an average level of around −0.25, with contemporaneous stock index returns (Figure 5.1 shows further evidence of this). Furthermore, under the null hypothesis that the stock market and the CDS market are equally quick to incorporate firm-specific information, we would expect the cross–serial (as opposed to contemporaneous) correlations to be equal to zero. However, lagged (1 day) stock returns are found to be almost as significantly (negatively) correlated with current CDS spread changes as current stock returns. Lagged CDS spread changes, on the other hand, are not at all related to current stock price changes. This one-way cross–serial correlation is interesting since it is a possible indication of information flowing from the stock market to the CDS market and not vice versa.

Overall, the patterns for the various sectors are very similar. One slight but interesting exception to this, though, is the consistently larger correlation between the subordinate financials CDS index and the stock market than between the senior financials index and the stock market. This, we think, is in line with subordinate CDSs being more equity-like in their character than senior ones.

5.5.3 OLS Regressions

In addition to correlation estimates, we also regress daily CDS spread changes on yesterday's CDS spread changes and on today's and yesterday's stock returns. The regression results are presented in Table 5.3 and the earlier correlation-based results are further strengthened. All the contemporaneous stock return coefficients are significant,

TABLE 5.3 OLS Regressions: 5 Year iTraxx CDS Index Spread Changes Regressed on Lagged iTraxx CDS Index Spread Changes and Current and Lagged Stock Index Returns

	Industrials	Autos	TMT	Energy	Consumers	Senior Financials	Subordinated Financials
$a_{0,t}$	0.0015	0.0026	0.0001	0.0004	0.0003	0.0005	0.0008
$a_{1,t}$	0.28***	0.07	0.25***	0.19***	0.28***	0.20***	0.35***
$a_{2,t}$	−1.02***	−0.84***	−0.69***	−0.58**	−0.64***	−0.58***	−0.75***
$a_{3,t}$	−0.42*	−0.63***	−0.72***	−0.20	−0.33	−0.27	−0.55***
R^2	0.18	0.12	0.17	0.07	0.15	0.11	0.25
$F_{[3,d.f.]}$	15.3***	9.3***	14.4***	5.5***	11.7***	8.9***	22.6***

Note: Period—June 21, 2004 to April 18, 2005. 1%, 5%, and 10% significance levels are indicated by ***, **, and *, respectively.

$r\text{CDS}_t = a_{0,t} + a_{1,t} r\text{CDS}_{t-1} + a_{2,t} r_t + a_{3,t} r_{t-1} + \varepsilon_t$.

indicating a strong negative link between CDS spread changes and stock returns, while roughly half of the lagged stock return coefficients are significant. This at least partly supports the earlier evidence from the correlation study of the stock market driving the CDS market, and this is important information for arbitrageurs and other cross-market investors. The $F_{[3,d.f.]}$ statistic is significant for all the sector regressions and the degree of explanation, R^2, varies between 0.07 and 0.25 with most R^2s lying in the range between 0.10 and 0.20.

When it comes to the possible autocorrelation in the CDS market, the regressions in Table 5.3 seem to confirm the earlier evidence given by the significant Ljung-Box statistics in Table 5.1. While the Ljung-Box statistics, Q(6) and Q(12), look at the entire correlation structure up to 6 and 12 lags, respectively, the regressions in Table 5.3 indicate significant positive first-order autocorrelation in the European CDS market. Only for one sector, the autos sector is the otherwise consistent pattern broken. Again, this is caused by the sudden and very significant spread widening in this sector toward the end of the sample. The significant positive autocorrelation across the various sectors is interesting since it is an indication of an inefficient European CDS market; predictable CDS spreads could mean large profit opportunities for investors who can take positions large enough to cover the transaction costs.

We can compare our results regarding the link between the CDS market and the stock market with those from earlier studies looking at single-name CDS contracts. Norden and Weber (2004), looking at CDSs from 58 firms from Europe, the United States, and Asia, also find stock returns to be negatively associated with CDS spread changes. Furthermore, they also find stock returns to lead CDS spread changes. Longstaff et al. (2003) look at U.S. firms and find both the CDS market and the stock market to lead the bond market. They find no clear cut lead–lag relationship between the stock and the CDS market, however. The latter result can be a consequence of the U.S. CDS market being more efficient than the European and Asian CDS markets. Blanco et al. (2005) also investigate the determinants of CDS spreads and find that stock returns have more of an impact on CDS spreads than they have on corporate bond spreads.

Our results are also similar to those of Kwan (1996) in their study of the relationship between corporate bonds and stocks. Kwan (1996) finds the same relationship between bonds and stocks that we find between CDSs and stocks; contemporaneous and lagged stock returns have a significant negative impact on credit spread changes.

The relationship between stock return volatilities and credit spreads was first investigated by Campbell and Taksler (2003) who find that firm-level volatility can explain much of the variation in bond yields in the United States. We dare claim that we find the same thing for historical volatilities in the European iTraxx CDS index market despite relying solely on sample correlations between stock return volatilities and CDS spread levels. Investigating credit markets in Europe and in the United States, Blanco et al. (2005) also find a significant link between implied stock volatilities and CDS spreads, but no link between implied volatilities and bond spreads. The latter is also found by Collin-Dufresne et al. (2001) who find only weak evidence of implied equity volatility (VIX) explaining corporate bond spreads in the United States. Finally, touching at the issue of stock volatilities and CDS spreads, Berndt et al. (2004) find that KMV Expected Default Frequencies (that are functions of stock volatilities) explain a large share of the cross-sectional variation in CDS spreads.

5.6 CONCLUSION

In this chapter, we have studied the European iTraxx CDS index market, particularly the relationship between iTraxx sectoral indexes and corresponding sectoral stock indexes. We believe knowledge regarding the link between the CDS market and the stock market to be important for anyone involved in hedging, speculation, or arbitrage activities in the CDS market.

One interesting finding in this chapter is the significant positive autocorrelation present in all the studied iTraxx indexes. This is possibly an indication of an inefficient iTraxx CDS index market where index changes are predictable. The economical significance of profits from trading strategies exploiting such regularities, however, is an interesting issue left for future research.

Moreover, significant correlations between iTraxx CDS index spreads and spread changes on the one hand and stock prices and stock returns on the other hand reveal a close link between the two markets. CDS spreads have a strong tendency to widen when stock prices fall and vice versa. Furthermore, in OLS regressions, both current and lagged stock returns are found to explain much of the variability in CDS spreads. This suggests that firm-specific information is embedded into stock prices before it is embedded into CDS spreads. Hence, the stock market seems to lead the CDS market in transmitting firm-specific information and this is important information for arbitrageurs and others. Again, it is a possible indication of an inefficient European CDS market.

Stock index return volatility is also found to be significantly correlated with iTraxx CDS index spreads; the spreads are found to increase (decrease) with increasing (decreasing) stock volatilities. The link is particularly strong for 3 month historical volatilities. These results are in line with the theoretical literature on credit risk that emphasizes the

importance of stock volatility for default probability predictions and it is further evidence of the importance of volatility forecasting in credit risk modeling.

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CHAPTER 6

The Determinants of Credit Default Swap Prices: An Industry-Based Investigation

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6.1 INTRODUCTION

Since its birth in 1996, the credit derivatives market has been experiencing an exponential growth with a compound growth rate of ~46% per year. According to the British Bankers' Association, a trend growth is likely to be sustained until at least 2008 (Barrett

and Ewan, 2007). Among credit derivatives, the credit default swap (CDS) is the most widely traded instrument.

The CDS is an agreement between a protection buyer and a protection seller whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event (such a certain default) happening in the reference entity. The periodic fee (also known as premium price or default swap spread) is quoted in basis points per annum of the contract's notional value. Usually, predetermined premiums are paid by the buyer of protection to the seller on a quarterly basis, with the contract terminating either at maturity or at the time of a credit event occurring.*

The size of the single-name CDS market accounted for 33% of the total credit derivatives market in 2006. Nevertheless, index-linked instruments are expected to match the single-name CDS by 2008, with an estimated equal 29% proportion of the total market by 2008.

Meanwhile, gross market values, which measure the cost of replacing all existing contracts and thus represent a better measure of market risk at a given point in time than notional amounts, had experienced a year-to-date (YTD) increase of 3%, to $10 trillion at the end of June 2006.

In parallel with the market growth, the recent years have experienced a shift in the average quality of underlying issuers in CDS and indices. The share of investment grade names (rated BBB and above by Standard and Poor's) fell by 6%, from 65% to 59%, from 2004 to 2006, with a sustained trend. As Figure 6.1 indicates, the degradation of the credit quality of issuers is witnessed both in the upper tier (BB-B) and the lower tier (below B) of speculative-grade issuers. This tendency can mostly be explained by the increased attention paid by hedge fund investors toward the high-yield market (Anson, 2006). This evolution partly explains not only the increase in notional amounts but also a greater focus toward high-risk instruments, which are able to generate extra returns.

Even though the CDS is a very standard and liquid type of contract, and considering the recent evolution toward a higher average credit risk premium, little empirical work has been done on the determinants of the observed premia, and most of this literature has focused on issuer-specific determinants.

Cossin and Hricko (2001) test for the influence of the theoretical factors predicted by the reduced and the structural form literature by using a cross-section of CDS rates on a variety of underlyings. The determinants they considered are credit ratings, interest rate, slope of the yield curve, time to maturity, stock prices, variance or volatility of the firm's assets, leverage, index returns, and idiosyncratic factors. They show that ratings are the single most important source of information on credit risk. Nonetheless, all the other factors tested add significant information to ratings.

Ericsson, Jacobs, and Oviedo (2007) investigate the relationship between theoretical determinants of default risk and CDS premia. They find that leverage, volatility, and the spot interest rate are economically important. For instance, a 1% increase in annualized

* In the case of those distressed credits in which the CDS market remains open, however, it has become more usual for sellers of credit protection to demand the payment of an up-front premium as opposed to the standard running spread.

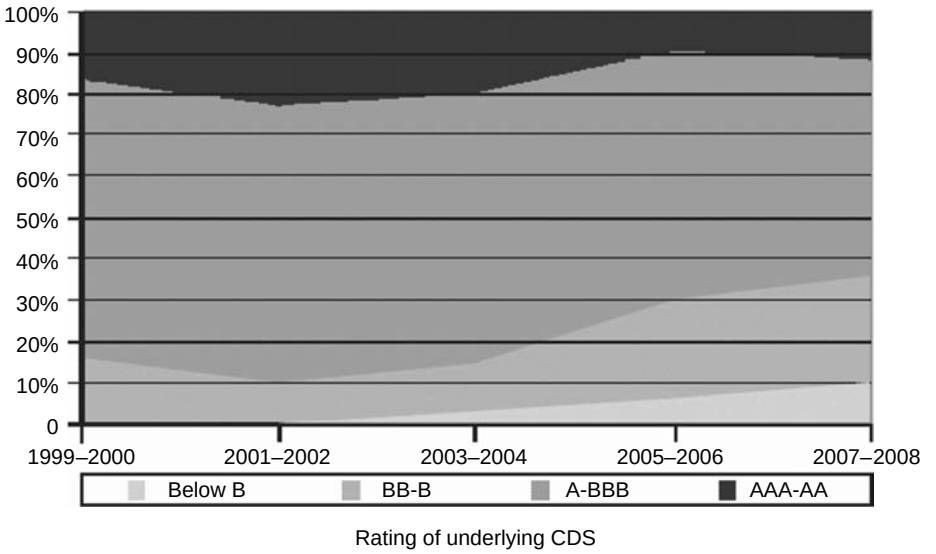


FIGURE 6.1 Evolution of credit default swap (CDS) issuer quality 1999–2008. Historical evolution of CDS issuer quality per credit rating during the period 1999–2008 (estimated). (From Barrett, R. and Ewan, J., 2007, British Bankers Association Credit Derivative Report 2006, Research report.)

equity volatility raises the CDS premium by 1–2 basis points. A 1% change in the leverage ratio raises the CDS premium by approximately 5–10 basis points.

Zhu (2006) compares credit spreads between the bond market and the CDS market. He extends the existing studies by also looking into the underlying factors that explain the price differentials and exploring the short-term dynamic linkages between the two markets. His results suggest that the CDS market often moves ahead of the bond market in price adjustment, in particular for U.S. entities. Liquidity also matters for their role in price discovery. But the terms of CDS contracts and the short-sale restriction in the cash market only have a very small impact.

For Amato (2005), CDS spreads contain a risk premium related to investors' aversion to default risk. The empirical findings confirm that CDS risk premia and default risk aversion are related to fundamental macroeconomic factors such as the stance of monetary policy and technical market factors, such as the issuance of collateralized debt obligations.

In this chapter, we examine the empirical relationship between CDS spreads and a set of macroeconomic factors. Although these types of factors have not yet been exploited to explain CDS premia, several past studies have examined the empirical relationship between variables accounting for default risk at large and macroeconomic conditions.

Duffie, Saita, and Wang (2007) show there is a countercyclical relationship between corporate default risk and macroeconomic covariates. In their study on default and recovery rates, Altman, Brady, Resti, and Sironi (2005) estimate negative correlations between default rates, loss given default, and the business cycle. More recently, Amato

and Luisi (2006) provide new evidence on the impact of macroeconomic conditions on corporate bond spreads. They find that spreads are strongly related to real economic activity and financial conditions and are less related to inflation. Moreover, spreads on lower-rated debt are affected more by macroeconomic variables than those on investment grade bonds.

But apart from a few studies, the relationship between CDS spreads and macroeconomic variables has been left largely unexamined. In this chapter, we specifically analyze the influence of macroeconomic variables on a homogeneous cross-section of CDS quotes. We model time-varying exposures of CDS spreads to several macroeconomic factors, using a Kalman filter procedure. The main innovation of our approach is that the state vector is comprised of observable macroeconomic variables, and we can identify how these factors dynamically influence the exposures of CDS premia. Through our approach, we identify three factors influencing CDS spreads: the Dow Jones Stoxx 600 (or alternatively S&P 500 Composite), which proxies for the stock market behavior; the Consumer Price Index (CPI); and the Lehman Euro Aggregate Bond Index. More importantly, our results outline the superiority of a Kalman filtering approach over a multifactor model with constant coefficients. This finding confirms a well-grounded, but loosely validated market perception that the determinants of CDS premia vary over time and that the main sources of systematic risk exposures tend to be temporary and unstable.

Our study begins by presenting our dataset. We then discuss the dynamic state space model to identify time-varying exposures of CDS spreads. Finally, we analyze the results of our empirical investigation.

6.2 DATASET

6.2.1 Credit Default Swap Data

We construct a historical daily time series of spreads for a fixed set of firms using data from Bloomberg. The groups of firms we consider are the members of the iTraxx Europe Series 3.*

The Dow Jones iTraxx comprises 125 issuers. The highest ranking issuers in each sector below are selected: 10 autos, 30 consumers, 20 energy, 20 industrial, 20 technology, media, telecommunications (TMT), and 25 financials. A new series of issuers is reset every 6 months (March and September). Standard maturities are 5 and 10 years for the swap contracts. Our study focuses on four sectors: auto, consumer, energy, and industrial. The underlying reference entities are senior single-name contracts. Each quote is a mid 5 year price. Synthetic daily time series of sector spreads are constructed as equal-weighted averages of returns on single-name contracts for the period January 2002 to September 2005 included.

As most of macroeconomic variables are reported in monthly time series, we must aggregate CDS returns on a monthly basis as simple arithmetic averages of individual

* The constituents of this index can be found on Markit's Web site at http://www.markit.com.

TABLE 6.1 Descriptive Statistics of the Credit Default Swap Returns

	Auto	Consumer	Energy	Industrial
Observations	45	45	45	45
Mean	−0.0002	0.1906	0.0006	0.0003
Median	−0.0009	0.0126	−0.0007	−0.0014
Maximum	0.0196	2.0828	0.0238	0.0227
Minimum	−0.0141	−0.2835	−0.0115	−0.0155
Standard deviation	0.0059	0.4585	0.0068	0.00734
Skewness	0.7216	2.1574	1.5754	0.93493
Kurtosis	4.6992	8.0942	6.3214	4.9678

Note: All observations are monthly from January 2002 to September 2006. The returns are calculated from the equally weighted index of aggregated CDS premia of issuers belonging to the same sector.

returns within each category. Descriptive statistics for the monthly time series of CDS returns in each sector are reported in Table 6.1.

6.2.2 Macroeconomic Factors

Table 6.2 presents the set of 17 factors we use in our analysis that could impact CDS returns. These factors are selected according to the previously tested variables for credit risk.

TABLE 6.2 Set of Macroeconomic Factors

Type	Index
Equity markets	Dow Jones Stoxx 600
	S&P 500 composite
	Vix
Money	Exchange rate euro/$
Cash interest rate	Euribor 3 months
	Treasury bills 3 months
Yield curves	Treasury bonds 5 years minus treasury bills 3 months
	Treasury bonds 20 years minus treasury bonds 5 years
	Germany bonds 5 years minus euribor 3 months
	Germany bonds 20 years minus Germany bonds 5 years
Bond markets	Lehman U.S. Aggregate
	Lehman Euro Aggregate
Real activity	Goldman Sachs Commodity Index Consumer Price Index U.S.
	Consumer Price Index Europe
Financial activity	M3
	High-yield debt issue (value)
	Global bond debt issue (value)

Note: All factors are measured with a monthly frequency.

6.3 METHODOLOGY

We use a dynamic state space model similar to the model proposed by Swinkels and Van der Sluis (2006) to identify time-varying exposures of CDS returns. A more classical estimation procedure could be the standard ordinary least squares (OLS) regression techniques to decompose the CDS returns. But one of the implicit assumptions of this method is that the exposures stay constant over time. This contradicts the market perception of CDS credit exposures, which are in no way considered as stationary for long periods. Alternatively, time variation is sometimes implicitly accounted for by using rolling regressions. However, the ad hoc chosen window size of the rolling regression period is subjective and causes conceptual problems. The main advantage of the Kalman filter procedure is the more efficient use of available data, while allowing for time variation in exposures.

We model style exposures as a random walk and estimate the coefficient corresponding to the kth factor at time t, denoted $\beta_{\kappa,\tau}$ with the Kalman filter.

Here, we adapt the linear k-factor model by allowing the beta to vary through time. No restrictions are imposed on the betas. Our state space model estimated by the Kalman filter procedure is the following:

$$R_t = \alpha + \sum_{k=1}^{K} \beta_{k,t} F_{k,t} + v_t \quad \text{measurement equation}$$

$$\beta_{k,t} = \beta_{k,t-1} + w_{k,t-1} \quad \text{transition equation}$$

For $k = 1, \ldots, K$, where R_t is return on CDS spreads, K is the number of factors, $F_{k,t}$ is factor k at time t, $\beta_{\kappa,\tau}$ is the exposure to factor k at time t, and α_t is the unexplained systematic part of the return.

This intercept is set equal to 0 to avoid time variation of α_t, which should not take place as CDS contracts are not actively managed portfolios and thus should not earn abnormal returns.

The random variables $w_{k,t}$ and v_t represent the process and measurement noise, respectively. They are assumed to be independent (of each other), white, and normally distributed with joint pdf:

$$p(w) \sim N(0,Q)$$

$$p(v) \sim N(0,R)$$

6.4 EMPIRICAL INVESTIGATION

The R^2 of the stepwise OLS regression is used as a goodness-of-fit statistic to select the most significant factors and initialize the Kalman filter. Our procedure involves starting with no variables in the model, trying out the models one by one, and including the uncorrelated variables which increase R^2 in an important way.

TABLE 6.3 Results of the Stepwise Ordinary Least Squares and Kalman Regressions—Auto Sector

	OLS			Kalman			
	R^2	*t*-Stat	*p*-Value	Mean β	Min β	Max β	SD
Dow Jones Stoxx 600	19.91%	−3.27	0.0021	−0.5679	−0.5802	−0.5628	0.0048
Consumer Price Index U.S.	25.48%	1.77	0.0838	0.0018	−0.0086	0.0229	0.0058
M3	35.36%	−2.51	0.0164	0.562	0.5578	0.5688	0.0025
Lehman Euro Aggregate	41.69%	2.09	0.0435	−0.6115	−0.6176	−0.6065	0.0027
German bond 5 years − euribor 3 months	41.70%	0.05	0.9619	4.592	4.591	4.593	0.0006

Tables 6.3 through 6.6 report the results using the stepwise OLS procedure as well as the average loading computed with the Kalman filter, for the four sectors considered. The first column reports the adjusted R^2 obtained with stepwise OLS. The next columns present the *t*-stat of the factor coefficient and its *p*-value. The last four columns report the average, minimum, and maximum βs computed with the Kalman filter equation as well as the standard deviation of the time series of betas.

Overall, the most significant factors of CDS spreads are Dow Jones Stoxx (or S&P 500 composite), i.e., a proxy for the stock market return, the Consumer Price Index, and the Lehman Euro Aggregate Bond Index.

During the whole period, the exposure to Dow Jones Stoxx was negative. Thus, when the equity market is bullish, CDS spreads tend to decrease. It is not surprising that aversion to risk tends to decline during good times. The exposure to CPI and Lehman Euro Aggregate is positive. CDS spreads are closely and positively related to default risk aversion, consumer price index, the stance of monetary policy, and the risk premia in bond market.

For the auto, consumer, and energy sectors, risk exposures do not vary much over time. In general, the $\beta_{\kappa,\tau-1}$ seems to be a good estimator of $\beta_{\kappa,\tau}$ because $\beta_{\kappa,\tau}$ has a very small variation compared to its previous value. Consequently, a weak time variation in exposures makes it possible to strongly increase R^2. Nonetheless, for each sector, there exists a significant variation in one beta: the CPI for auto and industrial sectors, the high-yield debt issue for the consumer sector, and the Lehman Euro Aggregate Bond Index for the energy sector. Figures 6.2 through 6.4 display the evolution of the corresponding betas.

From Figure 6.2a and b, there appears to be a large degree of harmony between the time variations of the CPI beta and the values of returns. This behavior is quite natural due to the small variability of the CPI. The risk exposure shows two peaks, namely in

TABLE 6.4 Results of the Stepwise Ordinary Least Squares and Kalman Regressions—Consumer Sector

	OLS			Kalman			
	R^2	*t*-Stat	*p*-Value	Average β	Min β	Max β	SD
Lehman Euro Aggregate	9.36%	2.11	0.0410	80.1	80.1	80.1	1.19e-09
Consumer Price Index Europe	10.93%	0.86	0.3937	411	411	411	1.93e-10
High-yield debt issue	41.11%	−4.85	0.0000	3.0e-05	−4.8e-05	0.00081	0.00014

TABLE 6.5 Results of the Stepwise Ordinary Least Squares and Kalman Regressions—Energy Sector

	OLS			Kalman			
	R^2 (%)	*t*-Stat	*p*-Value	Average β	Min β	Max β	SD
S&P 500	11.01	−2.31	0.0260	−0.4887	−0.4893	−0.4879	0.0005
Lehman Euro Aggregate	14.08	1.23	0.2271	0.0019	−0.0027	0.0108	0.0024
Consumer Price Index U.S.	15.41	0.80	0.4260	0.2556	0.2552	0.256	0.0002
M3	25.29	−2.30	0.0268	−0.9181	−0.9189	−0.9172	0.0006
Euribor	27.84	1.17	0.2474	4.545	4.545	4.545	0.0001

months 10 and 40. The second one (May 2005) corresponds to the downgrade of GM and Ford, which could not be anticipated with constant betas as it appeared to trigger a fear of increased inflation risk.

From Figure 6.3, we observe much less variability corresponding to the Consumer sector. Without the single peak in month 8 (April 2003), there would be hardly any need for a time variability in betas. This result suggests that effectiveness of the Kalman filter approach may be limited in this class of CDS. Nevertheless, the predictive ability of time-varying betas still represents a substantial improvement over constant betas.

Figure 6.5a and b shows the time-varying betas for the industrial sector.

For the industrial sector, all exposures vary much more over time. The CPI beta closely follows the pattern of returns, as for the auto sector CDS, mostly because of the low variability of this index. Meanwhile, the bond market exposure appears very volatile, with an 8 month period of very strong negative values. This period roughly corresponds to a tightening of credit spreads in industrial countries, which may explain a much higher sensitivity of CDS returns to the bond market at that particular period, which abruptly ended with the GM–Ford downgrade event.

6.5 SUMMARY AND FUTURE WORKS

Using state space model and Kalman filter procedure, we examine the empirical relationship between CDS spreads and macroeconomic factors during the period 2002–2005. Evidence points to links to macroeconomic variables Dow Jones Stoxx, Consumer Price Index, and Lehman Euro Aggregate. However, the procedure appears to be best suited for the industrial sector. For the other three studied indexes (auto, consumer, and energy), only one macroeconomic beta appears to display some variability.

TABLE 6.6 Results of the Stepwise Ordinary Least Squares and Kalman Regressions—Industrial Sector

	OLS			Kalman			
	R^2 (%)	*t*-Stat	*p*-Value	Average β	Min β	Max β	SD
Dow Jones Stoxx 600	27.90	−4.08	0.0002	−1.047	−3.92	1.757	1.642
Consumer Price Index U.S.	29.36	0.93	0.3573	−0.2742	−6.601	4.531	1.838
Lehman Euro Aggregate	30.84	0.94	0.3541	−1.663	−11.28	5.965	5.068

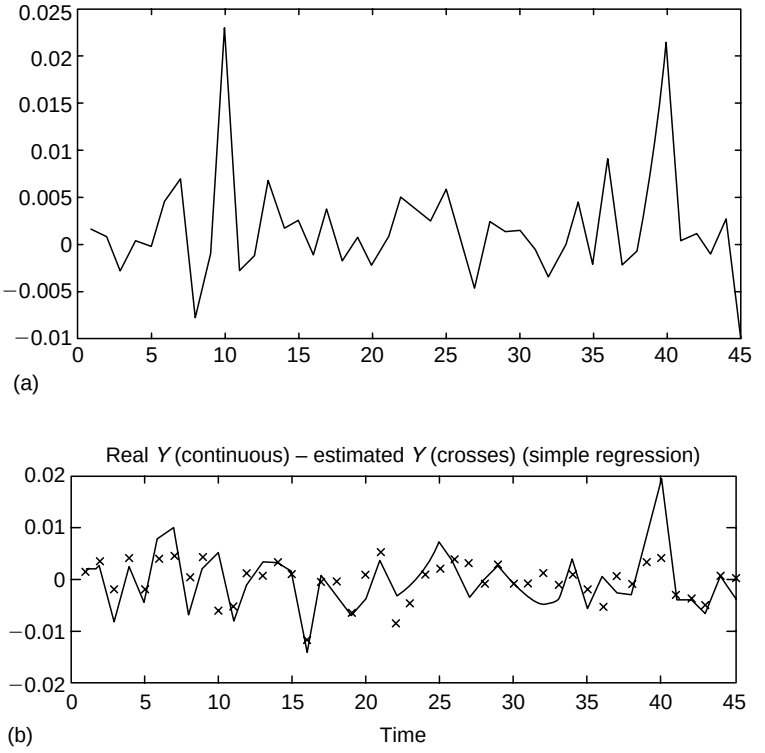


FIGURE 6.2 (a) Time variation of the Consumer Price Index (CPI) beta for the auto sector. (b) Real versus estimated credit default swap (CDS) returns with the ordinary least squares (OLS) regression for the auto sector. For each figure, the horizontal axis represents the month on the observation (start: Jan 2002—end: Sept 2006).

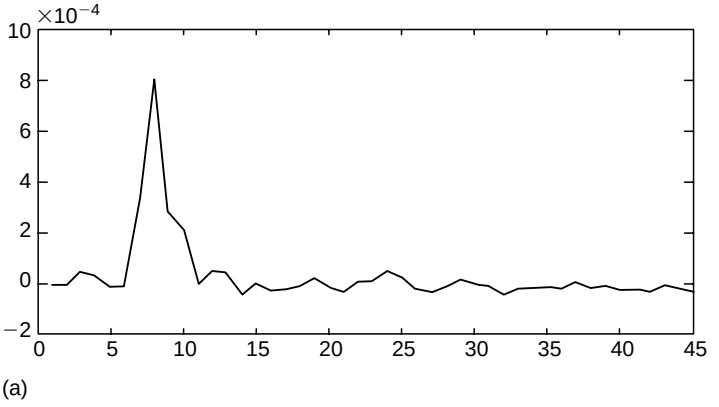


FIGURE 6.3 (a) Time variation of the high-yield debt issue beta for the consumer sector.
(*continued*)

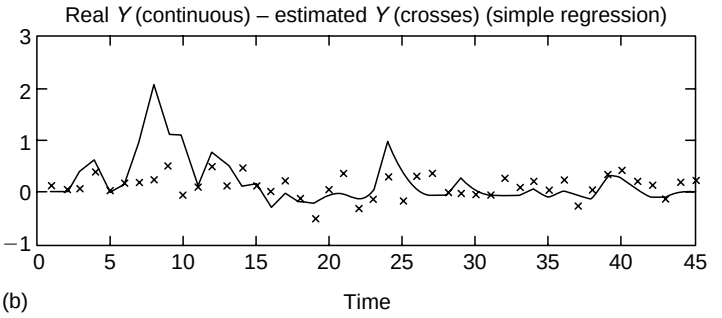


FIGURE 6.3 (continued) (b) Real versus estimated credit default swap (CDS) returns with the ordinary least squares (OLS) regression for the consumer sector. For each figure, the horizontal axis represents the month on the observation (start: Jan 2002–end: Sept 2006).

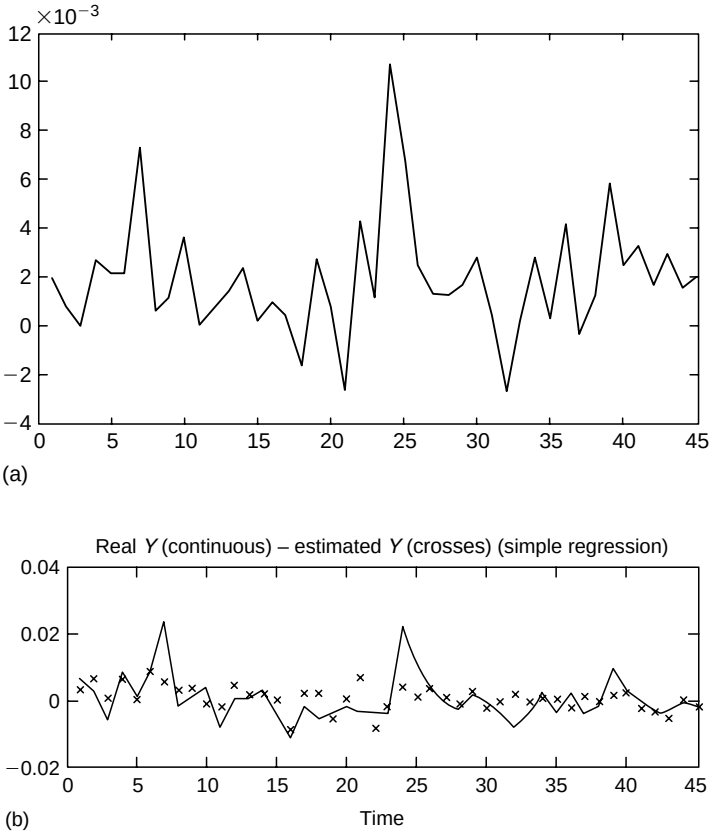


FIGURE 6.4 (a) Time variation of the Lehman Euro Aggregate Bond index beta for the energy sector. (b) Real versus estimated credit default swap (CDS) returns with the ordinary least squares (OLS) regression for the energy sector. For each figure, the horizontal axis represents the month on the observation (start: Jan 2002–end: Sept 2006).

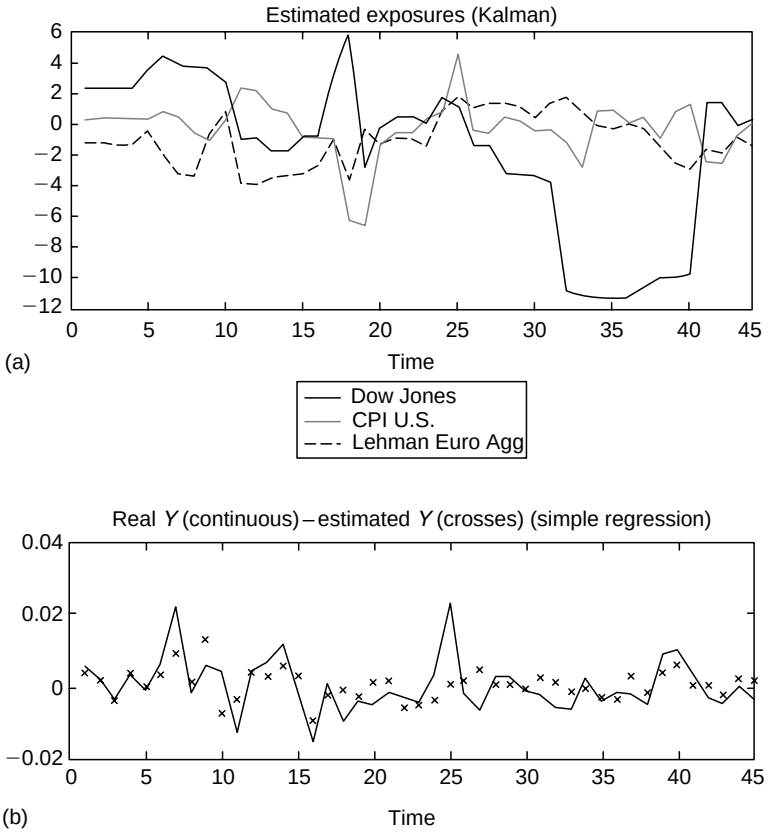


FIGURE 6.5 (a) Time variation of the high-yield debt issue for the industrial sector. (b) Real versus estimated credit default swap (CDS) returns with the ordinary least squares (OLS) regression for the industrial sector. For each figure, the horizontal axis represents the month on the observation (start: Jan 2002–end: Sept 2006).

There are several avenues to explore in future research. First, a more careful analysis would require constructing measures of firm's value as leverage, interest coverage, cash flow, assets volatility, and stock prices. Furthermore, factors that proxy for financial activity should be developed further.

Second, using the Kalman filter to forecast exposures to a wider set of variables would be beneficial for applications in CDS spreads analysis. Instead, the improvement of the predicted intercept would be largest when the exposures change slowly over time.

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CHAPTER 7

Credit Spread Dynamics: Evidence from Latin America

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7.1 INTRODUCTION

Fixed income instruments as an asset class play an integral role in portfolios where they substitute, or are held alongside, cash, stocks, and occasionally commodities. However, one key segment of this market—the market for risky and in particular noninvestment grade bonds—is characterized in secondary market trading by a relative lack of liquidity. Consequently, in order to make effective investment decisions, it is necessary for any analyst, or investor, to determine the fair price of these instruments. This requires an understanding of the complex dynamic relationships that exist between markets more generally and those factors that determine the pricing of bonds more specifically. In comparison with stocks, bonds have a well-defined set of cash flows over their term to maturity. However, these cash flows are subject to default, liquidity, and interest rate risks, which need to be considered when pricing the individual security.

The addition of a yield spread (or credit spread) over the equivalent near-maturity, risk-free benchmark, such as the on-the-run U.S. Treasury bond, or note, for U.S. dollar issues, is the industry approach to reflect the issuer's credit worthiness and the associated default and liquidity risks pertinent to the risky instrument being priced (De Almeida, Duarte, and Fernandes 1998). Structural models of default, such as the model proposed by Longstaff and Schwartz (1995), provide a simple and intuitive framework to capture the factors that drive yield spreads. Empirical evidence from the mature markets of the United States and Japan among others (Collin-Dufresne, Goldstein, and Martin 2001) points to two main factors—the *asset factor* and the *interest rate factor*—as the key drivers of changes in credit spreads. However, the generality of the developed market evidence to emerging markets and the understanding of the factors that drive these credit spreads, which are structurally and otherwise different from those in mature markets, is limited.

Emerging markets in general and Latin American markets in particular have witnessed a persistent decline in credit spreads over the last several years, yet the economic justification for this behavior remains largely unanswered. The objective of this chapter is to critically examine the economic reasons for the behavior of credit spreads in a key segment of the emerging bond market—the sovereign issues by Latin American issuers in the international bond market. The Bank for International Settlements (BIS 2007) reports that Latin American issuers comprise the largest single region (35% in 2006) of developing countries in these markets which comprised a total of US $994 billion in outstandings in 2006, followed by the Asia-Pacific (29%), European (26%), and Africa and the Middle East (10%) regions. Emerging market credit spreads also continued to decline prior to and after the near collapse of Long-Term Capital Management (LTCM) and the Asian financial crisis in 1997. The post-LTCM crisis also witnessed a declining trend on the credit spreads of other emerging market issuers.

In order to understand the behavior of emerging market credit spreads and to empirically establish the economic reasons for the persistent decline in credit spreads, we investigate the following important questions in this chapter predicted by the structural models of Longstaff and Schwartz (1995):

- Are the changes in credit spreads of U.S. dollar denominated Eurobonds of Latin America sovereigns driven by key asset and interest rate factors?
- Are the changes in credit spreads of U.S. dollar denominated sovereign Eurobonds of Latin America negatively related to changes in asset, and interest rate factors?

This study seeks to answer these empirical questions in a regional setting using the yield spreads between U.S. dollar sovereign Eurobonds by major Latin American issuers (Brazil, Chile, Colombia, Mexico, and Venezuela). The economic reasons underpinning the behavior of emerging market credit spreads are best understood by only investigating sovereign bonds, which have the benefit of being the most liquid and actively traded. Therefore, this study is limited to sovereign spreads and to a region that is likely representative of the behavior of credit spreads by other emerging market issuers. The sample period covers the daily yields from 25 February 2000 to 13 January 2006 of the Eurobonds issued by the governments of the above mentioned five countries, consisting of 1483 observation for each of the 18 bonds. Consequently, in addition to investigating the credit spread drivers in Latin American markets, we also investigate the change in behavior of these factors around the Argentine default which occurred in December 2001.

The chapter is organized as follows. In Section 7.2, we present some of the key issues surrounding the Latin American fixed income market. Section 7.3 presents the pricing issues related to credit spread and then Section 7.4 outlines data and methods. Empirical results are presented in Section 7.5, while Section 7.6 concludes the chapter.

7.2 PERSPECTIVE ON LATIN AMERICAN BOND MARKET

7.2.1 Introduction

Following the Asian financial crisis that began in 1997 emerging market spreads experienced a sharp increase, which continued up to the Russian financial crisis of July 1998. This is clearly evident from the JP Morgan Emerging Market Bond Index (EMBI) which rose to nearly 1600 basis points. After this sharp increase spreads stabilized and from the end of 1998 there was a systematic decline until 2007, where the EMBI is now around 200 basis points. Data provided by the BIS (2006) highlight the persistent reduction in emerging market credit spreads and comments that in first 2 months of 2006 there was a 70 basis point reduction in credit spreads of dollar denominated Latin American high yield bonds, while the European and Asian spreads declined by 20 basis points for the same period.

7.2.2 Financing Trends in the Latin American Region

The history of economic activities and the political leadership of the countries in the Latin American region are replete with economic mismanagement, inappropriate policy adoptions, and inappropriate allocation of funds. Bank intermediated financing and direct financing through the issuance of debt securities in the domestic and international markets are the two major sources that finance the activities of many emerging market governments and corporations.

Traditionally, Latin American governments rely more on the direct issuance of international bonds than their Asian or European counterparts. In fact excessive short-term financing from the international bank community was one of the key contributing reasons behind the 1997 Asian financial crisis, where nearly 60% of financing had a maturity of less than 1 year in 1997 (BIS 1999). Latin America, however, remains the major emerging market issuer in the international bond markets, which Jeanne and Guscina (2006) attribute to the absence of domestic savings. The structure of the fixed income markets

in the region also favors U.S. dollar denominated floating rate (short-term) instruments and consequently makes these economies more vulnerable to changes both in the financial conditions in the United States and contagion from U.S. dollar issuers in other markets. These features have contributed to the exacerbation of several crises in these economies (Turner 2002, Mihaljek, Scatigna, and Villar 2002).

Excess volatility in the financial markets is an inherent feature of the countries in the region. Weak international financial links and underdeveloped domestic financial markets are the prime candidates behind such excess volatility (Caballero 2000). Turner (2002) attributes the switching from international debt securities to domestic securities by emerging market issuers (especially in the Asia-Pacific region) to two main reasons: First, conscious efforts to improve the market infrastructure for bond trading by way of enhancing secondary market activities, taxation reform, and tailoring insurance policies have been undertaken. Second, the attractiveness of the domestic currency has been enhanced due to lower domestic inflation and declining domestic interest rates.

Classens, Klingebiel, and Schmukler (2003) survey of the government bond markets of 24 developed markets and 12 emerging markets found that the total size of the government sector amounted to $19.1 trillion, with 95% of this total comprising issues by governments in developed markets and only 5% is attributable to governments in emerging markets. Their analysis shows that greater importance is placed on foreign currency denominated bonds by emerging economies compared with those developed economies investigated. This finding is consistent with countries with larger economies, a larger domestic investor base and more flexible exchange rate regimes having a larger domestic currency bond market. In contrast, smaller economies with less flexible exchange rate regimes, weak economic fundamentals, and inadequate institutional frameworks rely more on foreign currency denominated bonds. The authors also show that countries that improve their institutional framework and economic fundamentals can enhance their domestic currency bond market.

7.2.2.1 Domestic Bond Market in Latin America

Economies in Latin America until recently have relied heavily on the foreign currency denominated international bonds as the main vehicle to finance their economic activities, with less reliance placed sourcing funds in domestic bond markets. On the other hand, lessons from other developed markets point to the importance of having an active and vibrant domestic bond market for the maintenance of a stable and healthy capital market. Progress has been made most by Mexico and Brazil where domestic bond markets have increased from $40.4 and $390.8 billion in 1998 to $251.5 and $623.5 billion, respectively. This increase, in excess of 100%, has resulted from key improvements in bond trading and clearing infrastructure as well as institutional changes, including a greater role for institutional investors.

It is notable that the significant portion of this growth in the domestic bond market is due to issues by the government sector. Jeanne and Guscina (2006) studied the government debt of 19 emerging markets including Latin American countries between 1980 and 2002 and find striking facts about Latin American domestic bond market. Their analysis reveals that domestic bond markets in Asia have a similar structure to that of more advanced

countries, where an overwhelming share is concentrated in local currency bonds with a fixed interest rate and a medium- to long-term maturity.

This, however, is strikingly different to the Latin America markets where the structure of the domestic debt is more concentrated in variable interest rate instruments. Fixed and variable interest rate long-term domestic currency denominated bonds are around 10% and 5% of the total domestic debt respectively. Similarly, long-term foreign currency denominated variable interest rate debt is around 10% of the total domestic market while the fixed rate foreign currency denominated bonds account for only around 2.5%. In addition, they also find that there has been a decline in the number of issues with a medium- to long-term maturity. This is attributed to weak economic fundamentals and monetary instability in Latin America.

The domestic debt structure between individual countries also reveals a diverse pattern. For example, the majority of Argentine domestic debt is denominated in foreign currency due to its economic and financial circumstances and stands out as country that relies heavily on foreign currency denominated bonds among the emerging market economies. Brazil, on the other hand, stands out as a nation with the largest domestic currency bond market in the region with 35% of its domestic debt denominated in local currency (variable interest). A substantial portion of domestic debt in Venezuela is denominated in local currency with a variable interest rate. Mexico, on the other hand, has a minor proportion of foreign currency denominated debt while spreading equal proportion across medium- to long-term fixed interest rate debt denominated in local currency.

7.2.2.2 International Bank Financing

Rapid structural reforms have been undertaken in the banking sector in Latin America in the recent past to overcome the shortcoming in the sector and to bring about a banking and financial system that is efficient and resilient. Compared to other emerging market regions, Latin America accounts for only 15% of the total international bank lending to emerging markets, with between 40% and 50% of this lending attributed to short-term financing (BIS, 2006). The Asia-Pacific region historically has been the major focus of international bank financing up until 2004 when Eastern Europe became the major beneficiary.

7.2.2.3 International Bond Issues

Table 7.1 provides information on the scale of international bonds issued by emerging market regions. Attention is drawn to the Latin American countries. The Latin American region was the single major issuer in the international bond market, occupying nearly 54% of the total size. However, this position of dominance is maintained by Latin American countries in 2006, but with the relative position of only 35% of the total outstanding of international bonds belonging to emerging markets for the last quarter of 2006. The data provided by the BIS (2007) show that the overall issues by Latin America in the international bond market have remained stable over the last 8 years. Emerging markets in the European region have increased their issues in the international bond market catching up with the Asian region. Brazil, Mexico, and Argentina are the sizable issuers that stand out in the Latin American region in terms of size.

TABLE 7.1 International Bonds and Notes Issued by Selected Latin American Countries' Amounts Outstanding (U.S. Dollars in Billions)

	1998 Dec	1999 Dec	2000 Dec	2001 Dec	2002 Dec	2003 Dec	2004 Dec	2005 Dec	2006 Dec
All countries	4181.1	5116.8	6004.4	7204.7	8841.4	11,156.9	13,311.2	13,964.0	17,561.6
Developed countries	3263.3	4146.6	4974.2	6140.7	7665.8	9820.5	11,811.8	12,415.8	15,791.2
Developing countries	508	542.5	584.9	588.1	634.3	705	795.8	841.2	993.9
Africa and Middle East	17.7	22	26.4	31.1	36.5	44.6	55.4	66.5	104.6
Asia-Pacific	142.6	142	137.9	135.6	153.8	176.3	211.4	240.4	285.3
Europe	74.2	79.6	104	105.5	117.1	140.8	172.1	198.8	255.3
Latin America and Caribbean	273.5	299	316.5	315.9	326.9	343.3	356.9	335.5	348.8
Argentina	72.2	79.6	86.7	89.8	90.9	94.1	92.7	58.0	60.8
Brazil	71.1	76.7	83.8	80.9	87.4	102.2	104.9	107.5	109.9
Chile	3.4	4.8	5	6.6	8.8	10.3	11	11.1	10.9
Colombia	6.2	7.7	8.9	12.1	12.5	13.6	14.4	14.7	16.2
Mexico	84.3	92.7	93.6	87.8	87	80.1	86.5	88.8	92.7
Peru	4.3	4.3	4.3	4.3	4.6	4.9	6.3	8	9.4
Uruguay	2.3	2.7	3.1	3.9	4.2	4.4	4.8	5.6	8.3
Venezuela	20.8	21.1	19.4	20.2	19.4	20.1	21.1	25.7	22.6
United States	796.4	1229.2	1653.5	2259.9	2656.5	2997.5	3269.9	3449.8	3972.1

Source: BIS International Financial Statistics—Securities (Table 15 B), Bank of International Settlements, available online at http://www.bis.org/publ/qtrpdf/r_qa0703.pdf/.

7.3 PRICING ISSUES

The building block of risky debt valuation consists of the interest rate and default cum asset recovery process. The term structure of interest rates defines the future evolution of interest rates through a probabilistic description. It essentially measures the relationship among the yields on default-free interest instruments that differ only in their terms to maturity. Vasicek (1977), Cox, Ingersoll, and Ross (1985), Ho and Lee (1986), Hull and White (1990), and Heath, Jarrow, and Morton (1992) are some of the popular studies that attempt to model interest rate process. The model by Vasicek* (1977) proposes that the interest rate follows a mean reverting Ornstein–Uhlenbeck process. Adopting the property of mean reversion, the short rate r rises when it is below the long-term mean, and falls when it is above the long-term mean. As far as the rate dependence on volatility is concerned, Vasicek (1977) assumes it to be constant, while Cox, Ingersoll, and Ross (1985) treat rate dependent volatility as proportional to the square root of the short rate. One of the theoretical problems associated with Vasicek's model is that it can generate negative interest rates. The key advantage is that it can be employed in a systematic manner to price interest rate sensitive interments.

The second building block of credit-risk pricing is the default process, which attempts to capture the possible implications of a credit-risk event. Sundaresan (2000) categorizes the literature on credit-risk pricing into three areas—structural models or firm value approach, reduced form models, and structural models with strategic behavior. The first category of structural models of default, or firm value approach, to credit-risk pricing assumes that default takes place as the forcing process reaches a reorganization boundary where the allocation of residual values takes place exogenously. Black and Scholes (1973), Merton (1974), Black and Cox (1976), Ingersoll (1977), Brennan and Schwartz (1980), John (1993), Kim, Ramaswamy, and Sundaresan (1993), and Longstaff and Schwartz (1995) are key authors that adopt this approach.

The second category of models examine the exogenous specification of default outcomes and recovery rates based on an arbitrage-free valuation by assigning probability of default and recovery rates exogenously while deriving pricing formula, which can be calibrated to data. Key examples include Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), and Duffie and Singleton (1999). The third category of models utilizes structural models together with game theory to study the strategic behavior. Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996), and Mella-Barral and Perraudin (1997) are some of the studies that endogenise the lower reorganization boundary while accommodating the behavior of firms in distress situations.

7.3.1 Structural Models

The structural framework captures credit risk based on the economic and financial fundamentals of the risky bond issuers by treating the equity of the firm as a call option

* $dr(t) = (a + br(t))dt + \sigma dW(t)$ where the interest rate follows an Ornstein–Uhlenbeck process and a, b, and σ are parameters of the process and $W(t)$ is the standard Brownian motion.

on the assets of the issuers. These models exogenously specify a particular firm value process and assume that default is triggered when the firm value falls to some explicit threshold. The structural approach views risky debt as a contingent claim on the value of the issuer. Interest rates are assumed to be constant and an option-pricing framework was used to model default risk for bonds. Merton (1974) formalized this theoretical base and developed a model for pricing risky debt by introducing the theory of a risk structure of interest rates. This perspective views the value of a risky debt as dependent upon (1) the required rate of return on the risk-free debt, (2) provisions on the debt and restrictions contained in the indenture, and (3) the probability of default.

Although the structural framework was widely used by subsequent researchers (Geske 1977, Ingersoll 1977), the major problem was the assumption concerning the occurrence of the default event. A default event is conditioned to take place only when the firm exhausts all its assets while in practice firms usually default long before assets are exhausted. Black and Cox (1976) significantly extended the previous work of Merton (1974) by explicitly modeling the effects of safety covenants, subordination arrangements, and restrictions on financing of interest and dividend payments. One of the important aspects of the Black and Cox (1976) study is that it conditions the default event to occur before the firm exhausts its assets. Longstaff and Schwartz (1995) incorporate many distinguishing features of the structural framework and formulate a dynamic continuous-time valuation framework which provides a simple closed form model. They relax the restrictive assumptions relating to the interest rate process and the default threshold through a dynamic interest rate process and by allowing an early default.

They make six important assumptions relating to the firm value process, the interest rate process, the capital structure of the issuer, the default process, the payoff in the event of default, and the market settings. The dynamic of the total value of the assets of the issuer (dV) is captured through a standard Weiner process: $dV = \mu V dt + \sigma V dZ_1$, where σ is a constant and Z_1 is a standard Weiner process.

The dynamics of the short-term interest rate (r) is accommodated through a Vasicek type of interest rate process and is given by $dr = (\zeta - \beta r)dt + \eta dZ_2$, where ζ, β, and η are constants and Z_2 is a standard Weiner process. They assume that the firm value V is independent of the capital structure of the firm and is in line with the Modigliani–Miller proposition (1958). This assumption implies that any cash outflows arising from existing debts are financed by issuing new debt and therefore the firm value is independent of the capital structure.

Solvency ratio X which is the ratio of threshold value K to firm value V takes care of the default process. The issuing firm is solvent when the firm value is above the threshold point and the firm enters bankruptcy if the ratio reaches 1 where $K = V$. Once the firm reaches insolvency position all debt contracts concurrently enter the default status. Longstaff and Schwartz (1995) assume that debt holders receive $1 - w$ times the face value of the debt at maturity where w represents the written down value of the bond.

Longstaff and Schwartz (1995) specify the price of a risky bond with maturity T as an explicit function of solvency ratio X, interest rate r, and maturity T where the price of the

risky bond is an increasing function of the solvency ratio X and a decreasing function of r and T.

$$P(X,r,T) = D(r,T) - wD(r,T)Q(X,r,T) \tag{7.1}$$

The first term $D(r,T)$ in the above equation represents the value of a benchmark risk-free bond and $wD(r,T)$ represents the present value of the loss to the risky bond holder in the event of distress. $Q(X,r,T)$ represents the risk-neutral probability of default. The above equation can be rearranged to represent the credit spread as the yield difference between a risky bond $P(X,r,T)$ and a risk-free bond $D(r,T)$.

Using the simple closed form model of Longstaff and Schwartz (1995), Batten and Hogan (2002) provide the following platform to test yield spread empirically. They differentiate Equation 7.1 and substitute the yield on a risk-free bond of $-\ln(D(r,T))/T$ to obtain the yield of a risky bond. The yield difference between the risky bond and the risk-free bond is the credit spread and is given by Equation 7.2,

$$S = -\frac{\ln(1 - wQ(X,r,T))}{T} \tag{7.2}$$

with the first difference of S, ΔS

$$= \frac{wQ \times X}{T(1 - wQ(X,r,T))}\frac{\Delta X}{X} + \frac{wQr}{T(1 - wQ(X,r,T))}\Delta r + \left(\frac{wQr}{T(1 - wQ(X,r,T))} - \frac{\ln(1 - wQ(X,r,T))}{T^2}\right)\Delta T \tag{7.3}$$

Having obtained the first difference of the credit spread (S) the regression form of Longstaff and Schwartz (1995) is given by simple Equation 7.4,

$$\Delta S_t = a + b\Delta Y_t + c\Delta I_t + \varepsilon_t \tag{7.4}$$

where $\Delta S_t = S_t - S_{t-1}$ is the change in credit spread between a risky bond and a risk-free U.S. T-bond with the same maturity. $\Delta Y_t = Y_t - Y_{t-1}$ is defined as the change in interest rate factor. $\Delta I_t = I_t - I_{t-1}$ is the change in the asset factor which is proxied by the return on the broader stock market index. Regression coefficients are represented by a, b, and c.

7.4 DATA AND METHODS

We use daily yield series belonging to five sovereign issuers in Latin America. For the purpose of this study only U.S. dollar denominated sovereign issues without a call provision are used. Latin American sovereign issues were searched in the Reuters fixed income database, with 18 bonds identified that fit this criteria. The sample period covers February 2000–January 2006 (1483 observations). The U.S. benchmark bonds with a similar maturity were used as the risk-free bonds to generate the spreads.

Following Longstaff and Schwartz (1995) and Batten, Fetherston, and Hoontrakul (2006) we proxy the change in the asset factor ΔI_t by the return on the stock market indices of Brazil, Chile, Colombia, Mexico, and Venezuela. Specifically, the Bovespa index for Brazil, the IPSA index for Chile, HSBC JCLACOL for Colombia, the Mexican Bolsa Index for Mexico, and the IBC index for Venezuela were used in this study. The Bovaspa index was chosen since it is a total return index and comprises the most liquid stocks from the Sao Paulo Stock Exchange. The Chilean IPSA index incorporates 40 actively traded stocks from the Santiago Stock Exchange and is regarded as the most popular market index. The IGBC index, from the Colombia Stock Exchange, commences on July 2001 and so could not be used given the February 2000 start date of our sample period. Instead, we utilize the HSBC JCLACOL index as the proxy for the Colombian asset factor. The Mexican Bolsa index from the Mexican Stock Exchange is the capitalization weighted index comprising of leading stocks, while the IBC index from the Caracas Stock Exchange of Venezuela comprises the most liquid and capitalized stocks.

The credit spread ΔS_t and interest rate variable ΔY_t were determined by first matching each of the sovereign bonds with a near maturity U.S. Treasury bonds, with the following bonds selected:

1. 6.25% coupon maturing on February 15, 2007
2. 6.625% coupon maturing on May 15, 2007
3. 5.625% coupon maturing on May 15, 2008
4. 9.125% coupon maturing on May 15, 2009
5. 6.0% coupon maturing on August 15, 2009
6. 7.5% coupon maturing on November 15, 2016
7. 8.5% coupon maturing on February 15, 2020
8. 6.75% coupon maturing on August 15, 2026
9. 6.625% coupon maturing on February 15, 2027
10. 6.375% coupon maturing on August 15, 2027
11. 6.25% coupon maturing on May 15, 2030

Table 7.2 reports the summary statistics for the spreads between the respective Latin American sovereign Eurobonds and the benchmark U.S. Treasury bonds. The mean spread for Brazil and Venezuela is higher than the Chilean and the Mexican spreads. Colombian spreads fall in between these two groups. The standard deviation of the spreads also reveals a similar pattern with higher volatility being associated with Brazilian and Venezuelan series. Higher spreads and higher volatility reflect the economic conditions that prevailed in Brazil and Venezuela during the sample period. The mean spread for Brazil ranges from 6.91% to 7.81%; on the other hand, the mean spreads for Mexican issues are in the range of 1.69%–2.68%. Similar to Brazilian spreads, the mean spreads for Venezuelan issues are in the range of 6.91%–7.9%. Colombian spreads are in the range of 4.27%–5.74%, which lies between the Brazilian and Venezuelan range. It should be noted that all of the mean

TABLE 7.2 Descriptive Statistics for Spreads on Latin American Sovereign Issues and U.S. Benchmark Issues

	BRA 08	BRA 09	BRA 20	BRA 27	BRA 30	CHI 09	COL 07	COL 08	COL 09	COL 20	MEX 07	MEX 08	MEX 09	MEX 16	MEX 26	VEN 07	VEN 18	VEN 27
Mean	6.91	7.65	7.77	7.25	7.81	1.36	4.27	4.59	4.87	5.74	1.69	1.92	2.08	2.43	2.68	7.30	7.91	6.90
Median	5.87	6.90	7.11	6.84	7.32	1.40	4.46	4.56	4.56	5.79	1.65	1.84	2.06	2.38	2.61	7.33	8.33	7.64
Maximum	25.40	26.15	23.12	19.32	21.82	2.60	11.69	11.42	11.48	10.32	3.97	3.97	4.29	4.31	5.06	17.97	13.55	10.62
Minimum	1.09	1.35	2.57	2.78	3.14	0.50	0.35	0.72	0.93	2.47	−0.21	0.40	0.54	0.83	1.29	1.46	2.82	2.63
Standard deviation	4.93	4.73	3.74	3.14	3.51	0.61	2.54	2.38	2.20	1.70	1.12	1.02	1.06	0.94	0.71	3.98	2.40	2.02
Skewness	1.67	1.54	1.58	1.38	1.55	0.11	0.33	0.35	0.38	0.27	0.19	0.19	0.16	0.08	0.14	0.32	−0.20	−0.43
Kurtosis	5.57	5.52	5.56	4.92	5.52	1.57	2.33	2.38	2.71	2.26	1.52	1.54	1.49	1.61	1.95	2.38	2.26	2.01

Note: The credit spreads are calculated as the arithmetic difference in the yields of the respective sovereign bond and the equivalent maturity of 11 U.S. Treasury bonds. The columns are coded BRA (Brazil), CHI (Chile), COL (Colombia), MEX (Mexico), and VEN (Venezuela). The maturity of the spread pairs ranged from 2007 (07) to 2027 (27).

spreads increase with maturity—that is the longer the maturity of the bond, the higher the mean credit spread for each country. There is also excess kurtosis on Brazilian spreads compared to other spread series in this study. A common feature that we observe in the descriptive statistics is that the standard deviations within each market consistently tend to decrease with increasing maturity.

During our sample period the capital reserves of Argentina were depleted (23rd of December 2001) and the country declared a moratorium on international debt repayments. Therefore, for the purposes of this study we choose the official default date as 23rd of December 2001. We then divide the sample into two subperiods—pre-Argentine default period and post-Argentine default period—in addition to the analysis comprising all observations for the full sample period. Accordingly our sample is subdivided as precrisis period (25 February 2000–23 December 2001) and postcrisis period (24 December 2001–13 January 2006) consisting of 457 and 1027 observations, respectively.

7.4.1 Changes in Credit Spreads

An important implication of Longstaff and Schwartz (1995) model is that credit spread, the yield difference between a risky bond and a risk-free benchmark bond, is driven by two major factors—an asset factor and an interest rate factor. Given the problems of heteroskedasticity and autocorrelation associated with the data in our sample we employ the Bollerslev (1986) GARCH(1,1) specification within an autoregressive moving average (ARMA) framework, with Equation 7.4 now specified as

$$\begin{aligned} \Delta S_t &= a + b\Delta Y_t + c\Delta I_t + AR(1) + MA(1) + \varepsilon \\ \sigma^2 &= \alpha + \beta\varepsilon_{t-1}^2 + \gamma\sigma_{t-1}^2 \end{aligned} \tag{7.5}$$

where $\Delta S_t = S_t - S_{t-1}$ is the change in credit spread between the risky Latin American bond and the risk-free U.S. T-bond with the same maturity. $\Delta Y_t = Y_t - Y_{t-1}$ is defined as the change in the interest rate factor. $\Delta I_t = I_t - I_{t-1}$ is the change in the asset factor which is proxied by the return on the broader stock market index of individual countries in our sample. a, b, and c are the regression coefficients of the mean equation. In line with the theory, we expect the regression coefficients b and c representing interest rate factor and asset factor to be inversely related to the dependent variable, ΔS_t. The conditional variance term (σ^2) in the variance equation is a function of the mean α; the ARCH term $\beta\varepsilon_{t-1}^2$ which is measured as the lag of the squared residuals from the mean equation effectively represents the information about the volatility from the previous period; and the GARCH term $\gamma\sigma_{t-1}^2$ representing the last period's forecast variance. The Bollerslev and Wooldridge (1992) procedure was applied to ensure that the statistical significance of the results was not affected by non-normally distributed residuals.

7.5 EMPIRICAL RESULTS

To provide an insight into the applicability of structural models in market settings that are often described as immature, highly volatile, and a region replete with default events, we investigate five important bond markets in Latin America in this study. We examine the

spreads between 18 sovereign issues matched with their U.S. benchmark bonds of the same maturity structure. For simplicity this study focuses only on sovereign Eurobond issues with no embedded options such as callable, puttable, and convertible bonds. Given the problems of heteroskedasticity and autocorrelation we employ a GARCH(1,1) specification within an ARMA framework, ARMA(1,1), to accommodate the time varying volatility structure of the return series and autocorrelation in the regression residuals at lag one. The regression tested the change in credit spread as a function of well-established asset factor and interest rate factor with AR(1) and MA(1) terms.

The theory on credit risk suggests that credit spreads are inversely related to interest rate factor and asset factor. Our regression of changes in credit spreads on changes in asset and interest rate factor confirms this theoretical proposition and the coefficients are highly significant (beyond 99.9%). Table 7.3 outlines the results for the whole sample (1483 observations), with the inverse relationship between the changes in credit spread and the interest rate factor clearly evident for all of the 18 bonds in our sample.

Argentina officially defaulted on the 23rd of December 2001 and to account for this credit event we divide the data into two subperiods with the analysis conducted for the whole period as well as subperiods. For the sake of brevity the results from this additional analysis are not reported. With the exception of two cases, the coefficient for the asset factor for the whole sample as well as the pre- and post-Argentine default period were negative as predicted by the model. This is consistent with the existing evidence both in emerging and developed markets. The ARMA terms were also found to be significant in most cases suggesting that there was pricing inefficiency in the market. The coefficients on the lagged squared error and lagged conditional variance in the conditional variance equation are also highly statistically significant (beyond 99.9%). In addition, the sum of the coefficients of the lagged squared error and lagged conditional variance is very close to unity which implies that the shock to the conditional variance will be highly persistent.

7.6 CONCLUSION

We test the Longstaff and Schwartz (1995) model in an emerging market setting using the spread between 18 sovereign issues matched with a U.S. benchmark bond of equivalent maturity. Brazil, Chile, Colombia, Mexico, and Venezuela were the specific markets in the emerging Latin American region that were included in this study. The purpose of this investigation was to provide an insight into the valuation issues surrounding emerging Latin American fixed income markets and the efficacy of Longstaff and Schwartz (1995) type structural model in market settings that are often described as immature and excessively volatile. Given the problems of heteroskedasticity and autocorrelation associated with the data we employ a GARCH(1,1) specification within an ARMA(1,1) framework. Tests were conducted for the whole sample as well as the subsample for the 2001 pre- and post-Argentine crisis periods. As predicted by structural models, changes in credit spreads of emerging Latin American markets were driven by interest rate and asset factors with these two factors negatively related to changes in yield spreads.

TABLE 7.3 Regression of Changes in Credit Spreads on Changes in Interest Rate Factor and Asset Factor—Whole Sample

	$\Delta S_t = a + b\Delta Y_t + c\Delta I_t + \mathrm{AR}(1) + \mathrm{MA}(1) + \varepsilon,\ \sigma^2 = \alpha + \beta\varepsilon_{t-1}^2 + \gamma\sigma_{t-1}^2$										
	a	***b***	***c***	**AR(1)**	**MA(1)**	***α***	***β***	***γ***	**AR^2**	**F**	**SSE**
					Brazil						
	−0.009	−0.738	−2.054	−0.991	0.992	0.000	0.146	0.870	0.031	7.801	212.886
BRA 08	−2.737	−11.088	−7.740	−125.170	121.030	1.526	5.090	34.349			
	(0.006)	(0.000)	(0.000)	(0.000)	(0.000)	(0.127)	(0.000)	(0.000)		(0.000)	
	−0.006	−0.620	−2.545	−0.994	0.994	0.000	0.107	0.912	0.038	9.348	177.377
BRA 09	−2.067	−10.109	−13.382	−50.025	54.724	0.291	3.800	52.592			
	(0.039)	(0.000)	(0.000)	(0.000)	(0.000)	(0.771)	(0.000)	(0.000)		(0.000)	
BRA 20	−0.007	−0.560	−2.255	−0.894	0.912	0.000	0.153	0.870	0.030	7.500	147.523
	−2.626	−10.818	−12.929	−9.472	10.616	1.173	2.890	25.269			
	(0.009)	(0.000)	(0.000)	(0.000)	(0.000)	(0.241)	(0.004)	(0.000)		(0.000)	
	−0.005	−0.504	−2.556	0.994	−0.997	0.003	0.244	0.680	0.030	7.606	139.838
BRA 27	−3.150	−5.329	−8.118	148.107	−190.255	4.796	3.126	11.382			
	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)		(0.000)	
	−0.003	−0.540	−2.449	0.209	−0.235	0.000	0.129	0.885	0.034	8.450	137.507
BRA 30	−1.176	−9.539	−10.752	0.165	−0.186	1.679	4.042	39.012			
	(0.240)	(0.000)	(0.000)	(0.869)	(0.852)	(0.093)	(0.000)	(0.000)		(0.000)	
					Chile						
	−0.001	−0.217	−0.235	−0.014	−0.300	0.000	0.175	0.855	0.197	52.991	4.390
CHI 09	−0.979	−12.929	−1.134	−0.089	−2.080	1.508	4.419	36.941			
	(0.328)	(0.000)	(0.257)	(0.929)	(0.038)	(0.132)	(0.000)	(0.000)		(0.000)	

					Colombia						
COL 07	−0.009	−0.863	−0.154	0.203	−0.385	0.000	0.249	0.801	0.088	21.383	57.836
	−2.933	−16.903	−0.743	0.967	−1.738	1.784	5.273	33.439			
	(0.003)	(0.000)	(0.458)	(0.334)	(0.082)	(0.075)	(0.000)	(0.000)		(0.000)	
	−0.006	−0.756	−0.487	−0.994	0.993	0.000	0.195	0.841	0.068	16.421	56.901
COL 08	−2.202	−15.237	−1.596	−43.546	39.654	1.272	5.997	28.652			
	(0.028)	(0.000)	(0.110)	(0.000)	(0.000)	(0.203)	(0.000)	(0.000)		(0.000)	
	−0.010	−0.695	−0.408	−0.809	0.858	0.000	0.223	0.830	0.013	3.734	82.016
COL 09	−4.035	−16.993	−2.226	−6.639	9.071	1.017	3.391	19.359			
	(0.000)	(0.000)	(0.026)	(0.000)	(0.000)	(0.309)	(0.001)	(0.000)		(0.001)	
	−0.011	−0.710	−0.686	−0.241	0.404	0.000	0.261	0.810	0.073	17.635	36.740
COL 20	−4.516	−8.754	−2.648	−0.962	1.731	2.372	2.537	22.395			
	(0.000)	(0.000)	(0.008)	(0.336)	(0.084)	(0.018)	(0.011)	(0.000)		(0.000)	
					Mexico						
	−0.001	−0.614	−0.839	0.181	−0.234	0.000	0.072	0.917	0.292	88.348	8.281
MEX 07	−0.298	−15.792	−4.204	0.506	−0.657	1.767	3.392	36.518			
	(0.766)	(0.000)	(0.000)	(0.613)	(0.511)	(0.077)	(0.001)	(0.000)		(0.000)	
	−0.001	−0.585	−0.744	−0.275	0.226	0.000	0.118	0.883	0.316	98.657	6.908
MEX 08	−0.720	−22.876	−4.821	−0.533	0.432	2.410	4.990	46.363			
	(0.472)	(0.000)	(0.000)	(0.594)	(0.666)	(0.016)	(0.000)	(0.000)		(0.000)	
	−0.001	−0.505	−0.969	−0.231	0.186	0.000	0.114	0.887	0.254	73.051	8.105
MEX 09	−0.903	−20.772	−7.113	−0.752	0.592	2.401	3.657	31.970			
	(0.367)	(0.000)	(0.000)	(0.452)	(0.554)	(0.016)	(0.000)	(0.000)		(0.000)	
	−0.002	−0.519	−0.763	0.663	−0.614	0.000	0.143	0.842	0.276	81.449	5.783

(*continued*)

TABLE 7.3 (continued) Regression of Changes in Credit Spreads on Changes in Interest Rate Factor and Asset Factor—Whole Sample

	$\Delta S_t = a + b\Delta Y_t + c\Delta I_t + \mathrm{AR}(1) + \mathrm{MA}(1) + \varepsilon,\ \sigma^2 = \alpha + \beta\varepsilon_{t-1}^2 + \gamma\sigma_{t-1}^2$										
	a	***b***	***c***	**AR(1)**	**MA(1)**	***α***	***β***	***γ***	**AR^2**	**F**	**SSE**
MEX 16	−1.233	−16.909	−5.730	3.715	−3.253	3.867	3.722	25.589			
	(0.218)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)		(0.000)	
	−0.002	−0.507	−1.137	−0.920	0.984	0.000	0.218	0.825	0.108	26.554	9.669
MEX 26	−1.077	−17.558	−5.273	−22.232	434.263	1.422	4.339	36.784			
	(0.282)	(0.000)	(0.000)	(0.000)	(0.000)	(0.155)	(0.000)	(0.000)		(0.000)	
					Venezuela						
VEN 07	0.073	−0.915	−0.008	0.076	−0.353	0.029	0.251	0.523	0.060	14.518	152.966
	1.096	−10.002	−1.210	0.479	−2.221	1.334	1.957	3.862			
	(0.273)	(0.000)	(0.226)	(0.632)	(0.026)	(0.182)	(0.050)	(0.000)		(0.000)	
	−0.008	−0.758	−0.225	0.913	−0.883	0.000	0.118	0.889	0.098	23.861	30.389
VEN 18	−3.074	−17.460	−1.246	14.085	−11.643	2.252	4.979	51.365			
	(0.002)	(0.000)	(0.213)	(0.000)	(0.000)	(0.024)	(0.000)	(0.000)		(0.000)	
	−0.006	−0.625	−0.458	0.275	−0.182	0.000	0.140	0.866	0.077	18.722	21.189
VEN 27	−2.593	−13.675	−2.641	0.876	−0.568	2.593	6.886	56.317			
	(0.010)	(0.000)	(0.008)	(0.381)	(0.570)	(0.010)	(0.000)	(0.000)		(0.000)	

Note: For each of the 18 Latin American bonds we estimate the following GARCH(1,1) specification of the regression $\Delta S_t = a + b\Delta Y_t + c\Delta I_t + AR(1) + MA(1) + \varepsilon$, $\sigma^2 = \alpha + \beta\varepsilon_{t-1}^2 + \gamma\sigma_{t-1}^2$. The term ΔS is the change in credit spread (spreads are generated by comparing the yields of risky Latin American sovereign with the yields of U.S. bench mark bond of the same maturity), ΔY is the change in the daily yield of U.S. government Treasury bonds with the same maturity, ΔI_t is the daily logarithmic return of Brazil (BRA), Chile (CHI), Colombia (COL), Mexico (MEX), and Venezuela (VEN) stock markets, AR(1) and MA(1) are autoregressive and moving average terms at lag one. Regression coefficients together with Z-statistics and their associated p values are reported.

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CHAPTER 8

Accounting Data Transparency and Credit Spreads: Clinical Studies

Umberto Cherubini

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8.1 INTRODUCTION

In a statement released in 2007, the U.S. Treasury secretary complained about the huge number of accounting data revisions in the U.S. market. He claimed that some 1500 firms revise their balance sheet figures every year, and that means, he claimed, that there must be something wrong with the system.

So, the debate on accounting transparency that arose at the beginning of the century is still open. Recent literature has focused on the impact of these events on the evaluation of corporate liabilities, namely equity and bonds. On theoretical grounds, Duffie and Lando (2001) were the first to attempt to model the impact of accounting noise on the credit spread term structure. On empirical grounds, Yu (2005) proved that accounting noise is actually priced in the market: a risk premium is charged to the credit spreads of firms that adopt less transparency. Fraud events that took place both in the United States (WorldCom, Tyco, Enron, etc.) and in Europe (Cirio, Marconi, Parmalat, etc.) in the first years of the century raised the issue of distinguishing between unbiased noise owing to measurement errors and cases of deliberate fraud. Inspired by these cases, Baglioni and Cherubini (2006) proposed a model adding the risk of deliberate fraud to measurement errors in accounting figures. The empirical issue whether the risk of running into a case of fraud is actually accounted for in prices, and how it impacts on the values of equity and debt remains an open question.

In this chapter, we begin to gather evidence to provide hints to the solution of this problem, and we do so by performing a sort of clinical studies on some of the cases quoted above. The analysis also provides material to go over the debate that followed those cases. That debate was (and still is) centered on two questions. Did the market know about, and care for, accounting data transparency? How did it perceive the problem? And what was the impact on equity and debt values? Structural models would predict that if lack of transparency results in an increase in noise, that would only redistribute value from bondholders to stockholders, while fears of deliberate misreporting would cause loss of value for both of them. The second question is whether someone in the market knew more than others. More precisely, were banks and professional investors endowed with superior information on the event of fraud, with respect to retail investors? A natural test to provide an answer to this question can be found in the dynamic relationship of the credit derivatives markets and the stock market. The credit derivatives markets, and in particular the CDS (credit default swap) market, are in fact reserved to financial institutions and professional investors, while the stock market is open to the general public of investors. If this is the case, some information advantage of the professional investors would result in misalignment of the two markets, and possibly in a lead–lag relationship between the two.

The plan of the chapter is as follows. In Section 8.2, we provide a brief review of the structural credit models' literature with particular focus on the contributions addressing the problem of accounting transparency. In Section 8.3, we discuss the empirical implications that one would expect to find if he allows for biases and noise in accounting data. In Section 8.4, we describe the data set and go over the chronicle of the transparency cases under study. In Section 8.5, we provide evidence of the joint dynamics of the value of equity and debt. Section 8.6 concludes the chapter.

8.2 CREDIT SPREAD PUZZLE

Credit risk models have in common the goal of explaining two figures, default probability and loss given default, whose product yields the expected loss figure. This represents credit risk in the evaluation of corporate bonds, credit derivatives written on the name of the issuer, counterparty risk in derivatives transactions against him, and so on. The main difference in the approach to this problem is in the pricing tool used. The general and well-known taxonomy of credit risk models denotes "structural models," those using option pricing theory to evaluate credit risk and "reduced form models," those using term structure theory to explain credit spread behavior. Reduced form models, that are nowadays largely used in credit risk pricing, own their development to the inadequacy of structural models to explain the observed behavior of credit spreads in the market.

In structural models, corporate liabilities are evaluated by decomposing their payoffs in linear and nonlinear products, and using standard option pricing theory to price them. The seminal work in this literature is due to Merton (1974), even though the world famous Black and Scholes (1973) were already targeted at the pricing of corporate liabilities. Default risk turns into a nonlinearity in the payoff

$$D(T,T) = B - \max\left[B - V(T),0\right] \tag{8.1}$$

where

$D(T,T)$ is the value of defaultable debt, assumed to be a zero coupon bond, at maturity
B is the nominal value of debt
$V(T)$ is the value of the firm

So, the value of the defaultable zero coupon bond at any time t would be given by

$$D(t,T) = P(t,T)B - \mathrm{PUT}(V,t:B,T) \tag{8.2}$$

where $P(t,T)$ denotes the risk-free discount factor and $\mathrm{PUT}(\cdot)$ denotes a put option written on the value of the firm, with strike price equal to the face value of debt: the default put option. As for equity, it is easy to see that its value ought to be equal to that of a call option with same underlying, strike price, and exercise date.

$$S(t) = \mathrm{CALL}(V,t:B,T) \tag{8.3}$$

Structural models are particularly elegant and informative, mainly because they impose an arbitrage relationship between equity and debt. From this point of view, they provide a natural guideline for relative value trades between the stock market and the credit derivatives market which is of utmost practical interest. Unfortunately, these models result in a generally poor fit to market data (see, Jones et al., 1984; Huang and Huang, 2003; Eom et al., 2004). This shows up in three main empirical regularities:

- Typically, reasonable values for leverage of the firm and volatility of assets produce lower credit spreads than those observed on the market.
- Undervaluation is particularly relevant for short-term maturities: a typical credit term structure in structural models shows a hump and zero intercept.
- Undervaluation is particularly relevant for high credit standing obligors.

Several answers have been proposed as possible solutions to these problems. Anderson and Sundaresan (1996) suggest that the owner of the firm may engage in a strategic debt rescheduling process to exploit the bankruptcy costs at the expense of bondholders. Along the same lines, Leland (1994) and Leland and Toft (1996) allow the owner of the firm to terminate the investment in such a way as to optimize the value of equity, again at the expense of debt. An alternative explanation for the failure of structural models to fit the data stems from the fact that the value of the firm is not directly observed and this lack of transparency may affect the prices in the market. This route was first followed by Duffie and Lando (2001), who propose a model with endogenous bankruptcy in which the market is assumed to observe a noisy signal of the value of the firm at discrete times. Several extensions have been provided to this model. Coculescu et al. (2006) generalize the approach by incorporating noise in a continuous time model that precludes arbitrage from insiders. Herkommer (2006) introduces correlation between the value of the firm and noise, and finds evidence of negative correlation. Other approaches based on imperfect information, concerning both the value of the firm and the default thresholds, have been proposed (CreditGrades, 2001; Giesecke and Goldberg, 2004). Cetin et al. (2004) and Guo et al. (2006) propose a different approach in which the market is assumed to only partially observe, and possibly with a lag, relevant information concerning the state of the firm.

Other models have proposed a distinction between noise and deliberate misreporting from managers and accountants. Baglioni and Cherubini (2006) model the bias in the signal as a hidden state to which the market attributes some probability. Differently from the Duffie and Lando approach, allowing for biased accounting information in addition to noise enables to explain undervaluation of credit spread of high credit grade obligors. Furthermore, while noise results in a transfer of wealth from bondholders to equity holders, biased information leads to a general reduction of the value of the firm. The goal of this literature is to explore the relevance of corporate governance and other institutional features to explain credit spread behavior. Along these lines, Capponi (2007) extends the model by making misreporting endogenous.

8.3 ESTIMATING CREDIT RISK UNDER INCOMPLETE ACCOUNTING INFORMATION

A key claim of the literature on accounting transparency surveyed above is that incomplete information may have a relevant impact on the evaluation of credit risk and corporate liabilities in general. To illustrate the topic in an orderly fashion, consider first the standard case with perfect accounting information. Assume balance sheet data to be truthfully reported at market value. Then, innovations on such values should be immediately reflected in the value of equity and bonds, and should result in movements in the same direction. Changes in the volatility of assets would instead cause a transfer of wealth from bondholders to shareholders or vice versa. This should be reflected in prices and returns on the most liquid markets for the corporate liabilities issued by the obligor and contingent claims written on its name. Nowadays, the markets that are considered for their liquidity the most are as follows:

- Stock market
- CDS market

These markets are used to back out information about credit risk of the firm. As for the stock market, two main strategies are available. One follows the so called KMV approach, which is based on the idea of extracting information about the value of the firm and its volatility from a system of two equations, one determining the value of equity and the other the corresponding volatility. If one assumes that volatility of assets does not change much across the sample, a more efficient way to estimate volatility and the value of the firm is to use the maximum likelihood approach on transformed data proposed by Duan (1994, 2000). Further elaborations on this subject were provided by Brockman and Turtle (2003), Duan et al. (2004), and Bruche (2004).

The CDS alternative uses the premia of contracts arranged in an increasing maturity order to recover the term structure of the probability of default maturity by maturity. This technique is called bootstrapping, a term borrowed from the estimation of the term structure in the fixed income market. Namely, the formula that we are going to use in the analysis below is

$$Q(t_n) = Q(t_{n-1})\left[1 - \frac{\mathrm{CDS}(t_n)}{\mathrm{LGD}}\right] - \frac{\mathrm{CDS}(t_n) - \mathrm{CDS}(t_{n-1})}{\mathrm{LGD}} \sum_{i=1}^{n-1} P(t, t_i) Q(t_{i-1}) \qquad (8.4)$$

where

DS(t_i) is the CDS spread for maturity t_i paid on a running basis
LGD is the loss given default figure
$Q(t_i)$ is the survival probability beyond time t_i

Actually, the formula is based on the simplified assumption that the premium is paid at the end of a period even if the underlying name has defaulted in that period. However, accounting for payment of the accrued premium and payment of protection at the exact time of default (which is the market practice) would have a very limited impact on the estimates.

In the ideal marked-to-market world depicted above, extracting implied credit risk figures either from the stock market or the CDS market would lead to the same answers, apart from frictions and institutional features. Even though the CDS market is made of institutional and professional investors, which represent a subset of the pool of investors pouring money on the stock market, nevertheless inconsistent implied probabilities in these markets would be arbitraged out by them. If they are not, some other mechanism must be in place.

The presence of incomplete accounting information may impact on this representation in two ways:

1. Unbiased noise information may cause further redistribution of wealth from bondholders to stockholders. In fact, the impact of noise on the dynamics of the value of the firm may be simply considered as an increase in the volatility of assets.
2. Probability of fraud and deliberately biased reporting leads instead to underreaction of both stocks and debt securities. If the accounting data were totally reported at market value of the assets, we would observe a discount of the sum of debt and equity with respect to the reported value of assets.

So, unbiased accounting noise would predict debt and equity to move in opposite direction, while market perception of a strong bias in the reporting would affect debt and equity in the same direction. Adding to that, it may be that in front of a situation in which the market fears the possibility of fraud, the insiders could engage in asset substitution. They may do so by increasing volatility of assets, even at the cost of lower expected returns, trying to rescue the firm with a "go for broke" strategy: this would actually result in the presence of both the effects of reduction of value for both equity and debt, with some redistribution from debt to equity, and this chance would be more likely in the presence of massive stock option plans for the managers.

A straightforward strategy to find evidence of the impact of accounting distortions would then be to evaluate equity and debt at market value and to analyze their stochastic dynamics, possibly along with their relationship to the dynamics of the value of the firm. This of course would require finding acceptable proxies for the market value of the firm, as well as for equity and debt. The only piece of information that we may be sure to recover for all the relevant obligors, at least those that have gone public, is the market value of equity. Evaluating debt at market value is more complex, as it would require finding information about the technical form of debt, which is generally very difficult to collect, and to allow for different liquidity of different tranches of debt. The development of the credit derivatives market, and particularly of the most liquid instrument, the CDS, suggests a way to address these problems and come up with a fair price of debt independently of frictions and liquidity issues. This is due to the fact that the underlying asset of a CDS is the name of the obligor, and any debt issue, independently of its liquidity and other features, is available for physical delivery in fulfillment of the contract.

In this chapter, we proxy the market value of debt by discounting it by the risk-free rate plus the CDS spread. The market value of debt computed in this way is subtracted from the quasi-value of debt, which is the value of debt discounted using the risk-free rate alone, to evaluate how much of the market value of debt is due to credit risk. This figure is compared with information embedded in the stock price. The task is to discover whether the two variables react at the same time and in the same direction.

8.4 DATASET

Here, we present the data on a set of some world famous cases of fraud. The main task is to compare the information content of the CDS market and the equity market, and to translate that in value of equity and debt. The cases we will investigate are Tyco, WorldCom, Enron, and Parmalat. For every case, we will use the following data set:

- Stock prices and number of pieces of stock available
- CDS quotes for different maturities (where available)
- Balance sheet data concerning short-term and long-term debt

Using this data, we compute the fair values of equity and debt, and the market value of the firm. The value of equity is computed as the number of pieces of stocks times the stock price. As for debt, we compute the value of quasi-debt by discounting short-term debt by

the 1 year risk-free rate and long-term debt by the 5 year risk-free rate. The fair value of debt is then computed adding the 1 and 5 year CDS premia to the discount rates for the short- and long-term debt, respectively. Only in the Parmalat case, in which the term structure of credit spread was not available, all debt was discounted applying the 5 year CDS figure.

In Section 8.4.1 we provide a brief sketch of the history of the cases, accompanying it with the data of equity and CDS collected and elaborated.

8.4.1 Tyco

Tyco is a diversified manufacturer and servicer of industrial and commercial products. Tyco failed to show transparency in its accounting documents. The crisis initiated when the company completed its acquisitions between fiscal year 1999 and 2001. Such acquisitions were valued at 30.5 billion dollars. Tyco dropped plenty of cash for those acquisitions, about 17.5 billion in all. The company was largely financed by debt as it is explained on its balance sheet. Long-term debt in 1998 was 5 billion, or about 26% of its revenues that year. By fiscal year 2001, long-term debt was 38 billion for the consolidated balance sheet, or 6% more than its revenues. This huge increase in debt was motivated by the need to fund the largest ever acquisition operated by Tyco: on March 13, 2001 Tyco announced a 9.2 billion cash and stock deal to take over the CIT group, a commercial finance company.

It was the CIT deal that also led to sudden discovery of management misconduct. Tyco shares dropped sharply at the end of January 2002, as a proxy report was filed with the Securities and Exchange Commission disclosing that Tyco CFO (chief financial officer) got a 10 million dollars fee on the CIT group deal, and that another 10 million went to a charity where he was a director. That was just the beginning: on January 30, 2002, *The New York Times* reported that Tyco CEO (chief executive officer) and CFO had sold more than 100 million dollars of their Tyco stock in the previous fiscal year despite public statements that they would not be selling their stock. The scandal became highly apparent on June 3, 2002, when CEO Kozlowski resigned unexpectedly as *The New York Times* reported he was the subject of a sales tax evasion investigation; the day after a criminal indictment accused him of conspiring to evade more than one million in state and city sales tax on fine art purchases. And that was not the end of the story: from 1996 to 2002, Kozlowski took more than 46 million dollars in interest-free relocation loans intended to assist Tyco employees who were required to relocate when Tyco moved its corporate offices from New Hampshire to New York city, and, subsequently, to Boca Raton, Florida. Kozlowski used at least 28 millions of those relocation loans to purchase, among other things, luxury properties in New Hampshire, Nantucket, and Connecticut as well as a 7 million dollars Park Avenue apartment. These loans had not been disclosed to shareholders, contrary to the requirements of the federal securities laws.

Figure 8.1 reports the CDS quotes for Tyco in the years 2001–2002. The crisis of end of January 2002 appears very clear.

A closer look at the data reveals an interesting behavior of the CDS term structure. The level of the CDS spread suddenly increased, from less than 80 bp for the 1 year maturity to

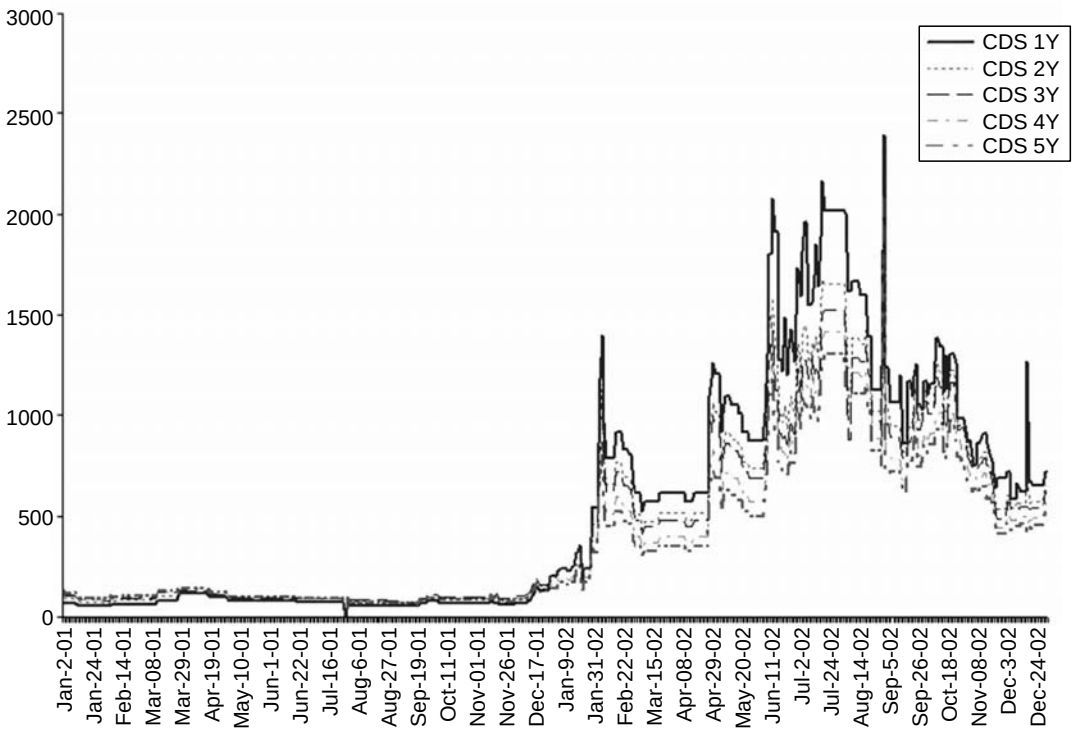


FIGURE 8.1 Tyco CDS (credit default swap) quotes: Values in basis points.

a double figure. The 1 year CDS spread rose again in the first quarter of 2002 to an average value of 570 bp. Furthermore, the term structure of the CDS spreads changed to an inverted shape, so that the 3 year CDS was quoting on average about 100 bp less than the 1 year contract, and the 5 year contract was quoting about 190 bp lower. The inverted shape persisted across all of year 2002 (the all year average 1 year CDS was 1000 bp, the 3 year 210 bp lower, and the 5 year 300 bp lower), and was only reversed in late 2003. An inverted structure of the CDS spread is indeed typical of periods of particular stress. This means that the market prices a very high probability of default in the first year, while the probability is lower for the years further in the future. This is confirmed in the data of the cumulative default probability which are portrayed in Figure 8.2.

To report some number, the probability of default in 1 year was 1.39% on average in the last quarter of 2001, while default probability in 5 years was more than five times, i.e., 7.51%. In the first quarter of 2004, the probability of default in 1 year climbed to 9.56% while the same figure on a 5 year horizon grew up to 22.90%, about twice as much as the 1 year figure. Furthermore, the probability of default in 3 years was very close to the 5 year figure, i.e., 21%. So, the market was implying a probability of default between years 2005 and 2007 of about 1.90%, somewhat consistent with the precrisis figure (if anything, precrisis figures were actually higher).

Figure 8.3 depicts the story of Tyco crisis in terms of the market value of balance sheet figures in 2001 and 2002. We see that the crisis was actually driven by a major fall in the

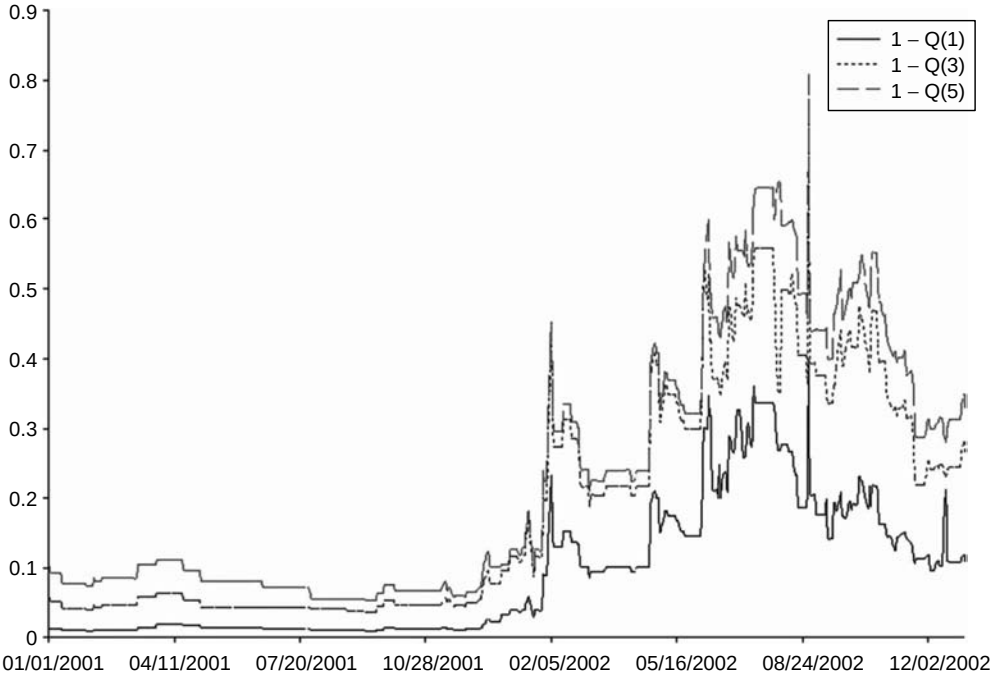


FIGURE 8.2 Tyco: Cumulative default probabilities bootstrapped from CDS (credit default swap) data.

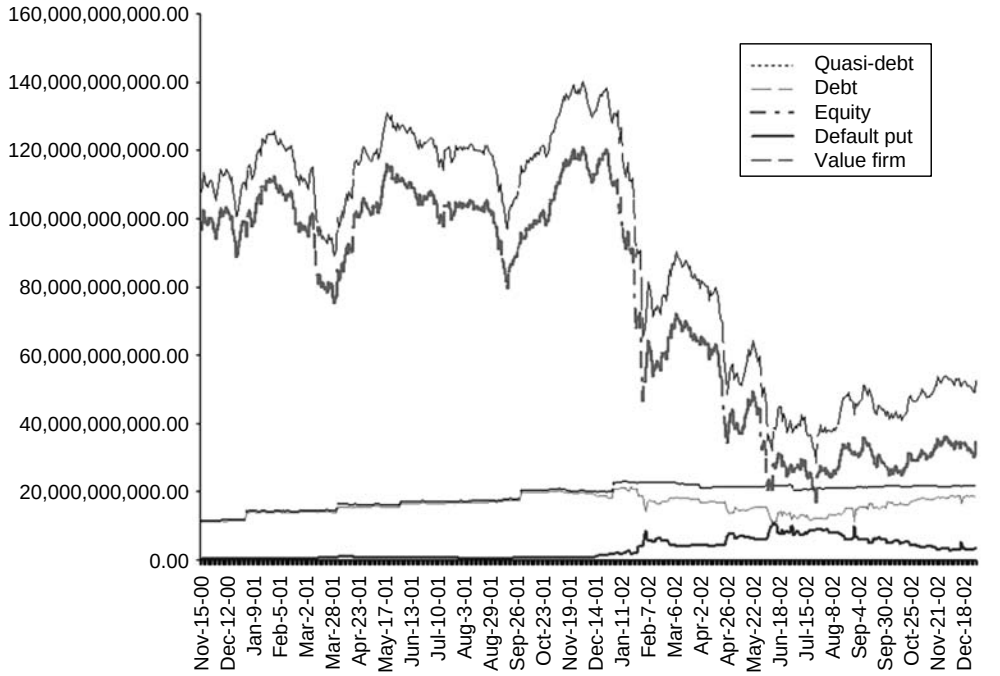


FIGURE 8.3 Tyco: Value of the firm, equity, quasi-debt, debt, and default put option.

value of equity. The picture seems to suggest that most of the action in the market was driven by the stock market rather than the CDS market. The crisis became particularly severe since June, following the resignation of the CEO, when the value of equity approached the value of quasi-debt: in those days, the CDS premia climbed up to 15% and beyond, while the stock price tumbled toward a floor between 10 and 15 dollars.

8.4.2 WorldCom

WorldCom was the biggest corporate bankruptcy in the history of the United States. WorldCom was a giant leading the telecommunication industry at the end of last century and was handling 50% of the U.S. Internet traffic and 50% of e-mails worldwide. WorldCom was deeply involved in acquisitions by purchasing over 60 firms in the second half of the 1990s, including the U.S. long-distance telephone company MCI, considered to be the largest acquisition, for an amount of 37 billion dollars.

Prosecutors say that the fraud began when it was discovered that the former CFO at WorldCom had directed employees to falsify balance sheets to hide more than 3.8 billion dollars expenses. In particular, WorldCom accounting executives were forced to record billions in operating expenses as capital expenses.

The fraud was discovered in March 2002, when the internal audit found accounting irregularities consisting of 400 million dollars that had been set aside to boost WorldCom's income. They started digging into that and found 2 billion dollars that the company announced for capital expenditures that were never authorized and an additional undocumented 500 million in computer expenses that were recorded as assets. Among other things, it was discovered that the CEO Bernard Ebbers had borrowed 366 billion dollars to cover losses on stock which was not repaid, and secured loans from WorldCom to fund personal investments including a 100 million Canada ranch, 658 million in Mississippi timberlands, and a 14 million Georgia shipyard.

After asking for documents supporting capital expenditures and discovering that no supporting accounting standards existed, the internal audit explained irregularities to the Audit Committee on June 20, 2002. Few days later, stock trading halted and the company was delisted by Nasdaq. WorldCom's story came to an end on July 21, 2002, when it filed for bankruptcy.

Figure 8.4 reports the CDS term structure in the year of crisis. The CDS spreads climbed up about at the same time as the Tyco crisis. Namely, the 1 year CDS spread averaged around 139 bp in January, with a positive slope of 15 bp for the 3 year maturity, and about 20 bp for the 5 year one.

At the end of the month, the spreads increased substantially, and the term structure of the spread became inverted. In the next 2 months, the 1 year CDS quoted 608 on average, and the 3 and 5 year maturities quoted 147 and 194 bp less than that, respectively. Again, this corresponds to a massive increase of default probability in the first year horizon, which does not seem to spread out to the years that follow. In January, the average default probability figure in 1 year was 2.30%, it was 6.30% over a 3 year horizon, and 12.60% over the 5 year one. So, the probability that WorldCom defaulted in the second and third year was 4%, and the probability of default in the next 2 years was the same as default in the first

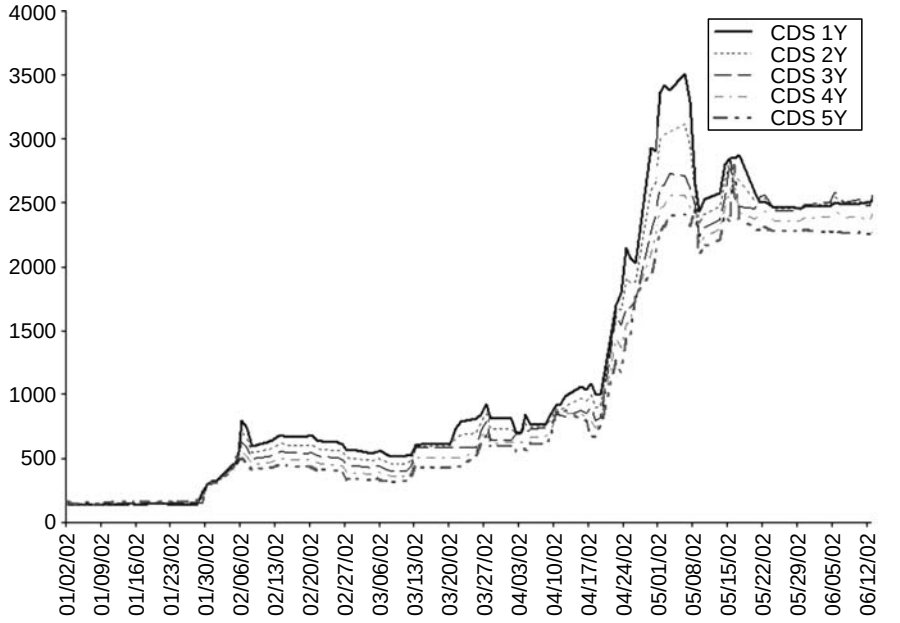


FIGURE 8.4 WorldCom CDS (credit default swap) quotes: Values in basis points.

3 years, which is 6.30%. In the following 2 months, the crisis brought about an increase in 1 year default probability up to more than 10%, while the probability over the 3 and 5 year horizons climbed to 22.66% and 27.95%, respectively. As in the Tyco case, the probability that the company defaulted between years 2005 and 2007 was about 5.30%, i.e., even less than in the precrisis period. Differently from Tyco, however, WorldCom did not manage to reach year 2005 alive.

As reported in Figure 8.5, the probability of default climbed in April and May, leading to the default of the company in June. Figure 8.6 finally reports the values of the balance sheet in the crisis in the first quarter of 2002.

The same regularities observed above apply to the WorldCom case as well. Namely, equity seems to start the crisis before the CDS market. While the CDS market was still calm in January, the stock undertook a progressive decline. The credit risk in the debt position began to rise when the value of equity reached the value of quasi-debt.

8.4.3 Enron

Enron was a large player in the natural gas and energy supply. In its traditional business, the company was running a gas and electricity transmission business, as well as a retail supply arm for end energy users. The company moved from this standard kind of business to a more innovative business strategy of trading power and communication commodities as well as the related derivatives products. The main event of this change of business was the creation of EnronOnline, a web-based transaction system that allowed buyers and sellers to trade commodities globally. The main commodities offered were natural gas and electricity although many others were traded. Thanks to this strategy Enron was prized as "America's

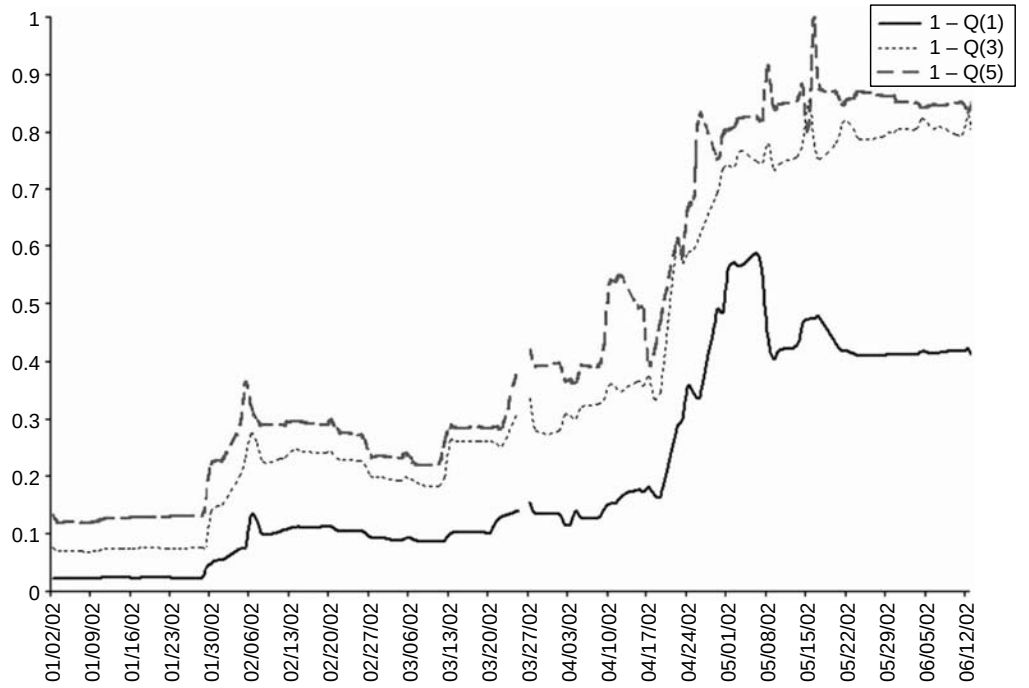


FIGURE 8.5 WorldCom: Cumulative default probabilities bootstrapped from CDS (credit default swap) data.

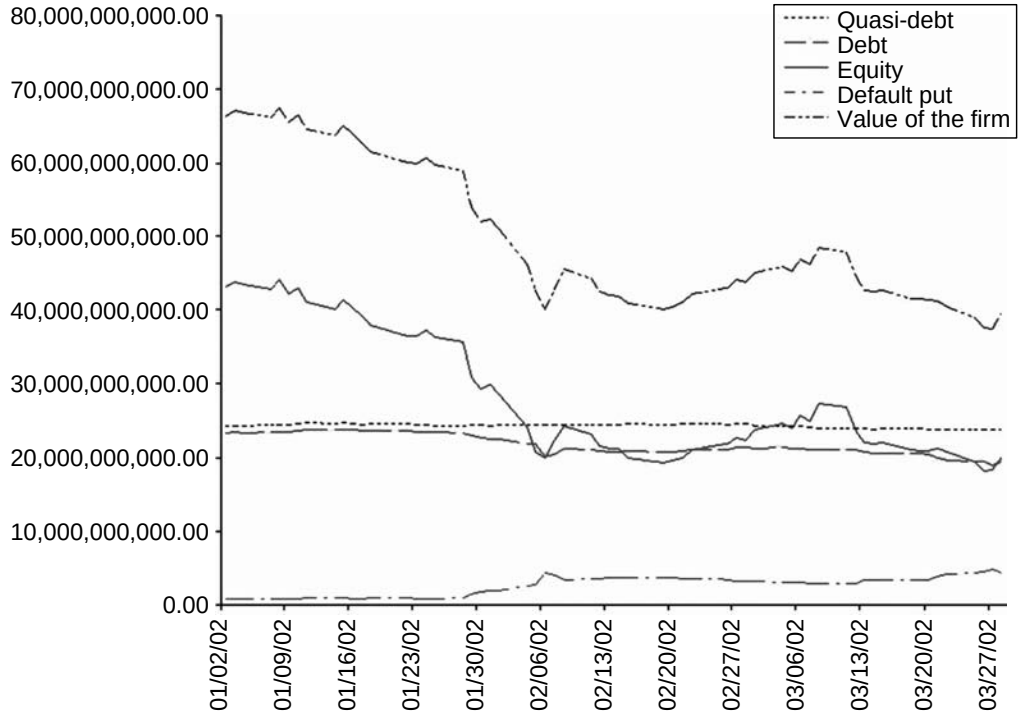


FIGURE 8.6 WorldCom: Value of the firm, equity, quasi-debt, debt, and default put option.

Most Innovative Company" by *Fortune* magazine for 6 years in a row, from 1996 to 2001. The keywords of this way of doing business were "asset lite" company or "virtual integration," describing a kind of synthetic business holding production and assets in small quantities (meaning low fixed costs) and substituting that with derivative transactions and trading (Deakin and Konzelmann, 2003). Accounting was synthetic as well. Enron was raising off balance sheet debt by resorting to almost a thousands of special purpose entities (SPE) funded by banks and outside investors. This practice allowed Enron to exploit a loophole in the United States generally accepted accounting principles (U.S. GAAP) to circumvent the principle of consolidation. Under the GAAP, the balance sheet of an SPE need not be consolidated with the originator if two conditions hold: outside investors must supply at least 3% of the funds, and the originator cannot control the disposition of the assets under control of the SPE. This technique enabled to increase leverage in the asset of Enron, without reporting this stock of debt in the company accounting data. Both from the economic and the accounting point of view, this had the same effect as synthetically creating a derivative contract on the assets of the firm. This leverage effect boosted growth of the company when the fundamentals of its business were good, but by the same token accelerated the crisis when the market turned against Enron. That happened between 2000 and 2001, following the unsuccessful attempt of Enron to move into the broadband communication business, and the reputation problems that the company faced in the California energy crisis of 1999 and 2000, where it was blamed to have gamed the market in the aftermath of deregulation. The rules had been accepted by the auditors, Arthur Andersen, even though they were to be considered on the edge of the acceptable practices. When they required the company to set back from that, it was too late, and the crisis reached and devastated Arthur Andersen as well.

Signals of crisis began when the CEO Skilling announced he was resigning from his position which he had held just 6 months, after having exercised stock options for several million dollars. Attempts by the former CEO and chairman Kenneth Lay to reassure the market and substitute the CEO did not succeed in bringing transparency to the market. On October 17, Enron announced a downward revision of its third quarter results by more than 1 billion dollars, and that were mainly due to investment losses, along with about 180 million dollars losses in the broadband unit. On October 22, the stock fell by about 25% in 1 day following the SEC (Securities and Exchange Commission) announcement that it was investigating several suspicious deals struck by Enron. Lay strove to convince the market of the availability of cash and liquidity both in words and in facts: he removed the CFO, and started a buy-back program of the commercial paper of the company, supported by banks. The move was not successful, and 2 days later a rumor spread that Enron was seeking further 1–2 billion funds from banks. The day after, Moody's lowered Enron credit rating to the lowest level of the investment grade scale, followed by Standard & Poor's. In November, Enron looked for a white knight to help it out of trouble. A Texas-based competitor, Dynegy, accepted and placed an offer for Enron. Bad news came however concerning debt and losses from the SPEs, such as JEDI (Joint Energy Development Project) and Chewco, and later on a disclosure that the company was facing repayment obligations in the range of 9 billion dollars by the end of 2002. The SEC also announced it had filed civil fraud

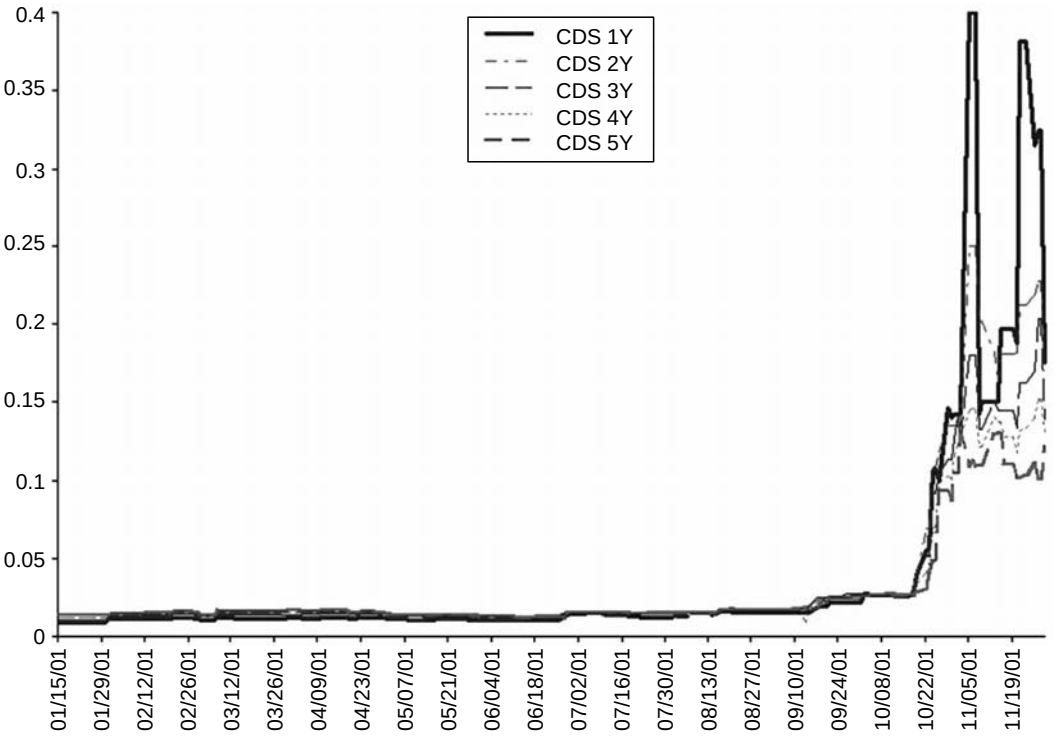


FIGURE 8.7 Enron CDS (credit default swap) quotes: Values in percentage points.

complains against Arthur Andersen, Enron's auditor. Following these events, Dynegy and Enron renegotiated the terms of the take over deal, with no success, and on November 28 Dynegy disengaged from the deal and Enron was downgraded to junk bond status. That was the end. Enron filed for Chapter 11 on December 2.

Figure 8.7 reports the history of CDS written on Enron's name. Even in this case, the crisis was announced by the inversion of the CDS spread term structure, which took place in mid-October 2001. In particular, corresponding to the SEC announcement on October 22 the 3 year maturity was trading at a price 44 bp lower than the 1 year maturity, and the 5 year one was quoted 243 bp lower. It was the first time when inversion of the CDS term structure took place, and it was the first day in which the 1 year CDS climbed 500 bp higher. In the first three quarters of the year, the 1 year CDS had remained between 100 and 150 bp, the 3 year CDS was about 22–25 bp higher, and the 5 year CDS in a range a little wider (between 20 and 30 bp higher).

Bootstrapped default probabilities remain between 1.6% and 2.4% for the 1 year, between 6% and 8% for the 3 year, and between 10% and 12% for the 5 year horizons (Figure 8.8). In particular, the probability of default between 3 and 5 years is very close to 4%. On the day of the crisis, October 22, the probability of default in 1 year climbed to 8.47%, while for the 3 year horizon was 20.8% and the 5 year one was 24.94%, with a probability of default between 3 and 5 year again stuck very close to 4%.

Finally, Figure 8.9 reports the values of the balance sheet items in the last year of operations of Enron. The figure shows that the crisis came at the end of a long period of

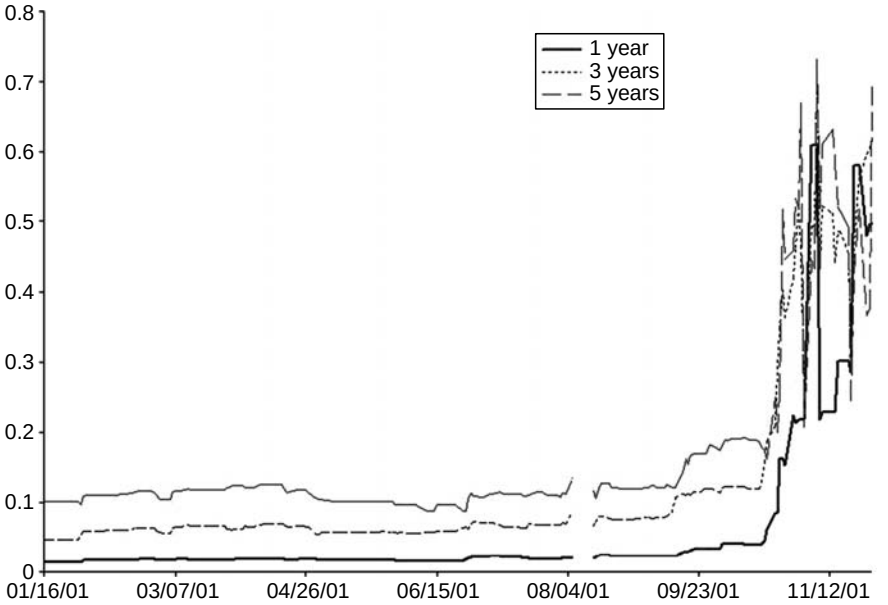


FIGURE 8.8 Enron: Cumulative default probabilities bootstrapped from CDS (credit default swap) data.

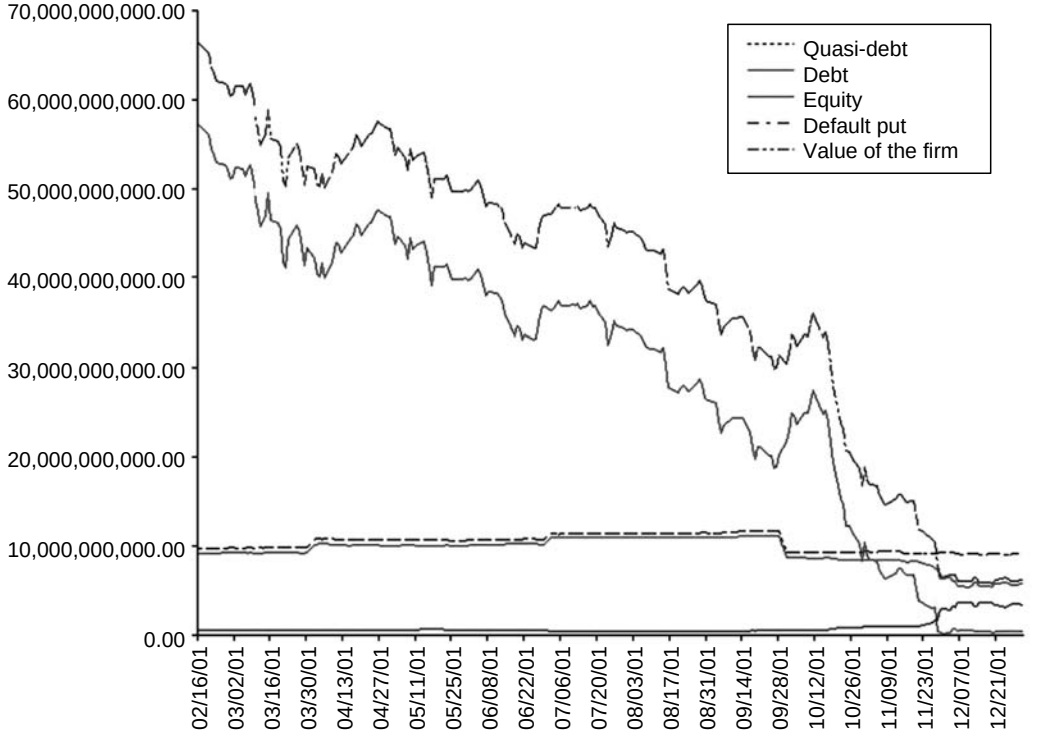


FIGURE 8.9 Enron: Value of the firm, equity, quasi-debt, debt, and default put option.

decrease of the stock price. The crisis began to involve the credit risk premium in the debt stock when the value of equity crossed the quasi-value of debt.

8.4.4 Parmalat

Parmalat is an Italian firm, located near Parma, which became a global player in the food industry. The trouble for Parmalat began in February 2003, when Parmalat announced issuance of a 300 million euros bond, despite the fact that the balance sheet was reporting a huge amount of liquidity available and that was not used. As a result of the announcement of the bond issue, the company stock fell by about 9% and the launch of the bond was withdrawn. In March, Assogestioni, the association of Italian fund managers, publicly complained against Parmalat for the lack of transparency of its accounts and reports. Following this, and the bond issue case, the company's CFO Fausto Tonna was removed, even though he was required to remain in the board of directors. Nevertheless, despite its reported liquidity, the company disclosed proposal to issue new capital to repay a large bond issue due in the end of 2003. On April 10, Parmalat spoke out (or claimed to do so) on its financial situation in a presentation in front of investors in Milan. In June, a debt issue of 300 million (the same amount as February's attempt) was privately placed with Nextra, an Italian fund manager. In September, the group announced a buy-back program for debt and at the same time went for a new private placement of a 350 million bond with Deutsche Bank. In the same day, September 15, Standard & Poor's revised its outlook on the company from positive to stable. In November, following the Cirio scandal, Commissione Nazionale per le Società e la Borsa (CONSOB), the Italian market supervisory authority, asked Parmalat to make clear how it was planning to repay the bonds coming due by 2004. The company's answer was that it would have resorted to liquidity cash. But existence of such liquidity buffer was going to be put into question just few days later. On November 11, Deloitte and Touche expressed doubts on Parmalat investment in Epicurum, a Cayman Islands-based hedge fund. Parmalat announced that it was going to unwind such investment. Even though such unwinding was approved by the end of November, it seems that the proceedings never arrived in Parmalat pockets. In fact, when in December 8 a bond for a principal value of 150 million dollars was not repaid, Parmalat explained to CONSOB that the proceedings from Epicurum, which were due by December 4, had not arrived. As a result of the failure to pay, Standard & Poor's downgraded Parmalat to junk status. The bond was repaid on December 12, thanks to the help of banks, for about 25 million. The plenty of cash and liquidity claimed to be available finally turned into a dismal mirage when on December 18 the negotiation with Epicurum went to a stop, and more so when on December 19, Bank of America denied existence of 3.9 million euros of cash that Parmalat claimed on behalf of one subsidiary. By then, fraud was clear and the company went through a reorganization procedure that took about 3 years to complete.

A specific feature of the Parmalat case is a conflict of interest within the banking system. Bank of Italy has recently disclosed that banks had reduced their holdings of Parmalat bonds and their exposure to the company before the crisis, while at the same time placing Parmalat bonds with retail investors. So, the argument is that they disposed of part of the trouble at the expense of retail investors.

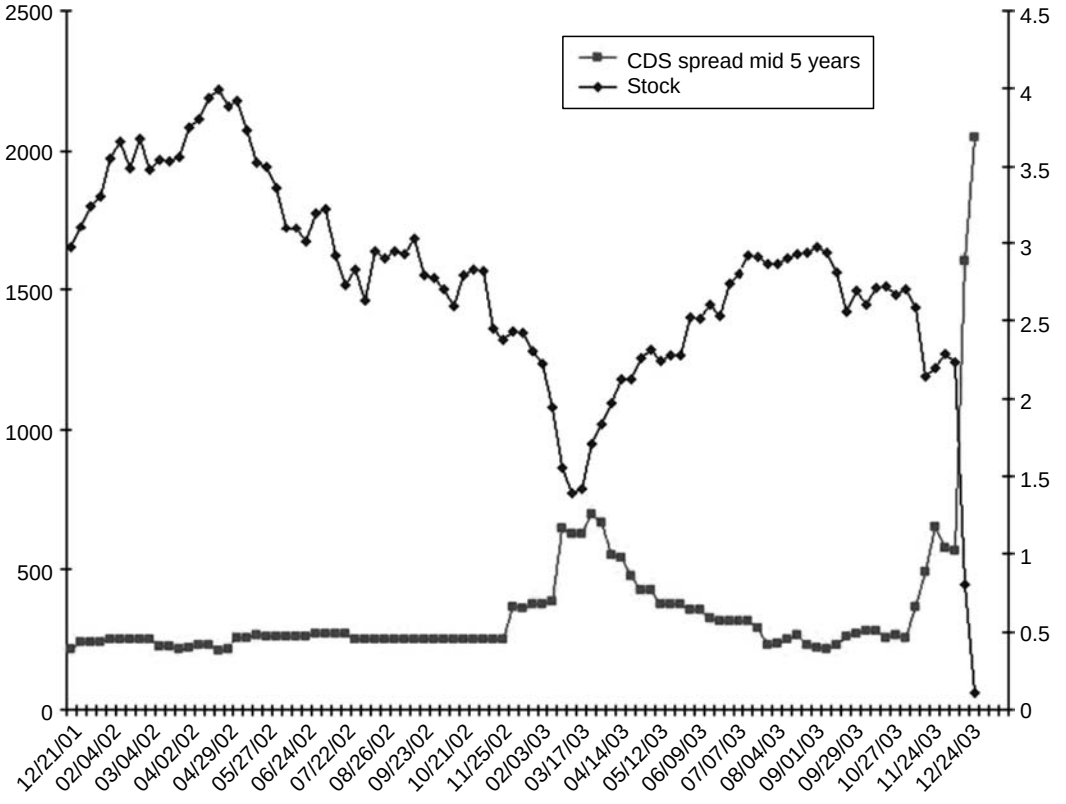


FIGURE 8.10 Parmalat: CDS (credit default swap) quotes (left axis, basis points) and stock price (right axis, euros).

For Parmalat, we do not have the term structure of CDS. In Figure 8.10, we report the dynamics of stock price and the 5 year CDS premium in the 2 years before Parmalat crisis.

The figure seems to suggest that the stock market leads crisis in the debt market as it happens in the U.S. cases, even though the evidence is less clear. Some difference in the Parmalat case can be gleaned from the analysis of the balance sheet items reported in Figure 8.11.

A look at the figure and comparison with the corresponding graphs for the U.S. cases shows that the credit premium in the debt is much more relevant that in former cases. It looks like bondholders participate in the crisis with losses similar to those of the stockholders. A possible explanation for this finding could be found in the different structure of the financial system in continental Europe, where investment in stocks is not diffused among retail investors as it is in Anglo-Saxon countries. Further discussion of such cross-country differences is however beyond the scope of this chapter.

8.5 EMPIRICAL INVESTIGATION

Using information on stock prices and CDS quotes, we computed the fair value of equity and debt for the companies in our sample throughout the period of financial crisis.

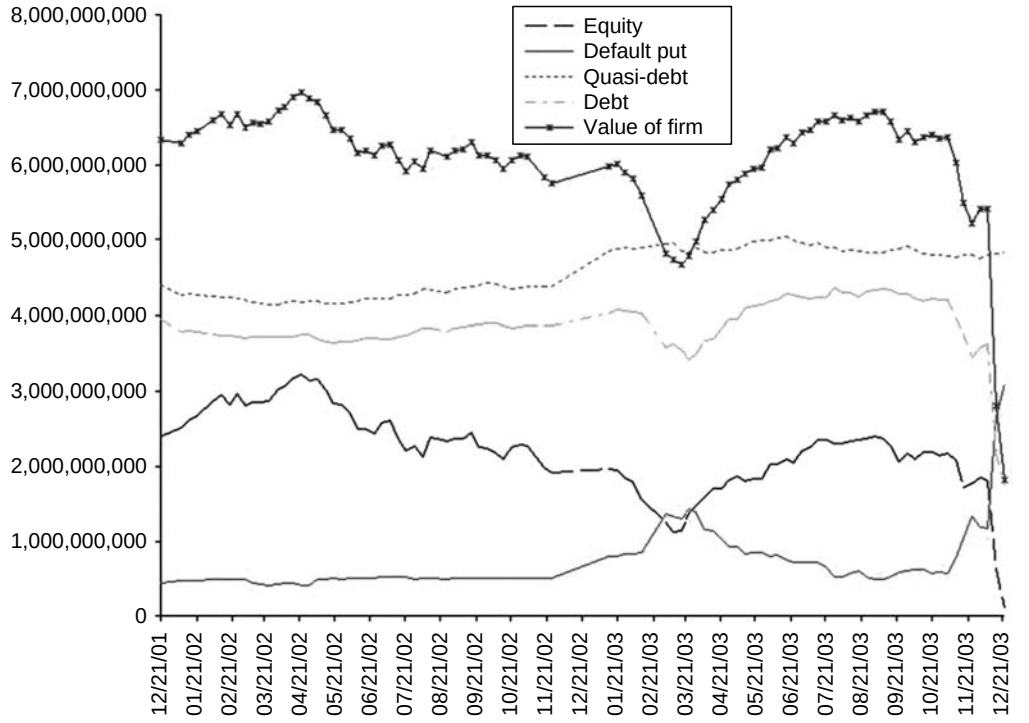


FIGURE 8.11 Parmalat: Value of the firm, equity, quasi-debt, debt, and default put option.

Of course, this representation of the balance sheet is simply a proxy, particularly as far as the value of debt is concerned. Nevertheless, it seems fair to believe that it may capture the fundamental dynamics of equity and debt. As for debt, we focus on the part of its value which is due to changes in the perception of credit risk. We consider the difference between the quasi-debt, i.e., the value of debt discounted by the risk-free rate, and the fair value of debt computed discounting debt with the risk-free rate plus the credit spread: this difference is actually a proxy for the default put option which is meant to represent credit risk in structural models.

In this section, we analyze the joint dynamics of equity, a call option, and the default put option representing credit risk. To make comparison easier, the variables were measured using a common numeraire, and the natural choice was quasi-debt. Summary statistics for the two variables are reported in Tables 8.1 and 8.2.

TABLE 8.1 Descriptive Statistics Value of Equity, as a Percentage of Quasi-Debt

	Tyco	**WorldCom**	**Enron**	**Parmalat**
Mean	4.273	1.148	2.805	0.493
Maximum	8.899	1.806	5.905	0.770
Minimum	0.801	0.761	0.020	0.018
Standard deviation	2.380	0.349	1.548	0.144
Skewness	0.055	0.747	−0.307	−0.233
Kurtosis	−1.393	−1.055	−0.749	0.446

TABLE 8.2 Descriptive Statistics Value of Default Put Option, as a Percentage of Quasi-Debt

	Tyco	WorldCom	Enron	Parmalat
Mean	0.138	0.108	0.089	0.154
Maximum change	0.501	0.203	0.376	0.641
Minimum change	0.024	0.035	0.032	0.099
Standard deviation	0.130	0.053	0.097	0.082
Skewness	0.870	−0.286	2.495	3.722
Kurtosis	−0.535	−1.395	4.759	17.848

A comparison at a glance shows the huge difference between the American cases and the Parmalat. Parmalat appears to be much less capitalized than the others: the value of equity capital is about half of quasi-debt, and if one considers that much of the debt was hidden from the books, that makes the evidence even more striking. As for the other cases, the same figure ranges from about one and a half up to more that four times as much as quasi-debt. Moreover, the average relevance of credit risk is highest for Parmalat, scoring more than 15% of the value of quasi-debt, while the same figure is around 9% and 10% for Enron and WorldCom and reaches almost 14% for Tyco.

Using the data series described above, we look for evidence to answer two questions. The first refers to the contemporaneous relationship of equity and the default put option. The second is about Granger causality between the two series, i.e., whether one of the two is able to predict the future dynamics of the other in a statistically significant way.

The question of the dependence structure between the value of equity and the credit risk enables us to speak out both on the structural determinants of credit risk and on the nature of the accounting opacity in the cases at hand. More specifically, a change in the value of the firm affects equity and credit risk in opposite ways—equivalently, it affects in the same direction as the value of equity and debt. This means that ominous prospects for the value of the firm as well as the scent of a possible upward bias in accounting figures should impact on the value of equity and debt in the same direction. On the other side, changes in volatility of the value of the firm impact the call option representing equity and the put option representing credit risk in the same direction: equivalently, changes in volatility redistribute wealth between bondholders and shareholders. This means that either (i) accounting figures are perceived as unbiased, but distorted by plenty of noise; (ii) or the market fears that in bad times the managers may be induced to select more volatile business lines; (iii) or the presence of stock option plans may provide a further incentive to make such a choice: in all these cases, the value of debt would be expected to decrease at the expense of the value of equity. Tables 8.3 and 8.4 provide answers to this question.

TABLE 8.3 Dependence Structure of Equity and Default Put: Levels

	Tyco	WorldCom	Enron	Parmalat
Correlation (%)	−87.21	−94.70	−66.74	−73.79
Spearman ρ (%)	−83.42	−93.08	−31.65	−79.08
Kendall τ (%)	−62.60	−79.00	−21.00	−61.90

TABLE 8.4 Dependence Structure of Equity and Default Put: First Differences

	Tyco	WorldCom	Enron	Parmalat
Correlation (%)	−1.24	−43.12	11.49	−71.86
Spearman ρ (%)	−9.18	−48.02	−9.25	−23.07
Kendall τ (%)	−5.90	−35.10	−5.90	−15.20

Table 8.3 reports correlation and association measures of the equity and credit risk. As we are trying to measure the co-movement of two options, a call and a put, it is well known that linear correlation may be subject to severe flaws. For this reason, we also report rank correlation measures such as Spearman's ρ and Kendall's τ. These measures of association are in fact nonparametric and robust to nonlinear relationships between the variables. The statistics concerning the levels provide strong evidence of negative co-movement of equity and the value of credit risk (and so positive co-movement of the value of equity and debt). The figures are very high, particularly for WorldCom, Tyco, and Parmalat. Enron also shows negative correlation between equity and credit risk, even though the values, particularly for the nonparametric measures, are much lower. In Table 8.4, the same analysis is applied to the first difference of the series. In this way, we want to check whether the evidence found is only due to changes in the trend or it is also present in short-run movements in the market. The evidence of co-movement for changes of equity and the default put option is actually much weaker, particularly for Tyco and Enron. The latter case is particularly interesting because the linear correlation figure turns actually positive, even though it remains pretty weak. Overall, we may say that there is evidence of negative association of the value of equity and credit risk in the value of debt, but most of the effect is due to the long-run components.

As most of the variation of the changes in equity and the default put remains to be explained, we proceed to investigate their relationship from a dynamic viewpoint. Namely, we ask whether there is some lead–lag relationship so that it would be possible to predict the future values of one variable given the current value of the other. The economic relevance of this question has to do with the degree of informational efficiency of the equity and CDS market. The topic has particularly important implications for regulators, mainly those in charge of defending fair and transparent behavior of institutional investors and intermediaries. Actually, if institutional investors may have some informational advantage over retail investors, one would expect that the CDS market should react to news more promptly than the equity market does. In other terms, if banks get to know that there is something wrong with the balance sheet of a firm, this would be expected to show up immediately in the quotes at which they buy and sell protection from each other on that "name," and this change in quotes would in turn induce a signal of crisis to the equity market. So, such informational advantage should result in the evidence that movements of the CDS market would help to predict future movements in the equity market. To provide an answer to this question, Table 8.5 reports Granger causality tests between the value of the default put option and that of equity.

The test was carried out considering lagged values up to 1 week. We ran vector autoregressions (VAR) estimates of the two variables with respect to five-lagged variables

TABLE 8.5 Granger Causality the Dynamic Relationship between Equity and Debt

	Tyco	WorldCom	Enron	Parmalat
Debt → Equity	2.52 (0.77)	2.89 (0.72)	6.38 (0.27)	1.62 (0.20)
Equity → Debt	41.43* (8E–08)	21.95* (0.0005)	8.11 (0.15)	0.55 (0.46)

Note: * denotes statistical significance at 5% probability level.

for Tyco, WorldCom, and Enron, for which we have daily data, and one-lagged variable for Parmalat for which we have weekly data. The tests refer to the joint statistical significance of the lagged values of a variable in the regression predicting the other. The tests are asymptotically chi-square distributed with a number of degrees of freedom equal to the number of lagged variables. The data reported in parenthesis below the test are the *p*-values denoting statistical significance, and asterisks denote cases in which Granger causality is statistically significant. A glance at the table shows that evidence against Granger causality from the default put option to equity, and so from the CDS market to the stock market is particularly clear-cut in all cases. On the contrary, there seems to be some evidence of Granger causality running in the opposite direction. In the Tyco and WorldCom cases, lagged changes in the equity market seem to predict future changes in the CDS market. So, this direction of Granger causality does not seem to support evidence of informational advantage of agents in the CDS market over those in the stock market, but rather some difference of liquidity may persist between the two markets.

8.6 CONCLUSION

The world famous financial scandals surveyed in this chapter share many common features. They are all inspired by the philosophy that transparency is a cost to evade rather than an opportunity to reduce the cost of capital. They are all more or less cooked with the same ingredients, even though in different proportions: management reluctant to disclose the plans of the firm, in terms of lines of business and the way to fund them, not to mention any information concerning management compensation schemes and exercise of stock options; accountants in conflict of interests; and banks facing the trade-off of losing money lent to the firm against losing reputation with the investors.

Histories of transparency crises also show many similar features. They typically start with rumors concerning some private interests of the management in some deal, or with the news that they have exercised stock options, or with some analyst reports. They continue with the firm changing some faces of the management, and trying to send some signal to the market, typically announcing some buyback of debt. Of course, the only way to end a confidence crisis would be to speak out on the accounting data and the business of the firm, but that is not an option, because a confidence crisis always intervenes on the background of some fundamental crises. In fact, it is lack of transparency itself that prevents any timely intervention to fix the problems.

Many of these features can be found in the cases of transparency crises that were surveyed in this chapter. The analysis of market data in these crises helped us to discover their main empirical regularities, as perceived by the market itself, and to answer some policy questions. We used data from the stock market and the CDS market.

The crisis is typically announced by some sudden drop in the equity market and by the inversion of the term structure of the CDS spread: the cost of protection increases for the short-term contracts, much more that for long-term contracts. The procedure of bootstrapping default probabilities from CDS spreads shows that this evidence means that the increase of default probabilities mainly affects the first year, or the first couple of years, while the probability of default in years further in the future remains unaffected.

Using CDS data, we computed a fair value of debt that may be subtracted from debt evaluated at the risk-free rate to provide an estimate of credit risk. We studied the joint dynamics of the value of equity and credit risk, built in this way. We found evidence of negative co-movement of equity and credit risk, and so positive co-movement of equity and debt, even though this was mainly due to co-movement of the long-term components of the variables. This suggests that most of the long-term, so to speak structural, innovations in these transparency cases have to do with the perception of changes in the value of the firm, rather than its volatility, and with biases in accounting data, rather than measurement errors and accounting noise. We finally provided a test of differential information content between the two variables to check whether the CDS market, limited to banks and professional investors, was reacting to news more promptly than the stock market, opened to the general public of investors. Actually this is a question that in one of the cases, Parmalat has been the main subject of legal litigation. We provide evidence that not only the CDS marked does not Granger cause the stock market, but also in two cases it is the other way around: in the Tyco and WorldCom cases, it is the stock market that Granger-causes the CDS market, probably because of better liquidity.

ACKNOWLEDGMENT

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CHAPTER 9

Anticipating Credit Events Using Credit Default Swaps: An Application to Sovereign Debt Crises

Jorge Antonio Chan-Lau

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9.1 INTRODUCTION

Credit derivatives are derivative securities with payoffs contingent on the realization of a credit event, such as default on a reference bond or a ratings downgrade of a reference entity below a threshold level agreed in the contract. Because these instruments isolate credit risk from other sources of risk, such as market risk and operational risk, they can be used for transferring credit risk from one party to another at a relatively low cost. Market prices of these instruments, especially credit default swaps (CDS), should reflect market assessments of the likelihood of the credit event and the expected value of the reference security after the credit event.

Conceptually, a CDS can be interpreted as purchasing insurance against the risk of default on a reference security. Hence, the price of a CDS is a function of both the probability of default of the issuer and the post-default expected recovery value. In reduced-form pricing models it is usually assumed that the recovery rate is equal to a fixed fraction of par value or market value of the reference security. Given a cross-section of CDS prices for different maturities and a fixed recovery rate, it is possible to extract the corresponding default probabilities by reverse engineering the pricing model.

This chapter proposes an alternative risk measure: the *maximum recovery rate* or upper bound on feasible recovery rates that match the cross-section of CDS spreads. For policy-making and risk management purposes, the use of the maximum recovery rate provides the most conservative estimate of the probability that the issuer will default on the reference security. Indeed, CDS spreads are uniquely determined by the magnitude of expected losses. For a given CDS spread, higher recovery rates imply higher default probabilities. Using the maximum recovery rate, hence, delivers the maximum default probability.

When the maximum recovery rate methodology is applied to cross-sectional time series data on CDS prices on Argentina's sovereign bonds, a remarkable empirical pattern emerges. During normal periods, the maximum recovery rate and implied default probabilities exhibit positive correlation. However, the correlation turns sharply negative in advance of the default event. This finding suggests that the joint behavior of the maximum recovery rate and implied default probabilities can be used to anticipate credit events.

The rest of the chapter is structured as follows. Section 9.2 provides useful background information on the credit derivatives market and CDSs, and explains how default swaps' spreads may help to forecast credit events. This section may be skipped by the reader already familiar with the credit derivatives market and the instruments. Section 9.3 explains how to obtain the maximum recovery rate and corresponding default probabilities with cross-section data on CDSs. While this method is tailored to a specific pricing model (Duffie 1999), it can be easily adapted to different CDS pricing models. Section 9.4 applies the maximum recovery rate methodology to the recent Argentina debt crisis. It also analyzes contagion in Latin America during the crisis episode. Section 9.5 compares the behavior of CDS spreads and bond spreads during the Argentina debt crisis. Section 9.6 concludes the chapter.

9.2 BACKGROUND

This section introduces some facts about the credit derivatives market and describes in detail the main characteristics of CDSs. Although CDSs are not the only credit derivatives available in the market, their relative simplicity and increased contract standardization have rapidly made them the "plain vanilla" contract in the credit derivatives universe.* Furthermore, standardization makes prices comparable across issuers and sectors at different aggregation levels, facilitating relative credit risk assessment.

9.2.1 Credit Derivatives Market

Credit derivatives provide corporations, financial institutions, and institutional investors with flexible tools to create synthetic risk exposure tailored to their specific needs. It is not surprising, then, that the use of credit derivatives has accelerated since the market inception in 1993. By the end of 2002, the credit derivatives market accounted for only 2%–3% of the $100 trillion derivatives market. However, its growth has been nothing but spectacular.

According to the British Bankers' Association's 2002 survey,† the credit derivatives market increased more than fivefold to $1189 billion by the end of 2001 from $180 billion by the end of 1997. The BBA survey suggests that the market has doubled to $1952 billion by end 2002, in line with recent estimates reported in the 2002 International Swaps and Derivatives Association (ISDA) survey, and that will grow to $4799 billion by end 2004. Emerging market issuers comprised a small share of the global credit derivatives market. As of mid-2001, estimates range from a low of $40 billion to $200–$300 billion, roughly between 4% and 20% of the credit derivatives market.‡

9.2.2 Credit Default Swaps

Credit default swaps are the most liquid instruments in the credit derivatives markets, accounting for nearly half of the total outstanding notional worldwide, and up to 85% of total outstanding notional of contracts with reference to emerging market issuers. In a CDS, the protection buyer pays a premium to the protection seller in exchange for a contingent payment in case a credit event involving a reference security occurs during the contract period (Figure 9.1).§

The premium (default swap spread) reflects the credit risk of the bond issuer, and is usually quoted as a spread over a reference rate such as LIBOR or the swap rate, to be paid either up front, quarterly, or semiannually. The contingent payment can be settled either

* See Tavakoli (1998) for descriptions and examples of other credit derivatives contracts such as total return swaps, credit spread options, and credit-linked notes, among others.

† British Bankers' Association, 2002, *Credit Derivatives Report 2001/2002*.

‡ See IMF (2002b), Chapter 4, for a comprehensive discussion of derivatives markets in emerging markets, including credit derivatives.

§ The 1999 ISDA credit derivatives definitions include the following six types of credit events: (1) bankruptcy, (2) failure to pay, (3) restructuring, (4) repudiation/moratorium (for a sovereign entity), (5) obligation default, and (6) obligation acceleration.

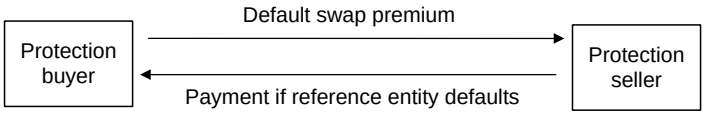


FIGURE 9.1 Credit default swap (CDS).

by physical delivery of the reference security or an equivalent asset, or in cash. With physical settlement, the protection buyer delivers the reference security (or equivalent one) to the protection seller and receives the par amount. With cash settlement, the protection buyer receives a payment equal to the difference between par and the recovery value of the reference security, the latter determined from a dealer poll or from price quote services. Contracts are typically subject to physical settlement. This allows protection sellers to benefit from any rebound in prices caused by the rush to purchase deliverable bonds by protection buyers after the realization of the credit event.

In mature markets, trading is highly concentrated on 5 year contracts, and to certain extent, market participants consider these contracts a "commodity." For emerging market issuers, CDS is available for a relatively large number of sovereigns and some selected corporations. Usual contract maturities are 1, 2, 5, and 10 years.

9.2.3 Default Swap Basis

The coexistence of markets for default swaps and bonds raises the issue on whether prices in the former merely mirrors market expectations already reflected in bond prices.* If credit risk were the only factor affecting the CDS spread, with credit risk characterized by the probability of default and the expected loss given default, the CDS spread and the bond spread should be approximately similar, as a portfolio of a default swap contract and a defaultable bond is essentially a risk-free asset.†

However, market frictions and some embedded options in the CDS contract, such as the cheapest-to-deliver option, cause CDS spreads and bond spreads to diverge. The difference between these two spreads is referred to as the *default swap basis*. The default swap basis is positive when the CDS spread trades at a premium relative to the bond spread, and negative when the CDS spread trades at a discount. For ease of reference, the text will refer to the default swap basis simply as basis henceforth. Reasons behind movements in the basis are explained in detail next.

Several factors contribute to the *widening of the basis*, either by widening the CDS spread or tightening the bond spread. Factors that tend to *widen the CDS spread* include: (1) the *cheapest-to-deliver option*, since protection sellers must charge a higher premium to account for the possibility of being delivered a less valuable asset in physically settled

* See Marsh (2002) for an in-depth discussion of the default swap basis. Rieu (2001) conducts an empirical investigation of what factors underlie basis changes for France Telecom both in time and along the term structure. Rule (2001) and IMF (2002a) analyze the financial stability implications of credit derivatives markets.

† Duffie (1999) and Hull and White (2000) formalize these arguments for floating par notes and analyze several extensions.

contracts;* (2) the *issuance of new bonds and/or loans*, as increased hedging by market makers in the bond market pushes up the price of protection, and the number of potential cheapest-to-deliver assets increases; (3) the ability to *short default swaps rather than bonds* when the bond issuer's credit quality deteriorates, leading to increased protection buying in the market; and (4) *bond prices trading less than par*, since the protection seller is guaranteeing the recovery of the par amount rather than the lower current bond price.

Factors that tend to *tighten bond spreads* include: (1) bond clauses allowing the *coupon to step up* if the issue is downgraded, as they provide additional benefits to the bondholder not enjoyed by the protection buyer and (2) the *zero-lower bound* for default swap premiums causes the basis to be positive when bond issuers can trade below the Libor curve, as is often the case for higher rated issues.

Similarly, factors that contribute to the *tightening of the basis* include: (1) existence of greater *counterparty risk* to the protection buyer than to the protection seller, so buyers are compensated by paying less than the bond spread; (2) the removal of *funding risk* for the protection seller, as selling protection is equivalent to funding the asset at Libor. Less risk demands less compensation and hence, a tightening in the basis; and (3) the *increased supply of structured products* such as CDS-backed collateralized debt obligations (CDOs), as they increase the supply of protection in the market.

Movements in the basis depend also on whether the market is mainly dominated by high cost investors or low cost investors. A long credit position, i.e., holding the credit risk, can be obtained either by selling protection or by financing the purchase of the risky asset. The CDS remains a viable alternative if its premium does not exceed the difference between the asset yield and the funding cost. The higher the funding cost, the lower the premium and hence, the tighter the basis. Thus, when the market share of low cost investors is relatively high and the average funding costs are below Libor, the basis tends to widen.

Finally, relative liquidity also plays a role in determining whether the basis narrows or widens, as investors need to be compensated by wider spreads in the less liquid market. Hence, if the CDS market is more liquid than the corresponding underlying bond market (cash market), the basis will narrow and vice versa.

9.2.4 Credit Default Swap Spreads and Credit Events

An analysis of the factors affecting the default swap basis provides an understanding on why CDS spreads react more strongly than bond spreads to a perceived deterioration of the credit quality of the issuer. For example, if shorting the bond is difficult, the buildup of short positions on the credit takes place in the CDS market through increased protection buying. In consequence, the price of protection, as measured by the CDS spread, rises. Coupon step-up clauses also become active when the credit deteriorates, and could further widen the CDS spread. If there is market segmentation, relative value trading between the CDS market and the bond market may not take place to arbitrage the basis widening away.

* This option ameliorates the pricing abnormalities that may arise when the number of CDS contracts outstanding exceeds the number of the reference bond in the contract.

During periods of distress, for mature markets liquidity in the underlying bond market may dry out while trading in the CDS market continues. Indeed, as noted by Fleming and Garbade (2002), in the aftermath of the events of September 11, 2001, price discovery migrated from the bond market to the CDS market, as a result of the serious disruption in the underlying bond market clearing mechanisms.* For emerging market issuers, the number of protection buyers increases relative to the number of protection sellers in the CDS market. Excess demand leads to higher CDS spreads. In the underlying bond market, imbalances between demand and supply are not as pronounced, so bond spreads do not increase as much as CDS spreads.

When CDSs with different maturities are traded in the same date, it is possible to construct a CDS term structure or CDS spread curve, a plot of CDS spreads against maturities. The CDS spread curve is normally upward sloping for creditworthy bond issuers: credit risk for a high quality issuer is not likely to deteriorate in the near term and explains why CDS spreads are low for relatively short maturities. As time passes, credit quality deterioration is more likely than credit quality improvements, i.e., the only rating movement for an AAA-rated issuer is downwards, and this is reflected in higher CDS spreads as maturity increases.

For lower-rated bond issuers, or bond issuers undergoing severe distress, the CDS spread curve can invert with short-term CDS spreads higher than long-term CDS spreads. Merton (1974) was among the first to explain this behavior, modeling it formally in an option-based framework. The intuition behind the curve inversion is simple: default risk for this class of issuers is very high in the near term, but it is believed that once the current difficulties are overcome, chances are that the bond issuer would be able to meet its obligations. Hence, default risk in the medium and long term is lower than in the near term, and is reflected in the downward slope of the CDS spread curve.

In 2002–2003, the behavior of CDS spreads on Venezuela, U.S. dollar-denominated sovereign debt, is illustrative of the points discussed above (Figure 9.2). While the 5 year CDS spread and the EMBI+ spread for Venezuela exhibit high correlation, the response of the CDS spread response to instability episodes in July 2002 and October 2002, and especially in January 2003, was stronger than the EMBI+ spread response.

During early 2002, the CDS spread curve shown in this figure exhibited an upward sloping shape. Increased market concerns about sovereign default risk caused the CDS spreads to widen all along the curve, but even more in the short end of the curve. As a result, the curve inverted in July 2002. The second half of 2002, though, was a relatively tranquil period, both bond spreads and CDS spreads compressed by around 300–500 basis points. Fears that Venezuela could default in the immediate future diminished, as reflected

* A widening of spreads does not necessarily reflect a fundamental deterioration of the credit quality of the bond issuer. Investment policy constraints requiring institutional investors to hold only investment-grade issues may cause speculators to build up substantial short positions on an investment-grade reference issue, either through the cash market or the CDS market, in the expectation that the ensuing widening of spreads and decline in equity prices would prompt a rating downgrade to noninvestment grade. Once the reference issue is downgraded, the forced sell-off allows short-sellers to cover their positions profitably.

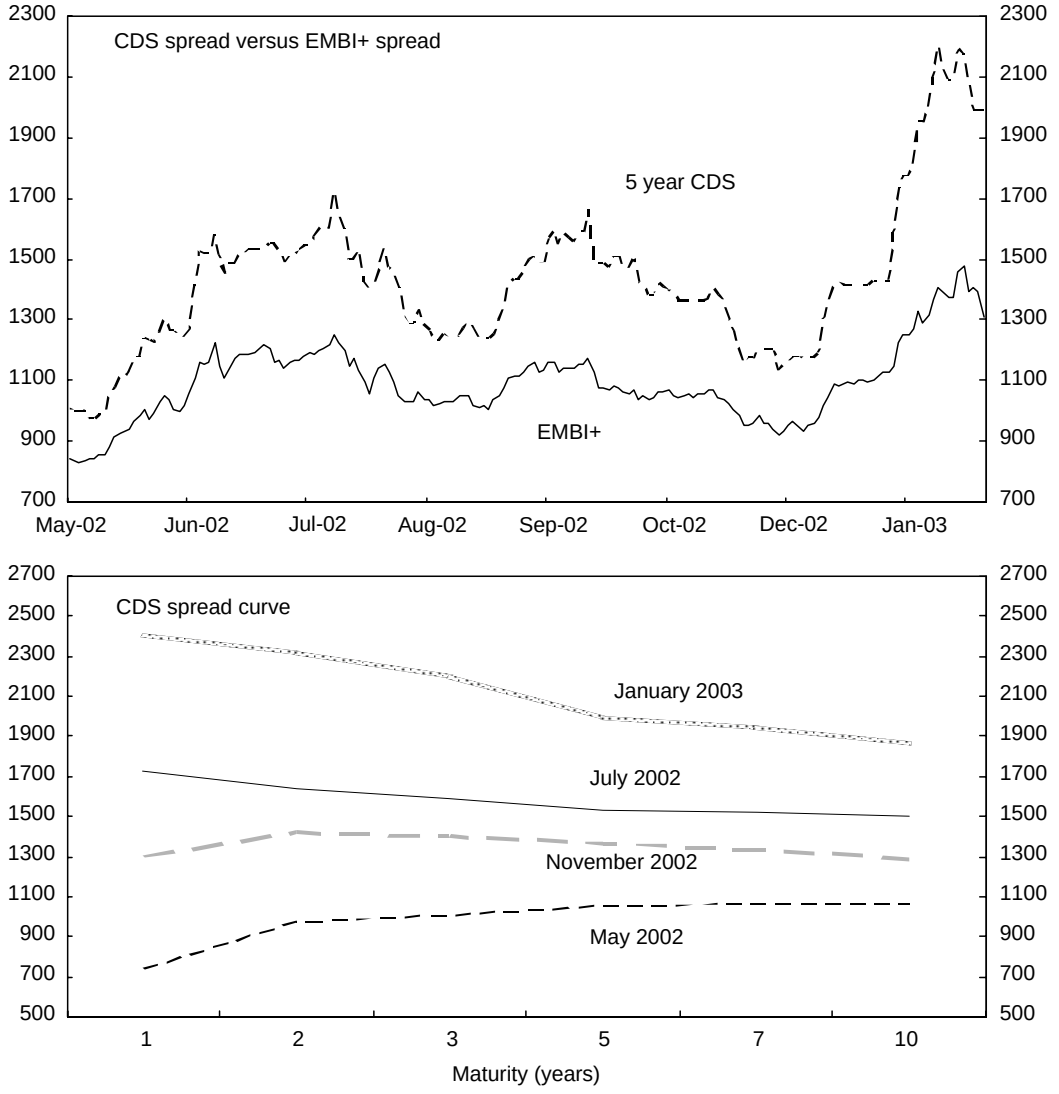


FIGURE 9.2 Venezuela: Developments in the CDS market (in bp).

in the fact that most of the compression took place in the short end of the curve. However, tranquility did not last long, as the political landscape in Venezuela deteriorated rapidly in January 2003. Again, there was a substantial widening of spreads, and the CDS swap curve inverted again.

More specific information on markets' expectations of default can be extracted from CDS prices. Namely, CDS pricing models can be used to recover default probabilities and recovery rates, providing an intuitive metric to assess an issuer's creditworthiness. The next section describes a methodology to extract two credit risk measures, the maximum recovery rate and its implied default probabilities, from CDS spreads.

9.3 MAXIMUM RECOVERY RATES AND IMPLIED DEFAULT PROBABILITIES

While some CDS pricing models are relatively complex, they rely on a simple foundation: at inception, the value of the default swap should be the same for both the buyer and seller of protection. The *premium leg* of the contract, payable to the protection seller, is the expected present value of premium payments either until the contract matures or the issuer defaults, whichever comes first. The *default leg*, payable to the protection buyer in case of default, is the expected present value of the loss given default before the maturity of the contract. The CDS spread is such that both legs are equally valued. Clearly, the terms structure of default probabilities and the recovery rate are important inputs to price both legs of the contract.

The interrelationship between these two elements is described in the context of the CDS pricing model of Duffie (1999). His model assumes that the time to maturity of the CDS, T, is equally divided into n periods with length T/n. The CDS spread, S, is paid at the end of every period at time $t(i) = iT/n$, $i = 1, \ldots, n$. Default can take place at any time. In case of default between times $t(i)$ and $t(i+1)$, the CDS is settled in period $t(i+1)$, with the protection seller paying the loss value of the defaulted security $F(1 - RR)$, where F is the notional value of the CDS contract, and RR is the recovery rate. For simplicity, it is assumed that the notional value of the contract, F, is unity.

The probability of survival is summarized by the term structure of hazard rates or default intensities, $\lambda = \{\lambda(i)\}$, $i = 1, \ldots, n$. Thus, the probability of surviving until period $t(i)$ given no default in period $t(I-1)$ is given by $\exp[-\lambda(i) \times (t(i) - t(i-1))]$, and the probability of survival until period $t(i)$ is simply

$$p(t(i)) = \exp\left(-\sum_{k=1}^{i} \lambda(k) \times (t(k) - t(k-1))\right) = \exp\left[\frac{1}{i}\left(-\sum_{k=1}^{i} \lambda(k)\right) t(i)\right] \tag{9.1}$$

Denote by $a(i)$ the value at time 0 of receiving one unit at time $t(i)$ when default occurs at $t > t(i)$. This value is equal to the probability of surviving until time $t(i)$ discounted to time 0 by the default-free yield $y(i)$ corresponding to time $t(i)$:

$$a(i) = \exp\left[-\left(\frac{1}{i}\sum_{k=1}^{i} \lambda(k) + y(i)\right) t(i)\right] \tag{9.2}$$

The premium leg, then, can be expressed as

$$A(\lambda, T)S = S\sum_{i=1}^{n} a(i) \tag{9.3}$$

Similarly, denote by $b(i)$ the value at time 0 of receiving one unit at time $t(i)$ if default occurs between times $t(i-1)$ and $t(i)$. Let τ be the default time. The probability of default in the interval $[t(i-1), t(i)]$ is given by

$$P(t(i-1) < \tau < t(i)) = \exp\left[-\left(\frac{1}{i-1}\sum_{k=1}^{i-1}\lambda(k)\right)t(i-1)\right] - \exp\left[-\left(\frac{1}{i}\sum_{k=1}^{i}\lambda(k)\right)t(i)\right] \tag{9.4}$$

and the value of $b(i)$ at time 0 is

$$b(i) = \exp(-y(i)t(i)) \times P(t(i-1) < \tau < t(i)) \tag{9.5}$$

The default leg can be expressed as

$$B(\lambda,T)(1-RR) = (1-RR)\sum_{i=1}^{n} b(i) \tag{9.6}$$

Equalizing the values of the premium and default legs yields the following formula for the CDS spread, S, given the term structure of default intensities, time to maturity, and recovery rate:

$$S(\lambda, T, RR) = B(\lambda, T)(1-RR)/A(\lambda, T) \tag{9.7}$$

Equation 9.7 allows reverse engineering the term structure of default intensities λ given a fixed recovery rate if the term structure of CDS spreads, or CDS spread curve, is known. Once λ is known, other defaultable contracts can be priced upon assuming a given recovery rate. Though it should be noted that different recovery rates imply different default probabilities, i.e., $\lambda = \lambda(RR)$, an indeterminacy problem arises if the recovery rate is not fixed a priori. This problem is usually dealt with by using historical data on recovery rates. For example, Moody's reports periodically average 1 year default rates and recovery rates for corporate obligors grouped by credit rating.

This study approaches the problem of determining the recovery rate and implied default probabilities from a different perspective. Rather than trying to pin down the recovery rate, the focus is placed on extracting the *maximum recovery rate*, the upper bound of feasible recovery rates, and their implied default probabilities from the CDS spread curve. Intuition suggests that using the maximum recovery rate provides conservative estimates of the probability that the issuer will default on the reference security. Because CDS spreads are uniquely determined by the magnitude of expected losses, higher recovery rates imply higher default probabilities. Thus, the maximum recovery rate delivers the highest feasible default probability. Intuition also suggests that the maximum recovery rate is likely to decline when the credit quality of the bond issuer is undergoing a serious deterioration. Indeed, the application of this methodology to historical data on CDS spread curves for Argentina, as described in the next section, validates this intuition.

Formally, the maximum recovery rate is defined as follows:

Definition (Maximum Recovery Rate). Given the CDS spread curve and the default-free yield curve for a set of maturities T, the *maximum recovery rate*, $RR\max$, is defined as

$$RR\max = \sup\left(RR : S(\lambda(RR), t, RR) = \frac{B(\lambda(RR), t)(1-RR)}{A(\lambda(RR), t)}, \quad \forall t \in T \right) \tag{9.8}$$

where $\lambda(RR\max)$ is its associated term structure of default intensities. Equation 9.8 can be solved using numerical methods. The default intensities can then be used for estimating the term structure of default probabilities using Equation 9.1.

9.4 APPLICATIONS

9.4.1 Anticipating Sovereign Debt Crises: Argentina (2001–2002)

The recent default of Argentina on its sovereign debt in January 2002 provides an interesting case study for assessing the behavior of the maximum recovery rate and its associated default probabilities. The data used in this study include daily mid-point quotes of CDS spreads on Argentina sovereign debt for 1, 2, 3, 5, 7, and 10 year maturities during the period August 3, 1998–December 12, 2001, and do not necessarily reflect transaction prices. Daily data on U.S. dollar swap rates were obtained from Primark Datastream.

The maximum recovery rate and the associated term structure of default intensities were obtained solving Equation 9.8 using the CDS spread curve for each date in the sample. The day-to-day estimation approach is conceptually equivalent to the extraction of daily implied volatilities from cross-section of options at any given date. Therefore, it does not account explicitly for the possible interaction between CDS spreads and the default-free short rate, or the dynamic behavior of the recovery rate. Figure 9.3 shows the maximum recovery rate, the 1 year forward default probability obtained from the term structure of default intensities, and the correlation between these two quantities. Some observations are worth discussing in detail.

The first panel in Figure 9.3 shows the behavior of the maximum recovery rate for the sample period analyzed. The average maximum recovery rate was 62%, with a standard deviation of 16%. These estimates seem to be justified by past historical experience. For example, Beloreshki (2003) noted that countries that completed their Brady plan restructuring between 1989 and 2000 were forgiven 30% to 35% of their debts, or equivalently, the recovery rate was about 65% to 70%. In some instances, though, the debt write-off was around 50% (Poland in 1991 and 1994, and Bulgaria in 1994). Hurt and Felsovalyi (1998) estimated that, for the period 1970–1996, the average loss in the event of default for Latin American defaulted loans was on average 32% that corresponds to an average recovery rate of 68%. On the other hand, the maximum recovery rate appears to overestimate the recovery rate of 25% of market value usually assumed by market participants in their estimates of credit risk (Beloreshki 2003).

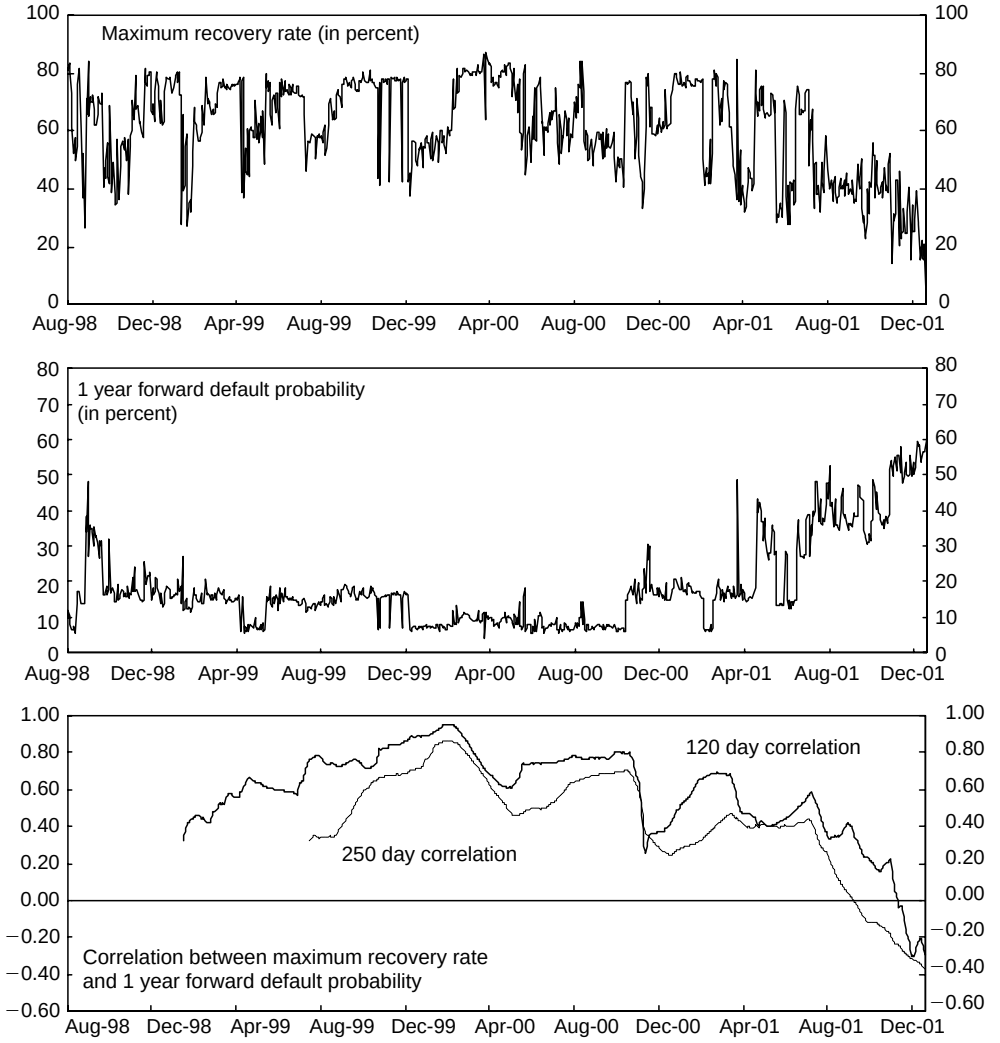


FIGURE 9.3 Maximum recovery rate and default probability.

The second panel in Figure 9.3 shows the evolution of 1 year forward default probabilities. Increased concerns about Argentina's ability to roll over its maturing debt were evident by the end of 2000. Indeed, default probabilities increased above the average level of 1998–1999, though the provision of IMF assistance helped to allay markets' negative sentiment. After the second half of 2001, default probabilities increased steadily, though there was a brief respite during September–October 2001. However, the continued fiscal deterioration and a lack of credible economic programs pulled the country towards the brink of default by the end of 2001. In January 2002, Argentina defaulted on its external debt.

The relationship between the maximum recovery rate and the 1 year forward default probability is illustrated in the last panel of Figure 9.3. Overall, the correlation

tends to be positive during normal periods, and is inversely related to the level of default probability. Both the 120 day and 250 day correlations fell sharply starting October 2000, but they rebounded after the approval of the IMF aid package to Argentina in December 2000. However, correlations declined steadily from July 2001. The 250 day correlation turned negative as early as September 2001, well in advance of the sovereign default.

9.4.2 Contagion

One year default probabilities associated to maximum recovery rate values can also be used to measure the degree of contagion between sovereign countries. Contagion in this context is defined as the extent to which credit events affecting one sovereign can influence market views on the creditworthiness of other sovereigns. Figure 9.4 shows 1 year default

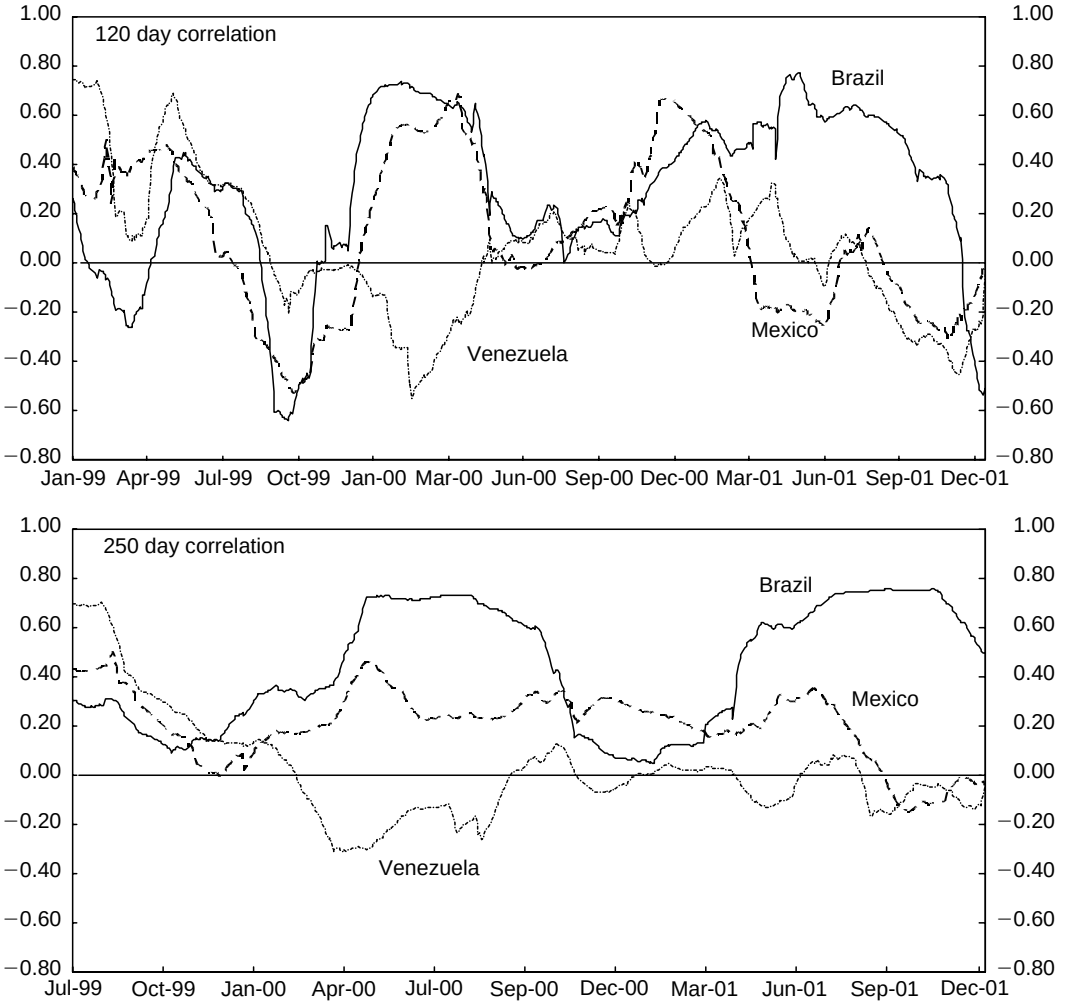


FIGURE 9.4 Cross-country correlations with Argentina: 1 year default probability.

probability cross-country rolling correlations with Argentina of different Latin American countries, including Brazil, Mexico, and Venezuela. The rolling correlations show that Brazil and Mexico experienced some spillovers from Argentina in the last quarter of 2000. The decoupling of Mexico and Brazil from Argentina occurred in early 2001 and the third quarter of 2001, respectively. Spillovers from Argentina to Venezuela were insignificant.

9.5 CDS SPREADS VERSUS BOND SPREADS

It is natural to inquire whether bond spreads and CDS spreads, and hence, the implied maximum recovery rate and default probabilities, contain the same information. Figure 9.5 shows that the EMBI+ stripped spread for Argentina was highly correlated during the sample period, with an average correlation of 0.94. However, the spread correlation declined substantially during the first half of 2000, and only returned to its average level in early 2001. Hence, it appears that CDS spreads contain some information beyond that already contained in bond spreads. Note that the additional information is not necessarily related to credit risk. Several of the technical factors affecting the basis described before may have caused the correlation breakdown without changes in default risk. Nevertheless, for both risk management and surveillance purposes, it is important to assess what factors underlie the breakdown of historical patterns.

Comparing spillover effects provides an alternative way to evaluate the information content in the bond and the CDS markets. Figure 9.6 shows the cross-country rolling correlations with Argentina for EMBI+ stripped spreads, 5 year CDS spreads, and 1 year default probabilities of Brazil, Mexico, and Venezuela. The 5 year CDS spread correlation

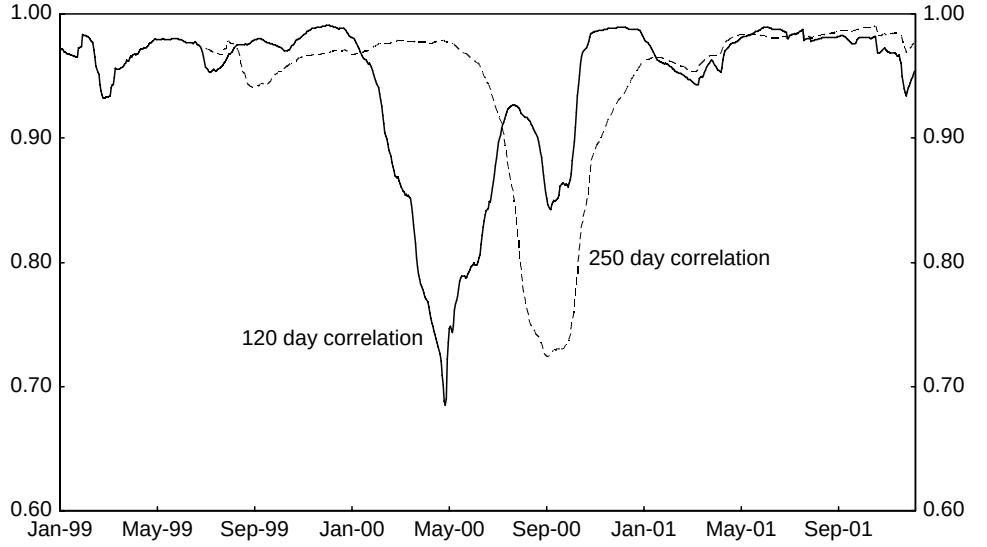


FIGURE 9.5 Argentina: Correlation between EMBI stripped spread and 5 year CDS spread.

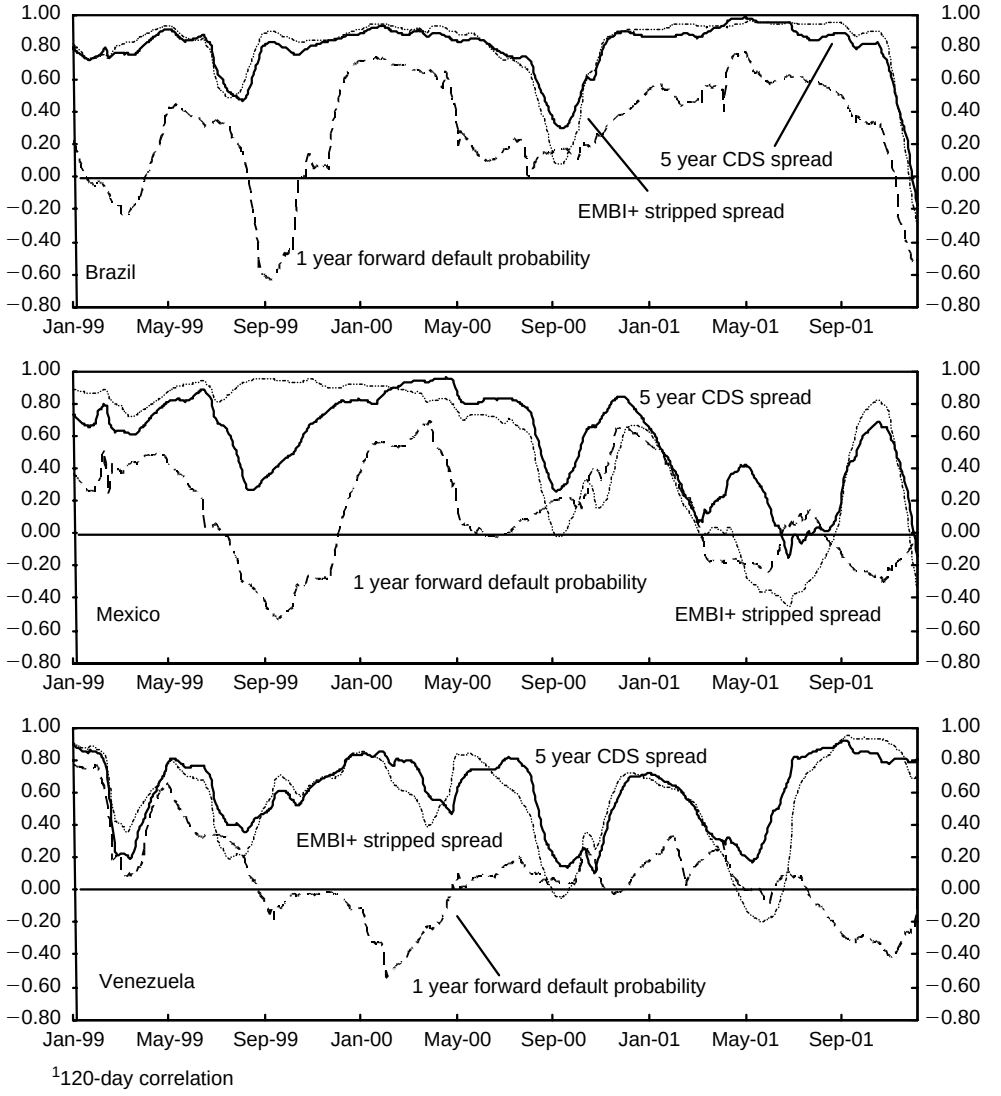


FIGURE 9.6 Cross-country correlations with Argentina: Spreads and default probability.

tracks the EMBI+ stripped spread correlation very closely. The only exception is Mexico in the second half of 1999, when the 5 year CDS spread correlation was substantially lower than the EMBI+ stripped spread correlation. The 1 year default probability correlation is, on average, lower than the both the CDS and EMBI+ spread correlation. More importantly, in many instances, probability correlations turned negative in advance of CDS and EMBI+ spread correlations. Thus, probability correlations may signal decoupling between countries faster than the other measures analyzed.*

* See Chan-Lau and Kim (2005) for a detailed analysis of price discovery across CDS and bond markets in emerging market countries.

9.6 CONCLUSION

Credit default swaps contain useful information about market views on the expected loss given default faced by a bondholder. However, in reduced-form pricing models, it is difficult to disentangle the contributions of the probability of the credit event and the recovery value in determining the expected loss given default.

This chapter suggests an alternative way to think about recovery values by introducing the notion of maximum recovery rate, i.e., the maximum recovery rate consistent with a given CDS curve in any given date, and its implied term structure of hazard rates (or default probabilities). The recent sovereign default of Argentina in January 2002 was analyzed using the maximum recovery rate methodology. The results indicate that the correlation between the maximum recovery rate and default probabilities turn negative in advance of the credit event, and can be used for constructing early warning indicators of debt default.

In addition, this chapter finds important differences between the information embedded in bond prices and CDS spreads that may be helpful to analyze credit events. This issue has been analyzed in more detail in Chan-Lau and Kim (2005).

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Part III

Credit Risk Modeling and Pricing

CHAPTER 10

Investigating the Role of Systematic and Firm-Specific Factors in Default Risk: Lessons from Empirically Evaluating Credit Risk Models

Gurdip Bakshi, Dilip Madan, and Frank Xiaoling Zhang

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10.1 INTRODUCTION

The paradigm that default occurs when a continuous process such as firm value reaches a default boundary has long-standing in finance (see Merton 1974 and the refinements in Longstaff and Schwartz 1995 and Collin-Dufresne and Goldstein 2001). In this conceptually elegant structural approach, default is modeled using a predictable stopping time, and default probabilities are systematically connected to firm leverage. There is wide consensus, however, that structural models are difficult to implement. One, the capital structure of the firm is often far too complex to specify recovery to all claimants in the event of default. Two, at the empirical level, structural models generate counterfactually low short-term credit spreads for high quality borrowers (Duffee 1999, Collin-Dufresne and Goldstein 2001, and Longstaff and Schwartz 1995).

Building credit risk models as the basis for evaluating default exposures remains a fundamental issue. Theoretical research continues to shed wisdom on the qualitative nature of credit spreads and their dependencies on essential features of the defaultable contract such as credit rating of the participating parties and firm-specific/systematic default characteristics. For instance, Jarrow and Turnbull (1995) propose a framework where the underlying asset or the counterparty may default. Duffie and Singleton (1997, 1999), Lando (1998), and Collin-Dufresne, Goldstein, and Hugonnier (2004) treat default as an unpredictable event governed by the instantaneous probability of default; Madan and Unal (1998) analytically decompose the risk of default into components related to timing and recovery. Each of these contributions view default as occurring at a surprise stopping time. In a related work, Jarrow, Lando, and Turnbull (1997) develop theoretical models where the bankruptcy process obeys a discrete-state space Markov chain in credit rating.

While theoretical advances have been made in interpreting credit risks and in parameterizing the price of credit sensitive securities, there is a relative paucity of empirical studies that investigate the relevance of leverage, distance-to-default, and other systematic/firm-specific factors in the surprise stopping time default approach. Which characteristics capture variations in default risk? Which credit risk model is suitable for marking-to-market defaultable securities? Which model performs the best in hedging dynamic credit exposures? Are single-name valuation errors correlated, and, if so, what is the possible source of this covariation? Empirical investigations of credit risk models attempting to analytically capture patterns of structural dependencies on theoretically interpretable grounds have become even more desirable in light of Basle committee recommendations on managing default risk.

The class of credit risk models we empirically analyze shares some features in common. One, our characterization of credit risk relies on the Duffie–Singleton (1999) assumption

that recovery is proportional to the predefaultable debt value. Two, complementing the empirical studies of Duffee (1999), Driessen (2005), and Jacobs and Li (2004), we develop a set of three-factor credit risk models that depend on systematic and firm-specific distress characteristics. In one specific model, we link the instantaneous likelihood of default to the short interest-rate, the stochastic long-run interest rate, and firm-specific leverage where the leverage dynamics is modeled in both level and logs. These models are empirically tractable and account for both interest rate risk and firm-specific default risk. The latter appealing feature inherits the flavor of the structural models of Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). To broaden the empirical content, we adopt alternative plausible models for firm-specific distress based on book-to-market, profitability, distance-to-default, and equity–volatility, and consider a general risk premium specification for the interest rate as in Duffee (2002). Having firm-specific characteristics in a default model, in theory, offers flexibility in explaining the behavior of single-name credit spreads.

Econometric implementation of default-theoretic models provides crucial insights into the sources of firm-level credit risk. First, incorporating a stochastic mean interest-rate feature into the working of credit risk models leads to more realistic credit curves. Thus, substantial care must be taken in specifying the factor structure of default-free bonds. Second, our decomposition shows that interest-rate risk captures the first-order impact of movements up to BBB-rated corporate credit yields. Third, controlling for interest-rate considerations in a credit risk model, the incremental pricing improvement with leverage is economically large and statistically significant for low-grade bonds. Differentiating our work from existing studies, the leverage generalization produces a 30% reduction in absolute yield errors relative to a model that excludes leverage. The estimation strategy shows that the credit risk model with leverage appears to be least misspecified among our three-factor default risk models. Overall, these conclusions hold irrespective of whether the performance of credit risk models is based on defaultable yields or on credit spreads. On balance, credit risk models have quantitative pricing performance that could be adequate for marking risk exposures. In the spirit of Collin-Dufresne, Goldstein, and Martin (2001), we investigate the relationship between default model mispricing and a set of economy-wide systematic factors, and find their joint explanatory power to be fairly small for the majority of single names.

Hedge portfolios that combine a short-position in the defaultable bond and a long-position in the risk-free bond of the same maturity using two Treasury discount bonds and the firm's equity remains sensitive to changes in debt and book values. Consistent with the pricing exercise, we observe that systematic risk exposures are crucial to the profit/loss accounts of the hedger. Given the mapping that exists between the price of corporate bonds, Treasury bonds and credit default swaps, our analysis potentially suggests that, for our sample of firms, the price of credit protection can be effectively hedged using Treasury bonds. This hedge is possible when credit exposures are tied to the Treasury markets.

The exact description of the models is given in the work by Bakshi et al. (2006) and in the Appendix. The defaultable coupon bond sample and empirical issues are described in Section 10.2. Section 10.3 employs a maximum-likelihood procedure jointly consistent

with risk-neutral and physical dynamics of firm-specific distress and the cross-correlation between bonds of distinct maturities. In Section 10.4, we present the pricing and hedging results and conduct specification diagnostics and generalized method of moments (GMM) tests. Concluding remarks are provided in Section 10.5.

10.2 SINGLE-NAME DEFAULTABLE COUPON BONDS

For this study, we use nonconvertible coupon debt with principal in excess of $1 million taken from Lehman Brothers Fixed Income Database. This database contains the amount of coupon and principal, the month-end flat price, the accrued interest, and the maturity date. Each bond is assigned a credit rating and debt issues are classified as callable, putable, subordinated, and by sinking fund provisions.

Several exclusion filters are imposed to construct the single-name bond sample. First, trader bid quotes are used (ask quotes are not recorded) and matrix quotes are eliminated. Second, we delete pass-through, asset-backed securities and debt with embedded options. Next, we include noncallable bonds with semiannual coupons that have maturity greater than 1 year. As few noncallable bonds were issued before March 1989, we limit attention to the sample between March 1989 and March 1998. Bonds with time-to-maturity closest to 2, 7, and 15 years are selected to represent short-term, medium-term, and long-term bonds and this classification produces a time-series of monthly bond prices. One hundred and eighty three firms satisfy all the above requirements with no missing observations.

To test the credit risk models outlined in the Appendix, five distress factor proxies are constructed:

Leverage: $\mathrm{LEV}_n(t)$, is long-term book value of debt (COMPUSTAT quarterly item 51) divided by firm value. The firm value is the sum of long-term debt and the market capitalization of common equity, M.

Book-to-market: $\mathrm{B}_n(t)/\mathrm{M}_n(t)$, is computed as the book value of equity (COMPUSTAT quarterly item 59) divided by the market value of equity.

Profitability: $\mathrm{PROFIT}_n(t)$, is one minus operating income (COMPUSTAT quarterly item 21) divided by net sales (COMPUSTAT quarterly item 2).

Volatility: To avoid overlapping observations, $\mathrm{VOL}_n(t)$, is the standard deviation of the daily stock returns during the month.

Distance-to-default: is based on estimates of asset value, default point, and asset volatility. The exact KMV procedure for constructing distance-to-default is proprietary, so we appeal to the measure in Vassalou and Xing (2004). The data is kindly made available to us by Yuhang Xing.

While matching firm sample with CRSP and COMPUSTAT, 80 firms did not have data on equity prices and 27 firms have missing data on leverage, book-to-market, or profitability. The final sample of 76 firms consists of 16,518 coupon bond observations with no violation of absolute priority rules.

Long-term debt and book values are recorded at the quarterly frequency while leverage and book-to-market series are monthly. To circumvent any look-ahead biases, we use debt and book values from the previous quarter to proxy leverage and book-to-market for the next 3 months. A cubic spline is used to approximate quarterly profitability measure into a monthly profitability measure.

Bonds are classified into credit rating categories using Standard and Poor's ratings: bonds with (numerical) rating score up to 9 are designated as A-rated and with score higher than 9 are designated as BBB-rated (or below). Since the fraction of bonds rated above BBB is 69%, our bond sample is skewed toward high-grade bonds.

10.3 MODELS AND ESTIMATION

10.3.1 Estimation of Interest Rate Process

On the basis of several considerations, Kalman filtering technique and a panel of Treasury yields are chosen to estimate the interest rate process. First, the unobservable nature of the instantaneous interest rate and its long-run drift makes Kalman filtering estimation appropriate for the task at hand. Second, the pricing equation (Equation 10.A3) and Bakshi, Madan, and Zhang (2006) suggest that interest rate parameters can be estimated using defaultable coupon bonds. However, Dai and Singleton (2003) and Duffee (1999) reason that the joint estimation of default and default-free process across individual firms can make the overall dimensionality of the optimization problem rather large, making estimation computationally difficult. Third, the full parameter set can be estimated separately for each firm but has the undesirable implication that the interest-rate process is different in the firm cross-section. The Kalman filter estimation procedure and empirical results can be found in Bakshi et al. (2006).

10.3.2 Estimation of Single-Name Defaultable Process

In the spirit of Chen and Scott (2003), Duffee (2002), Duffie and Singleton (1997), Duffie, et al. (2003), and Pearson and Sun (1994), we suppose that for each bond, the model price deviates from the true market price and pricing errors are correlated in the cross-section of bond maturities. To incorporate these restrictions into our econometric procedure, assume, under the physical probability measure, that the firm-specific distress variable $S_n(t)$ is governed by dynamics (Equation 10.A6) with $\lambda_s \equiv 0$:

$$\mathrm{d}S_n(t) = \kappa_s[\mu_s - S_n(t)]\,\mathrm{d}t + \sigma_s\,\mathrm{d}\omega_s(t) \tag{10.1}$$

where, ω_s is a standard Brownian motion. Since $S_n(t)$ is normally distributed conditional on $S_n(t-1)$, the transition density of $S_n(t)$ satisfies

$$f(S_n(t)|S_n(t-1)) = \frac{1}{\sqrt{2\pi Q_n(t)}}\exp\left[-\frac{(S_n(t)-\mu_n(t))^2}{2Q_n(t)}\right] \tag{10.2}$$

where, $\mu_n(t)$ and $Q_n(t)$ represent the first and second conditional moment of $S_n(t)$.

Recall by selection that there are three corporate coupon bonds outstanding each month with time-to-maturity close to 2, 7, and 15 years, respectively. Let the 3×1 vector

$$\varepsilon_n^\star(t;\,\vartheta) \equiv \overline{Y}_n^\star(t) - Y_n^\star(t;\,\vartheta_n) \tag{10.3}$$

be the difference between the market bond yields and model-determined yields of single-name bond n at time t and

$$\vartheta_n \equiv (\Lambda_0, \Lambda_r, \Lambda_s, \kappa_s, \lambda_s, \mu_s, \sigma_s, \rho_{r,s}) \tag{10.4}$$

denote the parameter vector characterizing the default process. Suppose $\varepsilon_n^\star(t;\,\vartheta)$ is serially uncorrelated but jointly normally distributed with zero mean and variance–covariance matrix Ω_ε. Under these assumptions, the log-likelihood function for a sample of observations on corporate coupon bond prices for $t = 2, \ldots, T$ is

$$\begin{aligned}\max_{\vartheta_n, C_n} \mathcal{L}^\star = &-2(T-1)\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\log(Q_n(t)) - \frac{1}{2}\sum_{t=2}^{T}\frac{(S_n(t)-\mu_n(t))^2}{Q_n(t)} \\ &- \frac{T-1}{2}\log|\Omega_\varepsilon| - \frac{1}{2}\sum_{t=2}^{T}\varepsilon_n^\star(t;\,\vartheta)'\Omega_\varepsilon^{-1}\varepsilon_n^\star(t;\,\vartheta)\end{aligned} \tag{10.5}$$

To ensure that the variance–covariance matrix Ω_ε is positive semi-definite assume, following Duffee (2002), that Ω_ε satisfies the Cholesky decomposition $\Omega_\varepsilon = C_n C_n'$ where C_n is a 3×3 matrix with non zero elements C_{11}, C_{22}, C_{33}, C_{21}, and C_{32}.

The estimation procedure (Equation 10.5) is applied to credit risk models with defaultable discount rate specification for firm n given by

1. Interest Rate Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t)$
2. Leverage Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n}\,\mathrm{LEV}_n(t)$
3. B/M Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n}\frac{\mathrm{B}_n(t)}{\mathrm{M}_n(t)}$
4. Profitability Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n}\,\mathrm{PROFIT}_n(t)$
5. Stock Volatility Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n}\,\mathrm{VOL}_n(t)$
6. Distance-to-Default Model $\quad R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n}\,\text{Distance-to-Default}_n(t)$

For each model, the corporate discount (coupon) bond price can be determined from Equation 10.A7 (Equation 10.A3). The econometric approach, thus, uses the desired information in the cross-section of corporate bond prices and the underlying transition density of S_n (t). For each firm we omit the last 12 months in the estimation to conduct out-of-sample inference. Motivated by the approach of Collin-Dufresne and Goldstein (2001), we also implement credit risk models that depend on the logarithmic transformation of leverage, B/M, and volatility.

Table 10.1 reports the results from applying the estimation procedure to the firm-sample and presents the average parameters for BBB-rated and A-rated bonds. Let us focus on the leverage model (in level) where the average Λ_s is 0.033 (0.015) with BBB-rated

TABLE 10.1 Estimation of Defaultable Bond Models

Model	Sample	Λ_0	Λ_r	Λ_s	κ_s	σ_s	μ_s	λ_s	$\rho_{r,s}$	$\mathcal{L}^*/T$	1_{χ^2}
Interest rate	BBB	0.013	1.018							10.28	
(base model)	A	0.008	0.985							10.84	
Leverage	BBB	0.006	0.799	0.033	0.465	0.100	0.345	−0.275	0.238	12.85	100
(level)	A	0.004	0.955	0.015	1.016	0.112	0.321	−0.631	−0.080	12.77	96
Leverage	BBB	0.010	0.890	0.013	0.376	0.125	0.227	−0.185	0.418	12.68	96
(logs)	A	0.007	0.969	0.001	0.806	0.090	0.307	−0.243	0.272	12.93	100
B/M	BBB	0.007	0.800	0.023	1.078	0.553	0.680	−0.956	0.022	11.15	75
(level)	A	0.006	0.902	0.010	1.175	0.609	0.630	−0.793	−0.116	11.56	76
B/M	BBB	0.005	0.836	0.026	0.406	0.111	0.373	−0.171	0.250	12.46	100
(logs)	A	0.005	0.952	0.007	0.859	0.094	0.449	−0.394	0.172	12.80	100
Profitability	BBB	0.007	0.840	0.012	0.983	0.148	0.709	−0.634	0.125	12.38	92
	A	0.005	0.961	0.005	0.889	0.061	0.710	−0.541	0.002	13.09	98
Volatility	BBB	0.010	0.851	0.036	2.084	0.244	0.171	−1.360	−0.068	11.63	96
(level)	A	0.007	0.927	0.026	2.887	0.299	0.245	−0.998	−0.056	11.18	86
Volatility	BBB	0.014	0.910	0.008	2.266	0.959	0.421	−1.509	−0.023	11.46	89
(logs)	A	0.007	0.964	0.003	2.238	0.961	0.289	−1.637	0.131	11.15	86
Distance-to-Default	BBB	0.006	0.767	0.032	0.268	0.097	0.325	−0.162	0.188	10.93	94
	A	0.005	0.966	0.006	0.879	0.091	0.371	−0.312	0.230	11.33	88

Note: This estimation procedure uses three corporate coupon bonds with maturity close to 2, 7, and 15 years. Let 3×1 vector $\varepsilon_n^*(t) \equiv \bar{Y}_n^*(t) - Y_n^*(t;\, \vartheta)$ measure the difference between the market observed corporate bond yields and the model-determined yields of firm n in month t. It is assumed that $\varepsilon_n(t)$ are (1) jointly normally distributed with zero mean and variance–covariance matrix Ω_ε and (2) the pricing errors are serially uncorrelated over time. The log-likelihood function to be maximized is

$$\max \mathcal{L}^* \equiv -2(T-1)\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\log(Q_n(t)) - \frac{1}{2}\sum_{t=2}^{T}\frac{(S_n(t)-\mu_n(t))^2}{Q_n(t)}$$
$$-\frac{T-1}{2}\log|\Omega_\varepsilon| - \frac{1}{2}\sum_{t=2}^{T}\varepsilon_n^*(t)'\Omega_\varepsilon^{-1}\varepsilon_n^*(t)$$

where $\mu_n(t)$ and $Q_n(t)$ represent the first two conditional moments of the distress factor $S_n(t)$. For each credit risk model with parameter vector ϑ, the reported values are the respective averages across BBB-rated bonds and A-rated bonds. 1_{χ^2} is the fraction of firms (in percent) for which the model restriction $\Lambda_s = 0$ is rejected based on a likelihood ratio test (distributed $\chi^2(6)$). The average log-likelihood value is shown as $\mathcal{L}^*/T$. The construction of distance-to-default is as outlined in Section 10.3. Results for (1) leverage model, (2) B/M model, and (3) stock volatility model are reported for both the level (denoted Level) and the logarithmic (denoted Logs) specification.

(A-rated) bonds. Hence, consistent with the work of Merton (1974), Longstaff and Schwartz (1995), and Collin-Dufresne and Goldstein (2001), the marginal impact of leverage is to enhance the magnitude of $R_n(t)$ and credit spreads and this effect is stronger for lower-rated debt.

We can observe that introducing leverage as an additional factor into the interest rate model tends to lower the magnitude of Λ_r: for BBB-rated and A-rated bonds the estimates

become 0.799 and 0.955 versus 1.018 and 0.985 with the interest rate model. As observed in Duffee (1999), the positive estimates substantiate the notion that an upward shift in the short-rate raises the defaultable discount rate (Longstaff and Schwartz 1995). The near-unity sensitivity coefficients suggest that bond yields are sensitive to $r(t)$ and more so for high-grade debt. With average Λ_0 close to 0.6% for BBB-rated and 0.4% for A-rated bonds, the instantaneous credit yield is higher with worsening credit quality.

Reported κ_s indicates that leverage is a mean-reverting stochastic process and $\lambda_s < 0$ has the interpretation that mean-reversion in leverage is stronger under the physical probability measure. Our estimates imply that the negative leverage risk premium increases the long-run drift of $S_n(t)$, which lowers corporate bond prices and increases credit spreads. Not at odds with intuition, the estimate of long-run leverage, μ_s, under the physical measure is 34.5% for BBB-rated bonds and 32.1% for A-rated bonds. Finally, the correlation coefficient $\rho_{r,s}$ is estimated as 0.238 for BBB-rated bonds. This estimate implies that leverage-induced distress is more severe in a rising interest rate environment possibly due to the impact of declining equity values.

In the last column, we examine the restriction that $\Lambda_s = 0$ via a likelihood ratio test computed as twice the difference between the unrestricted and the restricted model log-likelihood. This test statistic is distributed $\chi^2(6)$ and the reported 1_{χ^2} is the fraction of firms for which the likelihood ratio test is rejected. Providing support for the inclusion of leverage in the defaultable discount rate specification, our results indicate that the restriction $\Lambda_s = 0$ is rejected for 100% (96%) of BBB-rated (A-rated) single-name firms.

When distress is proxied by competing factors such as book-to-market, profitability, stock volatility, and distance-to-default, the parameter estimates have an interpretation similar to that described in the case of the leverage model. However, based on a comparison of average log-likelihood across models, leverage provides a superior fit to the objective function (Equation 10.5) and especially improves log-likelihood relative to interest rate and distance-to-default models. Fixing the distress factor, the log-likelihood values are typically higher with the level specification for BBB-rated firms while the log specification yields a slightly better description of A-rated firms.

10.4 MEASURES OF MODEL MISSPECIFICATION AND PERFORMANCE

10.4.1 Determinants of Credit Risk

Because credit risk models are often employed in marking-to-market other illiquid securities, evaluating model performance is of broad interest (Dai and Singleton 2003). Does a three-factor credit risk model with leverage improve upon a two-factor counterpart with firm-specific distress considerations absent? Is a credit risk model based on leverage less misspecified compared to the alternatives of book-to-market, profitability, volatility, and distance-to-default?

To address these questions and to model credit risk determinants, Table 10.2 presents four measures of performance for each model: (1) the absolute yield valuation errors

TABLE 10.2 Valuation Errors for Defaultable Coupon Bond Models

		Interest Rate		Leverage (Level)		Leverage (Logs)		B/M (Level)		B/M (Logs)		Profit Model		Volatility (Level)		Volatility (Logs)		Distance to Default	
		BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A	BBB	A
ALL	AYE	45	27	36	28	38	26	38	29	40	26	41	27	41	29	43	27	40	25
	APE	2.43	1.43	1.80	1.43	2.03	1.36	1.98	1.48	2.08	1.35	2.18	1.37	2.18	1.53	2.40	1.39	2.17	1.34
	SIC	−6.96	−7.80	−7.53	−7.90	−7.32	−7.93	−7.33	−7.84	−7.31	−7.95	−7.15	−7.91	−7.15	−7.82	−7.12	−7.91	−7.17	−7.94
	ACE	40	18	33	21	35	18	35	21	37	18	38	18	39	21	39	21	37	18
Short	AYE	49	29	38	30	39	28	41	31	41	28	39	29	41	29	41	29	42	28
	APE	1.21	0.77	0.99	0.78	1.03	0.74	1.15	0.79	1.05	0.73	0.98	0.78	1.02	0.78	1.13	0.75	1.12	0.74
	SIC	−8.59	−9.20	−8.83	−9.21	−8.79	−9.27	−8.65	−9.19	−8.73	−9.29	−8.77	−9.18	−8.78	−9.17	−8.73	−9.23	−8.65	−9.31
	ACE	43	22	33	22	35	19	38	22	31	22	35	20	40	22	39	22	39	20
Medium	AYE	45	27	36	28	38	26	39	28	40	26	42	26	45	26	45	27	40	25
	APE	2.40	1.47	1.88	1.54	2.05	1.43	2.02	1.58	2.19	1.42	2.25	1.45	2.42	1.44	2.45	1.45	2.17	1.39
	SIC	−7.17	−7.90	−7.63	−7.93	−7.46	−7.97	−7.43	−7.89	−7.41	−7.99	−7.28	−7.94	−7.20	−7.96	−7.23	−7.97	−7.36	−8.01
	ACE	39	16	30	18	32	15	33	19	35	15	38	16	40	16	37	17	35	15
Long	AYE	42	27	33	27	37	25	39	25	39	24	41	25	40	27	42	25	39	23
	APE	3.37	2.15	2.66	2.08	2.95	1.91	3.07	1.94	3.09	1.89	3.24	1.91	3.18	2.10	3.34	1.97	3.20	1.88
	SIC	−6.39	−7.18	−6.96	−7.34	−6.73	−7.39	−6.72	−7.36	−6.72	−7.41	−6.54	−7.37	−6.55	−7.28	−6.56	−7.37	−6.57	−7.38
	ACE	38	19	36	22	37	20	38	21	38	20	41	20	40	22	40	23	37	20

Note: For each credit risk model we compute absolute yield error (AYE, in basis points) as $\left|\overline{Y}_n^{\star}(t) - Y_n^{\star}(t;\vartheta)\right|$, where $\overline{Y}_n^{\star}(t)$ is observed yield and $Y_n^{*}(t;\vartheta)$ is model-determined yield, and a similar calculation applies to absolute percentage valuation error (APE, in %). The model defaultable bond price is based on the estimated interest rate process and the defaultable process. ACE is absolute credit spread error (in basis points) computed as the absolute difference between maturity-matched observed and model credit spreads. SIC is Schwarz–Bayes information criterion. Results for (1) leverage model, (2) B/M model, and (3) stock volatility model are reported for both the level (denoted "Level") and the logarithmic (denoted "Logs") specification.

(in basis points), (2) the absolute percentage valuation errors (in percent), (3) the Schwarz–Bayes information criterion, and (4) the absolute credit spread errors (in basis points) for maturity-matched spread of corporate over Treasury. Each measure provides a perspective on the effectiveness of a model to explain the term structure of defaultable yields and credit spreads.

On the basis of the results for yield errors on BBB-rated bonds, a three-factor model with leverage (in level) is attractive relative to the two-factor interest rate model. In the row labeled "ALL" note that incorporating leverage in the defaultable discount rate specification leads to yield errors of 36 bp with leverage (level) model compared to 45 bp with interest rate model. This strong improvement over the base model is, in fact, observed at all maturities with each model performing slightly worse at the short-end of the defaultable yield curve. From a different angle, the documented fitting errors suggest that systematic variables have a first-order impact on defaultable term structures with leverage providing a sizeable next-order explanatory ability.

Compared with the three-factor setting of Duffee (1999) and the single-factor intensity process specification of Longstaff, Mittal, and Neis (2005), the average yield error is -7 bp (not reported in the table to save on space) making the leverage model reasonably accurate for marking-to-market credit risks. Although the focus of academic work has mostly been on default factors and our results are encouraging, realize that omitted nondefault attributes such as illiquidity of corporate bonds and actual spread volatility can cause model prices to systematically deviate from observed market prices even in the presence of properly specified default factors (see the practitioner perspective outlined in Dignan 2003 and Kao 2000).

Table 10.2 imparts the insight that the interest rate model and the leverage (in level or logs) model are virtually indistinguishable for high-grade bonds at all maturities. This seemingly counterintuitive result can be understood to mean that high-rated bonds are mostly determined by systematic factors and rather insensitive to variations in firm-specific leverage. Although Λ_s has the theoretically correct positive sign in the leverage model that raises default probability for firms with higher leverage, the estimated magnitude is insufficiently strong to impact the pricing performance for high-grade bonds. Consistent with Λ_0 and Λ_r of the interest rate model, we may write defaultable discount rate $R_n(t) = 0.8\% + 0.985\, r(t)$ implying that fixed-income markets value high-grade bonds using a translated default-free process. Thus, a more complex model that accounts for firm-specific distress need not necessarily perform better for high-grade bonds. As would be expected, each credit risk model (in particular the interest rate model) is less misspecified for high-grade bonds than its low-grade counterpart. Shifts in interest rate risk play a fundamental role in the valuation of single-name high-grade bonds (Kwan 1996 and Collin-Dufresne et al. 2001).

According to empirical results for BBB-rated bonds, the leverage (in level) model performs the best followed in turn by the B/M (in level) model, the profitability model, the stock volatility model, and the distance-to-default model. The interest rate model is the last place performer. To fix main ideas, consider long-term bonds and the level specification: the percentage pricing errors (yield errors) are 2.66% (33 bp), 3.07% (39 bp), 3.24%

(41 bp), 3.18% (40 bp), 3.20% (39 bp), and 3.37% (42 bp), respectively, for the leverage, B/M, profitability, volatility, distance-to-default, and interest rate models. It must be noted that this model rank-ordering does not perfectly coincide with the average log-likelihood ratio based ordering presented in Table 10.1. The reason is that the objective function in the estimation is not to minimize the sum-of-squared pricing errors; instead it is derived from the conditional density of $S_n(t)$ and the assumed cross-correlation structure of the valuation errors to give realistic single-name empirical moments. Given the large notional value of fixed income securities, the documented superiority of the leverage model is economically meaningful.

Another way to judge performance is to compare the Schwarz-Bayes information criterion (SIC) statistic, which rewards goodness-of-fit while penalizing the dimensionality of the model. Across most maturities, the SIC of the interest rate model exceeds the SIC of the leverage model for BBB-rated bonds while the reverse is true for high-grade bonds. This empirical yardstick thus favors the more heavily parameterized leverage model over the interest rate model for BBB-rated bonds. However, for high-grade bonds, simplicity is a more desirable modeling element. It is worth noting that in the case of BBB-rated firms, the SIC of the B/M and distance-to-default models exceed the SIC of the leverage model and therefore perform no better. Issues related to model differentiation based on statistical significance are formalized in the next subsection.

Relevant to our findings on high-grade bonds, Collin-Dufresne, Goldstein, and Martin (2001) argue that while variation in corporate yields is predominantly due to Treasury yields, the credit spread is a much harder entity to predict. Are credit spreads mainly driven by the short interest rate or firm-specific distress variables? To answer this question, we provide additional information on the absolute difference between the observed and model generated credit spreads and report them as ACE in Table 10.2. The chief result that emerges from this exercise is that absolute credit spread errors have a large interest rate component, and accounting for leverage and other firm-specific distress is important. Consider BBB-rated bonds where the credit spread errors can be reduced from 40 bp in the case of interest rate model to 33 bp in the case of leverage model (in level). Sharing this feature with AYE, adding a firm-specific distress variable to the credit risk model does not help to lessen absolute credit spread errors for high-grade bonds. Thus, the performance metrics that rely on yields (i.e., AYE) and credit spreads (i.e., ACE) provide similar conclusions.

To gauge the robustness of the findings, Table 10.3 displays valuation results based on a holdout sample of 1 year. The standard argument favoring such an exercise is that model performance need not improve out-of-sample for a more complex valuation model: Extra parameters have identification problems and may penalize accuracy. In implementation, we compute theoretical prices evaluated at the Kalman filter forecast of $r(t+1)$ and $z(t+1)$ and the optimal forecast of $S_n(t+1)$ based on the parametric model (Equation 10.1). The results are more striking in the out-of-sample context for BBB-rated bonds. First, comparing the entries in Tables 10.2 and 10.3, there is a general deterioration in the performance of credit risk models and, particularly, the interest rate model. Thus, the interest rate model is most misspecified out-of-sample especially at the short-end of

TABLE 10.3 Out-of-Sample Valuation Errors Based on a Hold-Out Sample

		Interest Rate		Leverage (Level)		Leverage (Logs)		B/M (Level)		B/M (Logs)		Profit Model		Volatility (Level)		Volatility (Logs)		Distance-to-Default	
		BBB	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**	**BBB**	**A**
ALL	AYE	56	27	40	29	49	26	49	26	47	27	54	28	55	26	51	29	48	31
	APE	2.62	1.39	2.08	1.45	2.49	1.34	2.46	1.30	2.39	1.32	2.72	1.37	2.78	1.36	2.70	1.53	2.49	1.53
Short	AYE	66	28	38	26	47	24	46	23	43	25	48	26	49	25	45	27	44	29
	APE	1.34	0.66	0.80	0.60	0.98	0.56	0.98	0.53	0.92	0.57	0.98	0.61	1.03	0.60	0.94	0.62	0.96	0.68
Medium	AYE	56	28	40	30	49	27	46	27	46	25	53	26	55	28	50	30	47	33
	APE	2.69	1.45	1.97	1.57	2.39	1.36	2.33	1.38	2.29	1.31	2.60	1.36	2.80	1.42	2.58	1.55	2.46	1.77
Long	AYE	46	26	43	31	52	27	54	28	51	26	60	27	61	26	57	31	51	29
	APE	3.62	2.06	3.41	2.45	4.10	2.08	4.16	2.20	3.97	2.04	4.71	2.10	4.77	2.06	4.57	2.41	4.00	2.28

Note: The results are based on a hold-out sample of 1 year for each firm. For each credit risk model we compute absolute yield error (AYE, in basis points) as $\left|\overline{Y}_n^{\star}(t) - Y_n^{\star}(t;\vartheta)\right|$ and a similar calculation applies to absolute percentage valuation error (APE, in %). In implementation, the model defaultable bond price is evaluated at the Kalman filter forecast of $r(t+1)$ and $z(t+1)$, the optimal forecast of $S_n(t+1)$ based on the parametric model (Equation 10.22), and the estimated parameters of the interest rate and the defaultable process.

the low-grade yield curve. Second, the leverage model enhances the overall effectiveness of the interest rate model by 16 bp, and the B/M (distance-to-default) model by 8 (9) bp. Third, the results tell us that the relative ranking between credit risk models is unaltered in-sample versus out-of-sample.

On the basis of the in-sample and out-of-sample absolute yield errors and the SIC statistic, we generally observe that the model relying on the distress variable specified in level does slightly better than the long counterpart. For instance, the absolute yield error in Table 10.3 for BBB firms with leverage (in level) is 40 versus 49 bp with leverage (in logs). Similar improvement is documented for SIC in Table 10.3 which is −7.53 for leverage (level) and −7.32 for leverage (logs). To confirm whether the level or the log of leverage is more consistent with the assumed Gaussian physical dynamics, we also computed the Shapiro–Wilk normality statistic applied to firm-level leverage series. This test shows that moving from level of leverage to the logarithmic of leverage improves the Shapiro–Wilk statistic for 31% of the firms, but worsens for 69% of the firms. In summary, the level of leverage conforms to the normality hypothesis better than the log of leverage. However, the log B/M appears more consistent with the Gaussian dynamics for 63% of the firms.

10.4.2 Generalized Method of Moments Test of Model Comparisons and Specification Tests

The ask price is not reported in the Lehman database so we are unable to assess the extent of the improvement relative to the bid-ask spreads. However, for any two defaultable coupon bond models, define the disturbance terms

$$\mathcal{W}_t(\mu^{\text{int}}, \mu^{\text{leu}}) \equiv \begin{pmatrix} \frac{1}{3}\sum_{i=1}^{3} \left|\varepsilon_i^{\text{int}}(t)\right| - \mu^{\text{int}} \\ \frac{1}{3}\sum_{i=1}^{3} \left|\varepsilon_i^{\text{lev}}(t)\right| - \mu^{\text{lev}} \end{pmatrix} \tag{10.6}$$

and

$$E\left[\mathcal{W}_t(\mu^{\text{int}}, \mu^{\text{lev}})\right] = 0 \tag{10.7}$$

where μ^{int} and μ^{lev} represent the mean monthly absolute yield errors (averaged across short, medium, and long) for the interest rate model and the leverage model (in level).

Statistics presented in Table 10.4 are helpful for judging whether incremental explanatory power of a model is statistically significant relative to other models. First, we report the rejection rate for the null hypothesis $\mu^{\text{int}} = \mu^{\text{lev}}$ versus the alternative $\mu^{\text{int}} \neq \mu^{\text{lev}}$. Taking the weighting matrix from the unconstrained estimation this test is done by performing a restricted estimation with $\mu^{\text{int}} = \mu^{\text{lev}}$. Then, T times difference in the restricted and the unrestricted criterion functions is asymptotically $\chi^2(1)-$distributed. Second, we present the rejection rate for the one-sided test $\mu^{\text{int}} > \mu^{\text{lev}}$ using the Newey–West method to adjust for heteroskedasticity and autocorrelation. Last, we compute $\Phi_t \equiv \frac{|\varepsilon^{\text{int}}(t)| - |\varepsilon^{\text{lev}}(t)|}{|\varepsilon^{\text{int}}(t)|}$ (or, $\bar{\Phi}_t$),

TABLE 10.4 Generalized Method of Moments Tests of Statistical Significance

		Leverage Model (Level) versus					B/M Model (Logs) versus			
		Interest Rate	B/M (Logs)	Profitability	Volatility (Logs)	Distance-to-Default	Interest Rate	Profitability	Volatility (Logs)	Distance-to-Default
BBB	Two-sided rejection	70.8	50.0	54.2	75.0	46.7	45.8	54.2	54.2	33.3
	One-sided rejection	66.7	45.8	54.2	66.7	40.0	45.8	37.5	33.3	20.0
	Yield improvement, Φ_t (%)	30.1	14.3	25.2	33.8	15.2	18.4	12.7	22.8	−2.1
	Spread improvement, Φ_t (%)	29.3	12.7	25.8	37.1	7.8	19.0	14.9	28.0	−0.4
A	Two-sided rejection	24.5	22.4	18.4	12.2	13.0	20.4	10.2	14.3	21.7
	One-sided rejection	22.4	8.2	10.2	6.1	8.7	20.4	10.2	6.1	8.7
	Yield improvement, Φ_t (%)	−5.4	−11.4	−9.0	−6.1	2.9	5.4	2.1	4.7	2.2
	Spread improvement, $\overline{\Phi}_t$ (%)	−8.3	−15.1	−12.4	−1.8	8.7	5.9	2.4	11.6	10.4

Note: For each firm we conduct a pair wise valuation comparison between (1) the leverage model (level) versus other models and (2) the book-to-market model (logs) versus remaining models. Take the leverage model and fix the interest rate model as the base model. Define,

$$\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right) \equiv \begin{pmatrix} \frac{1}{3}\sum_{i=1}^{3}\left|\varepsilon_i^{int}(t)\right| - \mu^{int} \\ \frac{1}{3}\sum_{i=1}^{3}\left|\varepsilon_i^{lev}(t)\right| - \mu^{lev} \end{pmatrix}$$

and the moment condition

$$E\left[\mathcal{W}_t\left(\mu^{int}, \mu^{lev}\right)\right] = 0$$

where μ^{int} and μ^{lev}, respectively, represent the mean monthly absolute yield errors for the interest rate model and the leverage model. In the first row, we report the two-sided rejection rate that tests the null hypothesis $\mu^{int} = \mu^{lev}$ versus the alternative $\mu^{int} \neq \mu^{lev}$. Next, we report the one-sided rejection rate, which is the fraction of firms with $\mu^{int} > \mu^{lev}$ at the 95% confidence level. Last, we report a measure of relative model yield improvement, computed as

$$\Phi_t \equiv \frac{\left|\varepsilon^{int}(t)\right| - \left|\varepsilon^{lev}(t)\right|}{\left|\varepsilon^{int}(t)\right|}$$

and the spread improvement, $\overline{\Phi}_t$, is analogously based on credit spread errors. The Newey West method is used to adjust for heteroskedasticity and autocorrelation.

which reflects the average absolute yield error (credit spread error) reduction in moving from the interest rate model (the base model) to the leverage model.

Consider low-grade bonds: valuation improvement is statistically large by adding leverage (in level) to the defaultable discount rate specification of the interest rate model. Overall, leverage results in a 30.1%, 14.3%, 25.2%, 33.8%, and 15.2% relative improvement over the interest rate, the B/M, profitability, volatility, and distance-to-default models. Between leverage and interest rate models, the two-sided (one-sided) rejection rate is 70% (67%) implying statistical improvement for the majority of the single-names (and likewise for credit spreads).

The B/M (log) generalization to the interest rate model also delivers economically and statistically significant performance enhancement, albeit by a lesser extent relative to the leverage model (we focus on log as it is more consistent with the Gaussian hypothesis although the result is similar with level). The relative improvement is 18.4% over the interest rate model with a oneside rejection rate of 46%. With high-grade bonds, however, there is no discernible improvement over the interest rate model with either leverage or B/M. In this regard, Agarwal, Elton, Gruber, and Mann (2001) reason that differences in default are insufficient to explain yield spreads observed across rating classes. Such a result also parallels a calibration-based finding of Huang and Huang (2003) that default risk accounts for only a small fraction of aggregate credit spreads, and this result can now be extended to high-grade single-name credit spreads. Allowing for a richer stochastic structure beyond the interest rate model in the firm-specific dimension is of greater relevance for yield errors and credit spread error of low-grade bonds.

To further study properties of model mispricing, we appeal to Collin-Dufresne, Goldstein, and Martin (2001) and investigate whether pricing-errors from the leverage model are contemporaneously correlated. Specifically, we focus on four systematic factors: (1) default spread between low-rated and high-rated corporate bonds (denoted DEFAULT), (2) spread between the 10 year and 2 year Treasury yield (denoted SLOPE), (3) size premium (SMB), and (4) VIX volatility index from CBOE. The following regression is performed for each firm $\left(\text{letting } \varepsilon_n^{\text{lev}}(t) \equiv \frac{1}{3}\sum_{i=1}^{3}|\varepsilon_{n,i}^{\text{lev}}(t)|\right)$:

$$\begin{aligned}\varepsilon_n^{\text{lev}}(t) = a_0 + a_1\ \text{DEFAULT}(t) + a_2\ \text{SLOPE}(t)\\ + a_3\text{VIX}(t) + a_4\ \text{SMB}(t) + e(t), \quad n = 1,\ldots,N\end{aligned} \tag{10.8}$$

Three findings are relevant to our modeling approach. First, we find that the average adjusted-R^2 across our sample of firms is 9%, implying that single-name pricing-errors show little time-covariation with the chosen systematic factors (50% of the firms had $R^2 < 5\%$). Second, the sensitivity coefficients are typically positive. In economic terms, this means that model mispricing tends to be greater during periods of widening default spreads or steepening Treasury-curves (or equivalently when VIX/size premium is high). Third, the t-statistics (corrected for heteroskedasticity and autocovariance) are greater for 2 for at most 19% of the firms. For the majority of the firms, the pricing-errors are, thus, statistically unrelated to the systematic factors. Extant research has found that

corporate bond markets are subject to high transaction cost and low volume. Therefore, it appears likely that bond market illiquidity has contributed to model mispricing. More empirical work is needed to study the role of bond market liquidity along the lines of Collin-Dufresne, Goldstein, and Martin (2001). If there are market-wide default and liquidity factors, these are perhaps not adequately captured by the set of included systematic variables.

Additionally, we performed a principal component analysis of absolute yield errors and credit spread errors of each model by binning the data into nine groups (by short, medium, long, and by AA, A, and BBB credit rating). The results establish that there is a dominant component for each of the models, with first two components explaining the bulk of the variation. Take the leverage model as an example, where the first component explains 83.6% of the total variation. Together, the first two components capture 92.7% of the variation. As shown by the regression analysis, none of the systematic variables appear to be a good proxy for the predominant factors. Tables 10.2 and 10.3 have the unfortunate implication that credit risk models are misspecified albeit to various degrees. One key issue that remains to be addressed is whether cross-sectional variations in model misspecification are related to excluded firm-specific distress risk? Indeed, there is no theoretical basis for picking a firm-specific distress variable successively in a default model. In doing so, we were guided by parsimony and ease of implementation: a four-factor generalization driven by Equation 10.1 has four additional parameters and two correlation terms requiring the estimation of 22 parameters (plus five Cholesky decomposition constants). How can a four-factor credit risk model with two firm-specific factors expected to fare relative to a three-factor credit risk model? To informally attend to this concern, two regression specifications are explored using the entire cross-section. First, we regress the absolute yield errors of the leverage model on B/M ($n = 1, \ldots, N$) and obtained:

$$\underset{(15.2)\quad(4.86)}{\mathrm{AYE}_n|_{\mathrm{Lev}} = 0.262 + 0.08\frac{\mathrm{B}_n}{\mathrm{M}_n} + e_n, \quad R^2 = 2.7\%}$$

where the reported coefficients are time-series pooled averages and the *t*-statistics are based on GMM. Second, we regress B/M mispricing on leverage:

$$\underset{(14.7)\quad(3.13)}{\mathrm{AYE}_n|_{\mathrm{B/M}} = 0.288 + 0.098\ \mathrm{LEV}_n + e_n, \quad R^2 = 0.4\%}$$

On the basis of the sensitivity coefficients, one may conjecture that it is possible to lower the pricing errors of defaultable bonds: the valuation errors of B/M model are statistically higher for firms with high leverage even after accounting for B/M (and vice versa). Owing to burdensome MLE implementation of higher-dimensional models and the rather low R^2 explanatory power of $\frac{\mathrm{B}_n}{\mathrm{M}_n}$ and LEV_n in the aforementioned cross-sectional estimations, we leave a formal analysis of four-factor credit risk models to a later study.

10.4.3 Hedging Dynamic Credit Exposures

This subsection concentrates on hedging a portfolio consisting of a short-position in a corporate bond and a long-position in a Treasury bond with matched maturity. Notice there is an arbitrage link between the corporate bond market, the Treasury market, and the credit default swap market. According to Duffie (1999), Duffie and Liu (2001), and Houweling and Vorst (2005), holding a τ-period par corporate bond and buying credit protection through a credit default swap are approximately risk free (adjusting for floating-rate feature and coupon equivalence). For this reason, the price of single-name credit protection is the same as reference entity credit spread. Our approach, as shown, can be useful in understanding which traded fixed-income instruments can most effectively hedge dynamic credit exposures and, to first-order, potentially hedge the risks of writing credit protection. This connection arises since sellers of credit default swaps can hedge their exposure by shorting corporate bonds (Longstaff et al. 2005, p. 21 and Chen et al. 2005) or vice versa.

Fix the credit risk model as the interest rate model. In this case, the combined position can be written as

$$-\frac{1}{\tau}\log\left[P(t,\tau)/B(t,\tau)\right]+\left[w_0(t)+w_1(t)B(t,\tau_1)+w_2(t)B(t,\tau_2)\right] \tag{10.9}$$

which hedges credit exposure using two risk-free discount bonds with short-maturity τ_1 and long-maturity τ_2. Taking the target as the log relative of the corporate coupon bond price to the Treasury bond price and normalizing it by τ imparts it a yield interpretation. To dynamically hedge $r(t)$ and $z(t)$ risks, we take the required partial derivatives and derive the positioning as

$$w_1(t)=\frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}\right]\times\frac{\partial B(t,\tau_2)}{\partial r}-\frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_2)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}-\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}} \tag{10.10}$$

$$w_2(t)=\frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}\right]\times\frac{\partial B(t,\tau_1)}{\partial r}-\frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)}-\frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_1)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}-\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}} \tag{10.11}$$

with $\frac{\partial B(t,\tau)}{\partial r}$ and $\frac{\partial B(t,\tau)}{\partial z}$ determined from the default-free bond pricing formula (Equation 10.A16). The hedge portfolio is made self-financing by setting $w_0(t)=\frac{1}{\tau}\log\left[\overline{P}(t,\tau)/\overline{B}(t,\tau)\right]-w_1(t)B(t,\tau_1)-w_2(t)B(t,\tau_2)$. Local risk exposures for the defaultable coupon bond are the aggregated face and coupon exposures:

$$\Delta_r(t,\tau)\equiv F\Delta_r^{\star}(t,\tau)+\int_0^{\tau}c(t+u)\Delta_r^{\star}(t,u)\,\mathrm{d}u \tag{10.12}$$

$$\Delta_z(t,\tau)\equiv F\Delta_z^{\star}(t,\tau)+\int_0^{\tau}c(t+u)\Delta_z^{\star}(t,u)\,\mathrm{d}u \tag{10.13}$$

where $\Delta_r^{\star}(t,\tau)$ and $\Delta_z^{\star}(t,\tau)$ are displayed in Equations 10.A12 and 10.A13. The combined position in the defaultable bond and the replicating portfolio is liquidated at time $t+\Delta t$. Dollar hedging errors are defined as

$$H(t+\Delta t) = w_0(t)e^{r(t)\Delta t} + w_1(t)B(t+\Delta t, \tau_1 - \Delta t) + w_2(t)B(t+\Delta t, \tau_2 - \Delta t) - \frac{1}{\tau}\log\left[\overline{P}(t+\Delta t, \tau-\Delta t)/\overline{B}(t+\Delta t, \tau-\Delta t)\right] \quad (10.14)$$

Selecting τ as 2 and 7 years, the hedging strategy is executed each month and for each single-name bond. We compute absolute dollar hedging errors each month as

$$\text{AHE}(t+\Delta t) \equiv \frac{1}{K}\sum_{k=1}^{K}\left|H_k(t+\Delta t)\right| \quad (10.15)$$

and absolute percentage hedging errors as

$$\text{PHE}(t+\Delta t) \equiv \frac{1}{K}\sum_{k=1}^{K}\left|\frac{H_k(t+\Delta t)}{\frac{1}{\tau}\log\left[\overline{P}(t,\tau)/\overline{B}(t,\tau)\right]}\right| \quad (10.16)$$

Implementing a dynamic hedging strategy based on leverage and B/M models involves non-traded leverage and book values and only a partial hedge is feasible. To hedge $r(t)$, $z(t)$ risks we employ two Treasury discount bonds and single-name equity to dynamically hedge the market component of leverage. However, this leaves the long-term debt component of leverage unhedged. Notice that we can write leverage as $\text{LEV}(t) = \frac{D(t)/q_0}{\mathcal{M}(t)+D(t)/q_0}$ where q_0 is the number of shares outstanding and $D(t)$ is debt value. Assuming the stock price, $\mathcal{M}(t)$ follows geometric Brownian motion we derive $\omega_1(t)$ and $\omega_2(t)$ as shown in Equations 10.10 and 10.11 and $w_3(t) = -\frac{1}{\tau}\Delta_s \times \frac{D(t)/q}{[\mathcal{M}(t)+D(t)/q]^2}$ where $\Delta_s(t,\tau) \equiv F\Delta_s^{\star}(t,\tau) + \int_0^{\tau} c(t+u)\,\Delta_s^{\star}(t,u)\,\mathrm{d}u$.

The results reported in Panel A and Panel B of Table 10.5 reveal that leaving the debt (book) component of leverage (B/M) unaccounted in the hedging strategy can impair

TABLE 10.5 Hedging Credit Exposures

	Interest Rate		Leverage Model		B/M Model	
	BBB	**A**	**BBB**	**A**	**BBB**	**A**
Panel A: 2 year credit exposures						
AHE	\$0.106	\$0.099	\$0.107	\$0.100	\$0.107	\$0.100
PHE	4.18%	4.07%	4.20%	4.09%	4.19%	4.09%
Panel B: 7 year credit exposures						
AHE	\$0.033	\$0.057	\$0.034	\$0.057	\$0.034	\$0.057
PHE	4.76%	8.14%	4.87%	8.26%	4.87%	8.26%

Note: In what follows we fix the maturity, τ, of the target to be hedged as either 2 or 7 years. Choose the interest rate model and the target as $\frac{1}{\tau}\log[P(t,\tau)/B(t,\tau)]$. In the hedging strategy, as many instruments as sources of risks are employed to create a hedge in all dimensions, which leads to $w_1(t) = \frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial z}\right]\times\frac{\partial B(t,\tau_2)}{\partial r} - \frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_2)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r} - \frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z}}$ and $w_2(t) = \frac{\frac{1}{\tau}\left[\frac{\Delta_z(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau_1)}{\partial z}\right]\times\frac{\partial B(t,\tau_1)}{\partial r} - \frac{1}{\tau}\left[\frac{\Delta_r(t,\tau)}{P(t,\tau)} - \frac{1}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial r}\right]\times\frac{\partial B(t,\tau_1)}{\partial z}}{\frac{\partial B(t,\tau_1)}{\partial r}\times\frac{\partial B(t,\tau_2)}{\partial z} - \frac{\partial B(t,\tau_1)}{\partial z}\times\frac{\partial B(t,\tau_2)}{\partial r}}$ and $w_0(t) = \frac{1}{\tau}\log[\overline{P}(t,\tau)/\overline{B}(t,\tau)] - w_1(t)B(t,\tau_1) - w_2(t)B(t,\tau_2)$. For each credit risk model and single-name bond, compute the absolute dollar hedging errors (AHE) as $|w_0(t)e^{r(t)\Delta t} + w_1(t)\ B(t+\Delta t,\ \tau_1-\Delta t) + w_2(t)\ \ B(t+\Delta t,\ \tau_2-\Delta t) - \frac{1}{\tau}\log[\overline{P}(t+\Delta t,\ \tau-\Delta t)/\overline{B}(t+\Delta t,\ \tau-\Delta t)]|$ and a similar calculation applies to absolute percentage hedging errors (PHE). In the leverage model and the B/M model, the debt component and the book component of leverage and B/M are left unhedged for 1 month. All results are based on the face value amount of \$100.

dynamic ability of three-factor credit risk models. Despite the flexibility of an additional instrument, the absolute hedging errors from the leverage model and the B/M model are no better than that of the two-factor interest rate model. From the perspective of absolute hedging errors, interest rate movements are dominant. When averaged over BBB-rated single-name bonds, the absolute hedging error with the leverage (book-to-market) model is \$0.107 (\$0.107) compared to \$0.106 with the interest rate model (assuming \$100 face value and for $\tau = 2$ years). Credit risk models relying on purely traded factor tend to produce better hedging effectiveness. Regardless of maturity, the two Treasury instruments do a fairly reasonable job neutralizing dynamic movements in single-name credit exposures for firms in our sample. We also investigated the relation of single-name hedging-errors using Equation 10.8 and found these to be characterized by a lack of covariation.

Although not reported, a robustness check shows that the same conclusion applies when the target is a short-position in a corporate bond. Our results open up the wider implication that short-positions in credit default swaps could be equivalently hedged using Treasury instruments up to BBB-rated single-name reference entities.

10.5 CONCLUSIONS

Relying on a sample of single-name defaultable coupon bonds, the empirical investigation posed two questions of economic interest: Which credit risk models are suitable for marking-to-market default risks and for hedging credit exposures? What type of default factors (systematic or firm-specific) are relevant for explaining credit risk? Under appropriate assumptions on the interest rate process, the default process, and the market price of risks, our estimation strategy leads to several new insights. First, interest-rate consideration is of first-order prominence in explaining credit curves for our sample of investment grade bonds. Second, accounting for the impact of interest rates, the incremental pricing improvement from modeling leverage is economically large and statistically significant for low-grade bonds. Third, the credit risk model with leverage appears least misspecified and has pricing performance adequate for marking risk exposures. Last, the hedging analysis opens up the possibility that traded Treasury bonds may be effective in hedging dynamic credit exposures.

APPENDIX: REDUCED-FORM DEFAULTABLE COUPON BOND MODELS

Following the standard approach, the price of the defaultable coupon bond is

$$P(t,\tau) = E^{Q}\left[\int_{t}^{t+\tau} \exp\left(-\int_{t}^{t+u}\{r(s)+h(s)[1-\nu(s)]\}\mathrm{d}s\right)\mathrm{d}C(u) + F\exp\left(-\int_{t}^{t+\tau}\{r(s)+h(s)[1-\nu(s)]\}\mathrm{d}s\right)|\mathcal{G}_t\right] \tag{10.A1}$$

In the Duffie–Singleton approach, the effects of the hazard rate process $h(t)$ cannot be separated from that of the loss process $[1-\nu(t)]$, they define the aggregate defaultable discount rate as

$$R(t) \equiv r(t) + h(t)[1-\nu(t)] \tag{10.A2}$$

With deterministic and continuous coupon rate $c(t)$, the debt equation can be simplified as

$$P(t,\tau) = \int_0^\tau c(t+u) \times P^\star(t,u)\,\mathrm{d}u + F \times P^\star(t,\tau) \tag{10.A3}$$

where

$$P^\star(t,u) = E^Q\left\{\exp\left[-\int_t^{t+u} R(s)\,\mathrm{d}s\right]|\mathcal{G}_t\right\} \tag{10.A4}$$

is the price of the unit face defaultable zero-coupon bond with maturity $t+u$.

For the empirical investigation, we consider the family of aggregate defaultable discount rate models shown below:

$$R_n(t) = \Lambda_{0,n} + \Lambda_{r,n} r(t) + \Lambda_{s,n} S_n(t), \quad n = 1,\ldots,N \tag{10.A5}$$

where $S_n(t)$ surrogates firm-specific distress indexed by n and postulated either as the level or the logarithmic of the variable. We assume, under the risk-neutral measure, that the distress factor obeys

$$\mathrm{d}S_n(t) = [k_s\mu_s - (k_s+\lambda_s)S_n(t)]\,\mathrm{d}t + \sigma_s\,\mathrm{d}\tilde{\omega}_s(t) \tag{10.A6}$$

where

$\tilde{\omega}_s$ is a standard Brownian motions under the risk-neutral measure

$\lambda_s S(t)$ is the risk premium for $S(t)$

Let $\rho_{r,s} \equiv \mathrm{Cov}_t(\tilde{\omega}_r,\tilde{\omega}_s)$ and $\mathrm{Cov}_t(\tilde{\omega}_s,\tilde{\omega}_z) = 0$.

Consider the price of a unit face defaultable discount bond with τ periods to maturity. Using the risk-neutral dynamics of $r(t)$ and $S_n(t)$ and solving Equation 10.A4, we have:

$$P^\star(t,\tau) = \exp\left[-\alpha(\tau) - \beta(\tau)r(t) - \gamma(\tau)z(t) - \theta(\tau)S(t)\right] \tag{10.A7}$$

where

$$\beta(\tau) \equiv \frac{\Lambda_r[1-\mathrm{e}^{-(\kappa_r+\lambda_r)\tau}]}{\kappa_r+\lambda_r} \tag{10.A8}$$

$$\gamma(\tau) \equiv \frac{\kappa_r \Lambda_r [1 - e^{-(\kappa_z + \lambda_z)\tau}]}{(\kappa_r + \lambda_r)(\kappa_z + \lambda_z)} + \frac{\kappa_r \Lambda_r [e^{-(\kappa_z + \lambda_z)\tau} - e^{-(\kappa_r + \lambda_r)\tau}]}{(\kappa_r + \lambda_r)[(\kappa_z + \lambda_z) - (\kappa_r + \lambda_r)]} \tag{10.A9}$$

$$\theta(\tau) \equiv \frac{\Lambda_s [1 - e^{-(\kappa_s + \lambda_s)\tau}]}{\kappa_s + \lambda_s} \tag{10.A10}$$

and

$$\alpha(\tau) \equiv \Lambda_0 \tau - \frac{1}{2}\sigma_r^2 \int_0^\tau \beta^2(s)\,ds + \kappa_s \mu_s \int_0^\tau \theta(s)\,ds - \frac{1}{2}\sigma_z^2 \int_0^\tau \gamma^2(s)\,ds - \frac{1}{2}\sigma_s^2 \int_0^\tau \theta^2(s)\,ds$$
$$+ \kappa_z \mu_z \int_0^\tau \gamma(s)\,ds - \rho_{r,s}\sigma_r\sigma_s \int_0^\tau \beta(s)\theta(s)\,ds - \rho_{r,z}\sigma_r\sigma_z \int_0^\tau \beta(s)\gamma(s)\,ds \tag{10.A11}$$

We first observe that $P^\star(t, \tau)$ is exponential affine in three state variables: the short interest rate, the stochastic long-run mean interest rate, and the firm-specific distress factor. The bond price is driven by 18 structural parameters: there are 3 parameters in the defaultable discount rate specification, 10 in the interest rate process, and 5 associated with the dynamics of firm-specific distress. Second, under the positivity of Λ_r, the defaultable discount bond price is negatively related to $r(t)$ and $z(t)$:

$$\Delta_r^\star(t,\tau) \equiv \frac{\partial P^\star(t,\tau)}{\partial r} = -\beta(\tau)P^\star(t,\tau) < 0 \tag{10.A12}$$

$$\Delta_z^\star(t,\tau) \equiv \frac{\partial P^\star(t,\tau)}{\partial z} = -\gamma(\tau)P^\star(t,\tau) < 0 \tag{10.A13}$$

The bond price is also negatively associated with the distress factor, as seen by

$$\Delta_s^\star(t,\tau) \equiv \frac{\partial P^\star(t,\tau)}{\partial S} = -\theta(\tau)P^\star(t,\tau) < 0 \tag{10.A14}$$

provided $\Lambda_s > 0$. The expressions for the local risk exposures can be employed to develop hedges for marked-to-market risks.

Five candidates for $S_n(t)$ are selected for their empirical plausibility:

1. Assume S_n is the level or the logarithmic of firm-leverage. Leverage is also a key ingredient in the structural models of Merton (1974), Longstaff and Schwartz (1995), and Collin-Dufresne and Goldstein (2001). We will refer to this model as the leverage model.
2. Let S_n be the firm's book value divided by market value (i.e., book-to-market) or its logarithmic transformation. According to Fama and French (1992), firms with high book-to-market are relatively more distressed with poor cash flow prospects. This is the B/M model.
3. Next candidate for S_n is one minus profitability ratio and reflects the firms' ability to honor debt obligations out of its operating income (Titman and Wessels 1988). Credit risk models that incorporate profitability concerns are the profitability models.

4. For the fourth model, S_n is the level or the logarithmic of stock volatility and proxies asset volatility. Based on Merton (1974) this measure of business risk is included in practitioner models of default prediction such as KMV (i.e., credit spreads rise with volatility in Merton). Campbell and Taksler (2003) show that equity volatility explains cross-sectional variation in corporate bond yields. The resulting credit risk model is the stock volatility model.
5. Finally, S_n is distance-to-default model of KMV. It captures the number of standard deviations the firm asset-value is away from the default boundary (Crossby and Bohn 2003). KMV defines distance-to-default as net asset-value normalized by asset-volatility, where the default event is identified as the first time the firm asset-value falls below the default boundary. We call this credit risk model as the distance-to-default model.

Turning to default-free bond modeling, let $z(t)$ represent the long-run mean of the short rate and $X(t) \equiv \binom{r(t)}{z(t)}$. The two-factor model is adopted on empirical and theoretical grounds:

$$\mathrm{d}X(t) = \Pi^{\star}[\Theta^{\star} - X(t)]\mathrm{d}t + \sigma \mathrm{d}\widetilde{W}(t) \tag{10.A15}$$

where $\Pi^{\star} \equiv \begin{pmatrix} \kappa_r + \lambda_r & -\kappa_r \\ 0 & \kappa_z + \lambda_z \end{pmatrix}$, $\Theta^{\star} \equiv \begin{pmatrix} \frac{\kappa_r(\kappa_z\mu_z - \lambda_{0,z})}{(\kappa_r+\lambda_r)(\kappa_z+\lambda_z)} - \frac{\lambda_{0,r}}{(\kappa_r+\lambda_r} \\ \frac{\kappa_z\mu_z - \lambda_{0,z}}{(\kappa_z+\lambda_z)} \end{pmatrix}$, $\sigma \equiv \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_z \end{pmatrix}$, and $\widetilde{W}(t) \equiv (\tilde{\omega}_r, \tilde{\omega}_z)'$ are standard Brownian motions under the risk-neutral measure with correlation $\rho_{r,z}$. $\kappa_r(\kappa_z)$ is the mean-reversion rate for $r(t)$ ($z(t)$) and μ_z is the long-run mean for $z(t)$. The risk-neutral dynamics (Equation 10.15) results by assuming that the risk premium associated with $X(t)$ is $\binom{\lambda_{0,r}}{\lambda_{0,z}} + \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_z \end{pmatrix} X(t)$, and the $X(t)$ dynamics under the physical measure. Solving a standard valuation equation, the price of default-free discount bond is

$$B(t,\tau) \equiv E^{Q}\left\{ \exp\left(- \int_{t}^{t+\tau} r(s)\,\mathrm{d}s \right) \right\} = \exp\left[-\bar{\alpha}(\tau) - \overline{\beta}(\tau) r(t) - \overline{\gamma}(\tau) z(t) \right] \tag{10.A16}$$

where $\bar{\beta}(\tau) \equiv \frac{1 - \exp[-(\kappa_r + \lambda_r)\tau]}{\kappa_r + \lambda_r}$, $\overline{\gamma}(\tau) \equiv \frac{\kappa_r - \kappa_r \exp[-(\kappa_z+\lambda_z)\tau]}{(\kappa_r+\lambda_r)(\kappa_z+\lambda_z)} + \frac{\kappa_r \exp[-(\kappa_z+\lambda_z)\tau] - \kappa_r \exp[-(\kappa_r+\lambda_r)\tau]}{(\kappa_r+\lambda_r)(\kappa_z+\lambda_z-\kappa_r-\lambda_r)}$ and $\overline{\alpha}(\tau) \equiv -\frac{1}{2}\sigma_r^2 \int_0^{\tau} \bar{\beta}^2(u)\,\mathrm{d}u - \frac{1}{2}\sigma_z^2 \int_0^{\tau} \overline{\gamma}^2(u)\,\mathrm{d}u + (\kappa_z\mu_z - \lambda_{0,z}) \int_0^{\tau} \overline{\gamma}(u)\,\mathrm{d}u - \rho_{r,z}\sigma_r\sigma_z \int_0^{\tau} \overline{\beta}(u)\overline{\gamma}(u)\,\mathrm{d}u - \lambda_{0,r} \int_0^{\tau} \overline{\beta}(u)\,\mathrm{d}u$.

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CHAPTER 11

Pricing CDX Credit Default Swaps with CreditGrades and Trinomial Trees

Christian Stewart and Niklas Wagner

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11.1 INTRODUCTION

With the bond markets topping the stock market in total size, debt financing is the number one instrument used in corporate finance. In one of its purest forms, the heavyweight contract in the credit derivatives world is the credit default swap (CDS). It can be seen, very simplistically, as an insurance against the default of a corporation.

The present contribution discusses empirical issues in the pricing of CDSs from the Dow Jones CDX North America Investment Grade (DJ CDX.NA.IG) index universe. We focus on the debt–equity linkage, which is a major driver in explaining credit spreads by structural models. So far, only a few empirical studies focus on CDS pricing (Aunon-Nerin et al. 2002, Ericsson et al. 2004, and Houweling and Vorst 2005).

We perform a CDS pricing model comparison and consider ways of selecting companies where structural models perform above average. Model performance is measured by spread change correlation, Kendall's τ(tau), mean absolute mispricing as well as the standard deviation of price differences. Our approach of model comparison and out-of-sample performance testing is based on a broad data set. It applies identical input factors to both model approaches. We find that the trinomial model outperforms the CreditGrades model on average, while both structural models perform mediocre over our overall selection of companies. Applying the models to names that correlate above average with a market proxy (i.e., have above average R^2 in the sense of Roll 1988) yields vastly improved pricing results. This methodology is new to empirical credit derivatives model comparisons.

The remainder of this contribution is organized as follows. Section 11.2 describes the data set. The models are reviewed and their implementation is discussed in Section 11.3. Section 11.4 contains the empirical pricing results and considers the question of the drivers of model performance. Section 11.5 concludes the chapter.

11.2 CDS DATA

The data used in the empirical part of the paper come from the Open Bloomberg system. As a basis, the DJ CDX.NA.IG universe is used, which consists of 125 U.S. companies where CDSs are actively quoted.* The CDX index universe was used as a starting point to have companies where, on the one hand, credit spreads are traded, and on the other, a satisfying

* The DJ CDX.NA.IG index is recreated every 6 months, because of continuous market changes in the credit risk level of the index companies. Sixteen major international banks define the index and create a market with bid-ask spreads in the low 1/2 to 1/4 bp range (Jakola 2006).

level of liquidity is present.* Once available, stock prices and at-the-money implied stock volatilities are retrieved. In selecting the sample, we considered the length of the available data series as well as the resulting overall number of companies. Finally, the 125 index companies were reduced to a subset of 54 companies, as this was the number of companies where all the necessary data were available.†

The industry allocation by the number of companies in our sample is as follows: 31.5%—Basic Industrials, 27.8%—Consumer Products and Retail, 11.1%—Energy, 20.4%—Financials, and 9.3%—Telecommunication, Media and Technology. From the market capitalization point of view, the picture changes slightly. The total market capitalization of $2.14 trillion as of July 7, 2006 for the 54 companies is split up in the following industries. Basic Industrials are 19.0% or $408 billion, Consumer Products and Retail account for 29.5% or $632 billion, Energy captures 6.2% or $133 billion, Financials with 33.4% or $717 billion are the largest group, and finally Telecommunication, Media and Technology take the remaining 11.8% or $254 billion. The average company market capitalization is $39.7 billion, where the standard deviation is rather high with $57.9 billion. To shed more light on the sample, the top 10% mean is $138 billion market capitalization and the bottom 10% mean is $5.3 billion market capitalization. All currency related data (e.g., stock prices) are quoted in U.S. dollars.

Our sample consists of 540 trading days during the period May 14, 2004 to July 7, 2006. With CDS, stock, and volatility quotes, an overall of 87,480 market data points results. There were days of missing CDS data, although this was rare in the sample. Approximately 0.2% of the data were missing, mostly 1 or 2 days in a row, so data accuracy can be assured and there was no need to apply more advanced fitting methods like curves or splines. The missing data were linearly interpolated. Stock price and volatility data were obtained as complete series.

11.3 MODELS

Our first model follows the Merton (1974) structural approach, which has become a market standard. The simplified implementation that we use is the CreditGrades benchmark model as developed by the RiskMetrics group and published in Finger (2002). Our second model is a binomial pricing approach as commonly seen in equity option analysis, but with the added feature of a third branch representing the default of the company (Das and Sundaram 2000). This trinomial tree approach models the default probability via a three-parameter hazard rate function. In the following, we describe our method of model implementation, where the focus is on the less widespread trinomial tree model.

* We use the 5 year CDS spreads, as those are the ones traded the most frequently. There are 1, 3, 5, and 10 year CDS spreads quoted, although not all are liquid. It is worth noting that the CDS spreads used do not represent actual (i.e., traded) CDS contracts, but they are averaged from available bids and offers from institutional trades. Because of the fact that the CDS market is an OTC (over-the counter) market, the actual transactions are not publicly available. Nevertheless, the CDS data quality is decent and is the only continuous CDS data available in the open market. The data were all reported at trading days only. Weekends and U.S. holidays were left out and were not interpolated. This applies to all data used.

† In fact, there is selection bias in the sample using a specific index and then even reducing the number of companies. Although this chapter does not claim to fully represent the CDS universe, it represents the U.S. investment grade sector and goes well beyond calibration approaches that consider a few names only.

11.3.1 Trinomial Model

The trinomial model is originally based on the work of Das and Sundaram (2000) who formulate a discrete-time approach for valuing credit derivatives in an arbitrage-free setting. To implement the model, a modified version following Bandreddi et al. (2006) is used. Strictly speaking, the model is neither purely structural nor intensity based. The input data show close resemblance to a structural model. This is why calling the trinomial model "structural" refers to the data input and makes it in this sense directly comparable to the CreditGrades model. The trinomial model mainly differs from the CreditGrades model in that it uses an intensity model-type hazard function with three input parameters.

The trinomial model is built along the following steps. First, the stock price is modeled. The starting stock price S_0 is multiplied with up and down factors, which depend on the volatility of the stock as in the Cox et al. (1979) model: $u = e^{\sigma\sqrt{h}}$ and $d = 1/u$. The next step is the calculation of the probability of each branch:

$$S_{t+h} = \left\{ \begin{array}{ccc} uS_t & \text{with probability:} & q(1-\lambda_t) \\ dS_t & \text{with probability:} & (1-q)(1-\lambda_t) \\ 0 & \text{with probability:} & \lambda_t \end{array} \right\} \tag{11.1}$$

The probability that the company defaults in the interval from t to $t+h$ is λ_t, which is specified as

$$\lambda_t = 1 - \exp(-\xi_t h) \tag{11.2}$$

Here, ξ_t is the default intensity and three model parameters, namely α, β, and γ, come into play. The parameters are related to the level, slope, and curvature of the CDS spread curve

$$\xi_t = \exp(\alpha + \gamma t) S_t^{-\beta} \tag{11.3}$$

Since the discounted stock price will follow a martingale under the risk-neutral measure, we have

$$E(S_{t+h}) = e^{rh} S_t \tag{11.4}$$

Using Equation 11.1, it follows

$$q = \frac{e^{(r+\xi)h} - d}{u - d} \tag{11.5}$$

Each nondefaulted endpoint of the tree gets equipped with a coupon of one currency unit which is discounted back to time zero using the risk-free interest rate and the corresponding probabilities. For defaulted branches, the time zero risk-neutral expected losses can be calculated. Finally, the discounted expected losses are divided by the discounted expected coupons to get the CDS spread.

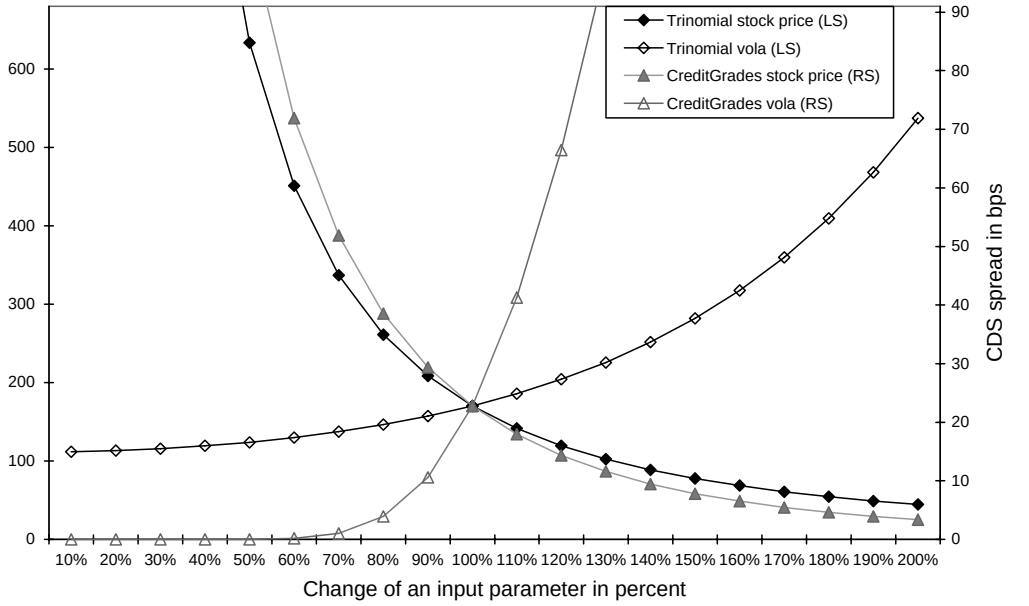


FIGURE 11.1 Parameter sensitivity of CreditGrades and the trinomial tree models.

11.3.2 Model Sensitivity

Understanding model sensitivities helps to judge how a model reacts to input changes. This is true for market-driven as well as for nonobservable model parameters. The following facts are noteworthy: (i) changes in stock price and stock price volatility exhibit a nonlinear effect on the CDS spread, (ii) the risk-free interest rate has a very small, nonlinear effect on the CDS spread, and (iii) the recovery rate exhibits a linear relationship to the CDS spread.

A comparison of the sensitivities of the two models to the stock price and its volatility is illustrated in Figure 11.1. For the stock price, the sensitivities are similar. However, the CreditGrades model shows a higher sensitivity to changes in volatility than the trinomial model.

11.3.3 Model Calibration

Every single company must run through the same model calibration process to ensure that each company is treated identically and that no bias arises. Both models, CreditGrades and the trinomial model, use the same market input data, namely stock prices, at-the-money implied stock price volatility, and the risk-free interest rate. Note that both models use the recovery rate as an input, which is not observable.

The calibration process for both models is conducted over the first 50 trading days of our sample, which is the fitting period as shown in Figure 11.2. In the calibration, we minimize squared differences between the model and the market spread.* An alternative way of fitting would correlate model and market spreads, but a main purpose here is to

* Model calibration uses the Solver Tool in Microsoft Excel as implemented in a macro to generate an automated process. For the optimizations, the following options were set: iterations = 1000, precision = 0.00001%, tolerance = 5, convergence = 0.0001. The estimates were tangent solutions in a Newton search pattern approach. In every case, convergence was given.

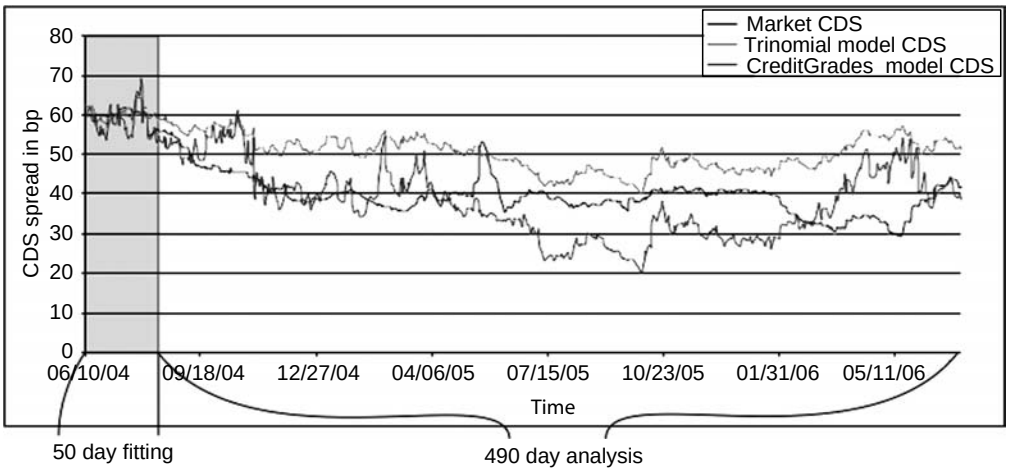


FIGURE 11.2 Pricing process: 50 days model fitting, 490 days out-of-sample analysis.

establish the level of spreads. The length of the calibration period surely has an impact on the quality of the out-of-sample model fit. To see the impact, the fitting period was extended to 100 trading days. Results, unreported here, show a hardly notable, marginal improvement in the overall out-of-sample fit of the models. The calibration process for the trinomial model additionally includes the parameters of the hazard rate function. When the calibration procedure for the models is completed, the values for all parameters are kept constant for any further analysis.

11.4 EMPIRICAL RESULTS

With our empirical results, we want to shed light on the connection of equity input factors to the market pricing of credit risk. We do this over a period of 490 trading days, i.e., roughly 2 years, which is an extent period for measuring out-of-sample model performance. Asking for the models' ability to adjust CDS model prices to new input data, our evaluation focus in Section 11.4.1 is on the correlation between changes in market and model spreads. We additionally consider mean absolute pricing error as well as mean squared pricing error. In the second part, in Section 11.4.2, we then focus on the question of how to explain the vast cross-sectional differences in the models' pricing performance. Finally, Section 11.4.3 discusses the CDS pricing of three example companies as taken from our sample.

11.4.1 Model Performance Evaluation

11.4.1.1 Correlation of Spread Changes

Our main focus in the credit models' evaluation is on spread change correlation between model prices and market quotes. An overview of the correlation results for the 54 companies is given in Table 11.1. Overall, the mean correlation is 0.496 for the trinomial model, and 0.454 for the CreditGrades model, i.e., in a similar range. The results show that there is a wide cross-sectional variability in the results with a range of company

TABLE 11.1 Spread Change Correlation, Trinomial Model and CreditGrades Model versus Market

	Trinomial	CreditGrades
Median	0.599	0.424
Standard deviation	0.337	0.312
Mean	0.496	0.454
Minimum	−0.373	−0.354
Maximum	0.924	0.925

correlations between −0.3 and 0.9. This suggests that there are companies for which the models perform well while at the same time there are companies for which the models do not give reasonable results at all.

When it comes to the distribution of the correlation results, for the trinomial model the median is 0.599, and hence larger than the mean of 0.496 suggesting a skewed distribution. This is not observed for the CreditGrades model results, as median and mean almost match.

When we look at the distribution of model correlation results in Figure 11.3, obvious differences between the two models appear. The CreditGrades model has most (16 out of 54) companies in the correlation range between 0.25 and 0.5. For the trinomial model, most companies (17 out of 54) fall in the correlation range larger than 0.75. One can also see that for the CreditGrades model 24 out of 54 companies fall into the category of 0.5 correlation or above, while for the trinomial model this number is higher with 30 out of 54 companies.

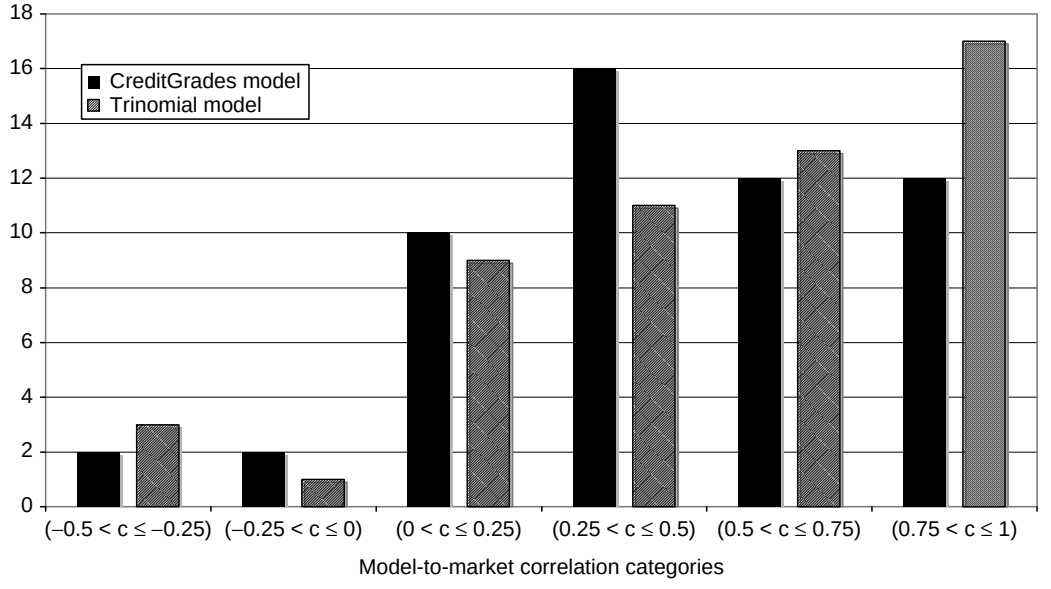


FIGURE 11.3 Spread change correlation distributions.

TABLE 11.2 Kendall's τ for Spread Changes, Trinomial Model and CreditGrades Model versus Market

Kendall's τ	Trinomial	CreditGrades
Median	0.356	0.307
Standard deviation	0.249	0.222
Mean	0.348	0.319
Minimum	−0.246	−0.197
Maximum	0.762	0.702

An alternative measure of the co-movement of the credit-spread changes is given by Kendall's τ (see Kendall 1976). The measure is based on ordinal scaling and helps us to evaluate if the model and market movements of the CDS spreads follow the same direction. Table 11.2 shows the results for both models. The results confirm those from the correlation measures. The trinomial model slightly outperforms the CreditGrades model.

11.4.1.2 Pricing Error Statistics

This section considers the differences between model and market spreads. Two measures of out-of-sample fit are used. We calculate MAPE, which takes absolute mis-pricings into account. We also calculate the standard deviation of the pricing error (SDPE), which accounts for mis-pricing variability only and reacts with higher sensitivity to potential outliers (e.g., a company where a pricing model fails). The results are reported in Table 11.3.

The statistics in Table 11.3 provide us with an understanding of how well the models are able to mimic the market spread over our 490 days analysis period. One can see that the average absolute mis-pricing of the trinomial model is lower than that of the CreditGrades model on average, considering both, the cross-sectional mean and the median measures. Also, pricing errors show lower variability in the case of the trinomial model. The mean overall standard deviation is 0.184 for the trinomial model and 0.325 for the CreditGrades model. This suggests that the trinomial model's deviations are relatively stable.

11.4.2 What Drives Model Performance?

11.4.2.1 Company-Specific Variables

Here we now consider the above mentioned cross-sectional differences in the models' pricing performance. We ask for company-specific variables that coincide with improved or weakened model performance. To clarify this question, we set up a regression analysis to explain the model-to-market correlation for the trinomial and the CreditGrades models.

TABLE 11.3 Mean Absolute Pricing Error (MAPE) and Standard Deviation of the Pricing Error (SDPE), Trinomial Model and CreditGrades Model versus Market

	Trinomial MAPE	Trinomial SDPE	CreditGrades MAPE	CreditGrades SDPE
Median	0.272	0.168	0.352	0.231
Standard deviation	0.147	0.074	0.274	0.341
Mean	0.315	0.184	0.413	0.325

The explanatory variables that we use should affect, in theory, the company's CDS spread. We choose the following six explanatory variables:

1. The "leverage" of the company measured by the debt-to-equity ratio as reported by the end of the second quarter of 2006.
2. The "debt-to-asset" ratio of the company as reported by the end of the second quarter of 2006. This number is closely related to the expected recovery as the assets are used to pay the debtors in the case of liquidation.
3. The "size" of the company as measured by market capitalization as of July 7, 2006. This is used as a proxy to see if larger companies might be less prone to speculative movements in CDS spreads.
4. The company's "average spread level" during the analysis period. This variable addresses the question whether a higher (lower) spread level yields better (worse) model performance.
5. The "rating" of the company as taken from Standard & Poor's (if not available, the implied rating is used). The rating varies from AAA to B and is numbered from 1 to 6 for the regression.
6. Finally, the annualized "stock market performance" of the company over the analysis period is used to see if, for good performing companies, the models work better or not.

The results of the regression analysis are presented in Table 11.4. We first point out the adjusted R^2 values. The explanatory variables explain 7.2% (11.5%) of the variability of the model performance as measured by correlation for the CreditGrades (trinomial) model.

TABLE 11.4 Regression Results for Company-Specific Variables

	Coefficient	**Significance Level**	***T*-Value**
Dependent Variable Correlation: CreditGrades Model			
Constant	0.618	0.017	2.464
Debt-to-equity	−0.133	0.041	−2.102
Debt-to-assets	0.451	0.337	0.970
Market capitalization	0.000	0.869	0.166
Average spread level	−0.002	0.540	−0.618
Rating	0.012	0.865	0.171
Performance (per annum)	0.026	0.863	0.174
Dependent Variable Correlation: Trinomial Model			
Constant	0.644	0.027	2.291
Debt-to-equity	−0.106	0.141	−1.498
Debt-to-assets	0.218	0.678	0.418
Market capitalization	0.000	0.970	0.038
Average spread level	−0.001	0.828	−0.219
Rating	0.009	0.911	0.112
Performance (per annum)	0.057	0.738	0.337

Both regression models unfortunately lack significance. This leads us to the conclusion that our company-specific variables hardly help to explain the models' accuracy in explaining the market CDS spread evolution.

In Table 11.4, the regression constant yields the highest significance for both models. Apart from one case, all variables are insignificant at the 5% level. The exception is the debt-to-equity ratio, which yields significance at the 5% level for the CreditGrades model. The coefficient is negative, indicating that a lower ratio may correspond to a better performance of the model. Interestingly, the average level of spreads or the company ratings, i.e., the implied risk class, shows no significance.

11.4.2.2 Cross-Sectional Correlation Clustering

The above findings demonstrate that the identification of company-specific variables that may yield superior model performance remains an open question. To improve our understanding of what drives credit-spread pricing performance, our goal is to identify homogenous groups, i.e. clusters, within the given CDS data set. To this aim, we propose a method of clustering according to R^2 measures with a market proxy. We thereby follow the idea of Roll (1988) who uses the R^2 measure to examine to what extent individual stock price movements are related to the market.

In the following, the correlation coefficient is used as a measurement tool as, in the simple regression case, the correlation is equal to R. We consider the following question: Is the individual CDS data correlated to each other via a synthetic market index proxy? The idea behind this is to find the relevant drivers of the credit risk of a company.* Note that there is never a clear separation between market risk (price volatility) and credit risk (default) of a company (Duffie and Singleton 2003). Here, we consider the index correlation of credit risk using CDS spread as well as stock price data.

Analyzing the correlation of the CDS data, we start with the correlation against a synthetic CDS index generated from the 54 companies in our data sample. We use three different kinds of indices. The first index equally weights the CDS data and generates an aggregated number for every trading day. The second index version normalizes the CDS data by using the average over the 540 sample period. This should ensure that companies having extremely high spreads do not bias the index too much. The third index version is weighted by the market capitalization of the respective companies. We neglect daily changes in market capitalization and use the fixed weights as of July 7, 2006 (the last day of the sample period) instead.

The mean correlation of the 54 companies to the equal-weighted index is 0.61 with a standard deviation of 0.28. The normalized index has a mean correlation of 0.63 and a standard deviation of 0.32. Finally, the cap-weighted index has a mean correlation of 0.57 and a standard deviation of 0.42. Standing alone, these numbers suggest that the CDS

* Of course, several spread components are reflected in the CDS quotes: (i) the default or credit risk, (ii) counterparty risk, a veritable issue in an over-the-counter (OTC) market, and (iii) the risk premium for taking on default risk. For example, Amato (2005) argues that CDS spreads in his sample include a risk premium of 1.42 bp. Finally, a liquidity premium may be considered. These components are not considered here. This is due to practical reasons and as they do not deliver the major driver of CDS spreads.

spread is not only solely dependent on the company but also on the other companies. This does not contradict structural theory, since the CDS spreads should be related to (correlated) stock prices and asset volatilities.

The second step is now to run the same correlation analysis using the stock prices. If the structural models do not fall short of their promise, the correlation should approximately match the CDS correlation for the models to work. This implies that the amount of index correlation in the CDS spreads should be roughly equal, in level, with the stock price correlation. The correlation of the stock prices using the same three indices as in the CDS correlation analysis results in the following numbers. The mean correlation of the 54 companies to the equal-weighted index is 0.56 with a standard deviation of 0.53. The normalized index has a mean correlation of 0.56 and a standard deviation of 0.53. Finally the cap-weighted index has a mean correlation of 0.55 and a standard deviation of 0.49.

As the CDS correlation data and the stock price correlation data do not result in any major improvement in understanding the valuation of credit risk, combining this information should yield new information. There are manifold clustering options. We propose a simple clustering of the cross-section of CDS names according to the following scheme.

11.4.2.2.1 Names in Correlation Cluster 1 (CC1) Both the correlation of the CDS spread with the index and the correlation of the stock price are positive and furthermore both correlations are above the average within the sample. In such a scenario, a structural model where the CDS spread of a company is based on its stock price and its volatility is reasonable. However, with such a high amount of market correlation, single-name CDS analysis might not be the perfect choice, because of the fact that not only the company-specific risk seems to be part of the CDS spread but also the overall market risk level.

11.4.2.2.2 Names in Correlation Cluster 2 (CC2) The correlation of the stock price is above the sample average, but the correlation of the CDS spread is not. In this scenario, the implications are more straightforward. When the stock price shows above average index correlation, but the CDS spread shows below average correlation with its peer group, then a structural model is quite difficult to imagine. The stock price movement can be based on a different market expectation than the CDS spread on the one side, or the price discoveries in the different markets move at different speeds. This means that, e.g., the issue of a new bond might not have an immediate effect on the stock price, while the additional debt burden makes the CDS spread move immediately.

11.4.2.2.3 Names in Correlation Cluster 3 (CC3) The CDS spread moves with the peer group while the stock does not, which may be the case if the CDS spread moves with the overall risk in the market and the stock price trades on different information. At first sight this scenario looks quite similar to CC2. Generally, the stock price and the CDS spread are decoupled, but in a different way. This can, e.g., happen when the stock price is increasing beyond high levels and the credit spread is already on a very low level and does not reach zero because of embedded various additional risks. For example, the CDS spread

TABLE 11.5 Distribution of Correlation Clusters CC1 to CC4 within the Credit Default Swap (CDS) Data Sample

	Equal-Weighted (%)	Normalized (%)	Cap-Weighted (%)
CC1	54	52	48
CC2	20	22	24
CC3	13	13	15
CC4	13	13	13

may approach the one-digit range and then correlates with the market (as the stock price does not affect the spread any more).

11.4.2.2.4 Names in Correlation Cluster 4 (CC4) Both the correlation of the CDS spread with the index as well as the stock price with the index are below the average index correlation. Here, structural credit risk valuation seems neither supported as in the CC1 case nor questionable as in the CC2 and CC3 cases. Hence, model performance prediction in this case is vague.

Table 11.5 shows the distribution of the four clusters in our sample. Roughly 50% of our 54 companies fall into the first cluster, CC1. CC2 is the second most probable classification and the remainder clusters account for less than 30% of the companies.

The pricing results for the correlation clusters CC1 to CC4 based on spread change correlations are given in Table 11.6 for the CreditGrades model and in Table 11.7 for the trinomial tree approach. The results show that pricing performance vastly increases to

TABLE 11.6 Spread Change Correlation According to Correlation Clusters CC1 to CC4, CreditGrades Model versus Market

CreditGrades	**CC1**	**CC2**	**CC3**	**CC4**
Equal-Weighted Index				
Mean	0.640	0.189	0.280	0.300
Standard deviation	0.217	0.328	0.244	0.212
Median	0.720	0.236	0.280	0.329
# (out of 54)	29	11	7	7
Normalized Index				
Mean	0.660	0.178	0.280	0.300
Standard deviation	0.191	0.313	0.244	0.212
Median	0.722	0.202	0.280	0.329
# (out of 54)	28	12	7	7
Cap-Weighted Index				
Mean	0.686	0.165	0.325	0.300
Standard deviation	0.183	0.259	0.259	0.212
Median	0.753	0.223	0.338	0.329
# (out of 54)	26	13	8	7

TABLE 11.7 Spread Change Correlation According to Correlation Clusters CC1 to CC4, Trinomial Model versus Market

Trinomial	**CC1**	**CC2**	**CC3**	**CC4**
Equal-Weighted Index				
Mean	0.647	0.221	0.245	0.544
Standard deviation	0.234	0.353	0.384	0.272
Median	0.719	0.239	0.311	0.651
# (out of 54)	29	11	7	7
Normalized Index				
Mean	0.684	0.171	0.245	0.544
Standard deviation	0.181	0.321	0.384	0.272
Median	0.736	0.192	0.311	0.651
# (out of 54)	28	12	7	7
Cap-Weighted Index				
Mean	0.717	0.147	0.295	0.544
Standard deviation	0.160	0.252	0.383	0.272
Median	0.757	0.205	0.338	0.651
# (out of 54)	26	13	8	7

above 70% average correlation for the cluster CC1, where both models now show similar performance. This finding indicates that, in cases with above average index correlation on the stock and on the CDS sides, the structural pricing approach is most appropriate.

As expected from our discussion above, both structural models indeed show weak pricing performance for the clusters CC2 and CC3. In cluster CC2, the mean pricing correlation is at a low level ranging from 15% to 22%. In cluster CC3, the results are somewhat better. Interestingly, Tables 11.6 and 11.7 show that the trinomial model performs better than the CreditGrades model particularly in cluster CC4, which may explain its overall superior performance as documented in Table 11.2. We may argue that the additional degrees of freedom in model calibration as given by the parameters of the hazard rate are especially advantageous in a setting where index correlation is weak.

11.4.3 CDS Pricing Examples

This section discusses the CDS pricing of three examples as taken from our overall sample of 54 companies. We consider these examples in the context of the correlation clustering results of Section 11.4.2 and provide a graphical inspection. The examples underline the practicability of our correlation clustering approach, which helps to preselect companies for which structural pricing models may prove to be suitable.

The first example is taken from the first correlation cluster, CC1, where the CDS and the stock price follow the synthetic index well. The company is Halliburton (HAL), a member of the energy sector with about $36 billion in market capitalization. From the given discussion, this indicates that the models' performance should typically be good. The results are a

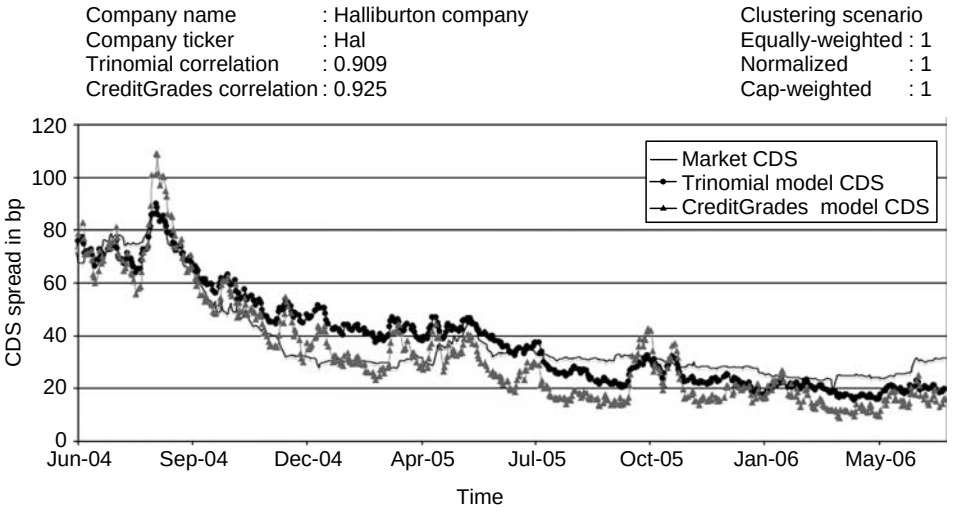


FIGURE 11.4 Halliburton: Market, trinomial, and CreditGrades spreads.

correlation of 0.909 and a Kendall's τ of 0.673 for the trinomial model and a correlation of 0.925 and a Kendall's τ of 0.623 for the CreditGrades model. Figure 11.4 shows that the model spread follows the market spread rather well, where the latter declines from around 80 bp to as low as 20 bp during the sample period.

The second example is a company from the second cluster, CC2. The company is Autozone Inc. (AZO), a member of the retail sector (about $6 billion in market capitalization). From the given discussion, model performance should be weak. The results for the

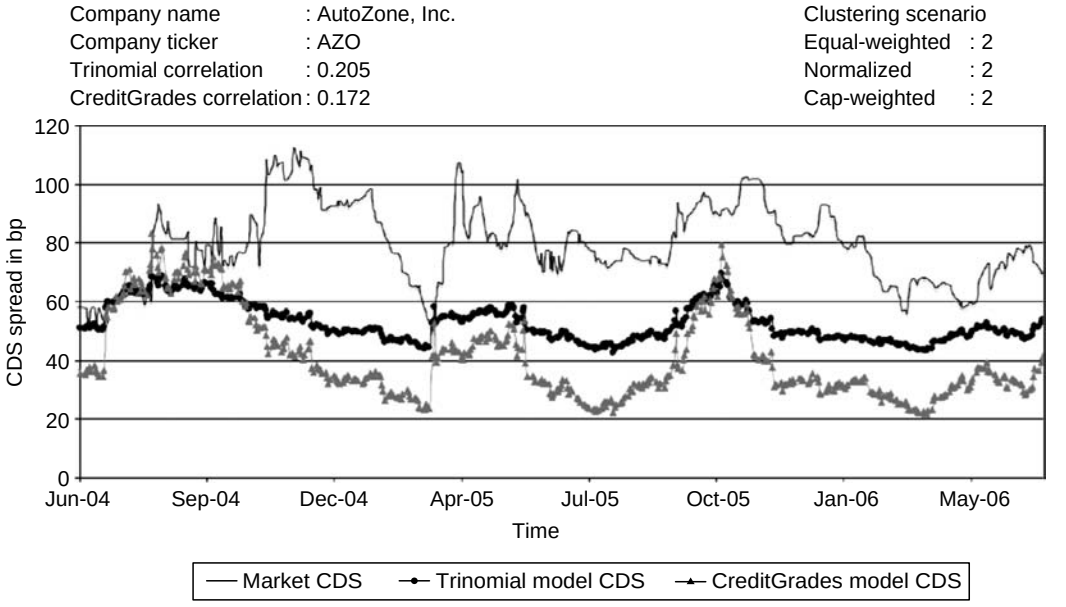


FIGURE 11.5 Autozone: Market, trinomial, and CreditGrades spreads.

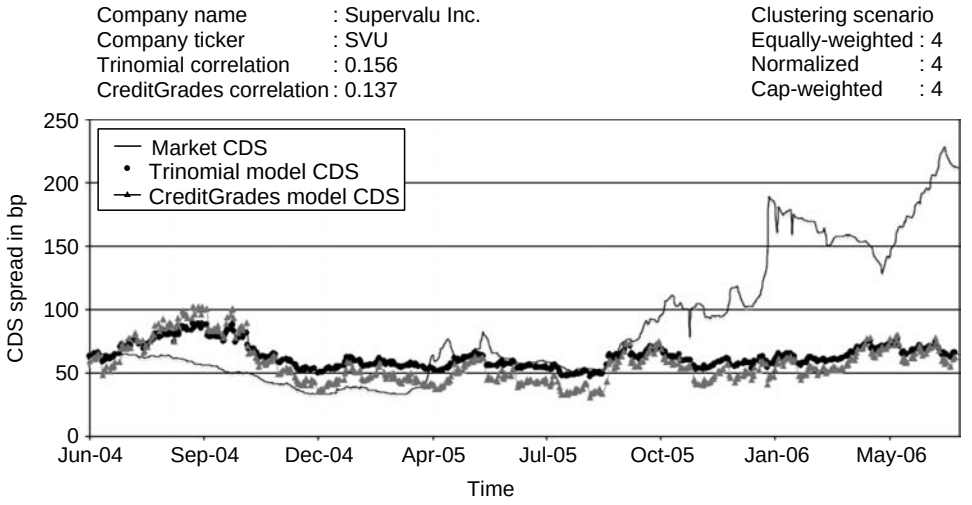


FIGURE 11.6 Supervalue: Market, trinomial, and CreditGrades spreads.

trinomial model (correlation: 0.205, Kendall's τ: 0.194) and for the CreditGrades model (correlation: 0.172, Kendall's τ: 0.195) are indeed disappointing. Figure 11.5 shows that models and market spreads do not exhibit synchronous moves, sometimes not even movement in the same direction is given.

The third and final example is part of the fourth cluster, CC4, where the CDS and the stock price do not follow the synthetic index. The company chosen is Supervalue Inc. (SVU), a company from the retail sector with about $6 billion in market capitalization. From the given results, we would predict average model performance. The results for the trinomial model (correlation: 0.156, Kendall's τ: 0.209) and for the CreditGrades model (correlation: 0.137, Kendall's τ: 0.200) are indeed mixed. Figure 11.6 illustrates that the model spread does not closely follow the market in the second-half of the out-of-sample period. The market CDS spread increases to over 200 bp, but the model CDS stays below 100 bp not capturing information obviously available to the market.

11.5 CONCLUSION

When compared to models based on historical default probabilities, structural credit risk models are in the unique position to use current market data. The possibility to calculate credit risk measures from equity market data—namely, stock prices and volatilities—is an opportunity for credit analysis, which should not be missed.

Unfortunately, we find that the degree to which two prominent model candidates are able to explain the given market spread is fairly limited. Admittedly, one could make a less strict judgment by reducing the length of our 2 year out-of-sample period and allow for a readjustment of the model parameters. Nevertheless, we go one step further in addressing the question of what are the drivers behind our observation that structural credit risk models in some cases work well and in others do not. Our main observation is that when companies' stock prices and CDS prices correlate with the overall CDS market,

the resulting model spreads for such companies confirm better to market spreads. This idea is puzzling to some extent as one would assume that an assessment of credit risk is mainly related to the performance and the financial standing of a single company, rather than to the correlation with the overall CDS market. Our findings would suggest that structural model spreads fit better to market spreads given that less company-specific credit risk is involved.

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CHAPTER 12

Pricing CDX Credit Default Swaps Using the Hull–White Model

Bastian Hofberger and Niklas Wagner

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12.1 INTRODUCTION

The market for credit default swaps (CDSs) shows an enormous growth as documented by the International Swaps and Derivatives Association (ISDA) in mid-2006. This is in line with the exponential growth of the market since the early years of this century. Taking a closer look on the structure of the various products, we see that two-third of the notional amount outstanding are products on single-name entities, while the remainder are multi-name instruments. Most single-name CDSs are traded on investment grade companies (84%), with a relatively large proportion (23%) referenced to sovereigns. The average rating for the underlying firms is BBB. The dominant maturity is 5 years, covering 80% of the market. The growth in the credit derivatives market since the year 2000 makes obvious that researchers like Hull and White (2000) not even had the chance to perform a reliable testing of their timely theory. Probably due to a lack in sufficient data, a broad empirical testing of CDS pricing models has not yet been conducted in the literature. Houweling and Vorst (2005) represent one of the largest studies of a reduced form model so far, with about 23,000 CDS quotes used, but the study includes CDS data only up to January 2001, when the CDS market was not yet fully developed. The last few years with increased bond market liquidity and a well-developed CDS market now provide sufficient time-series data for a comprehensive empirical study of CDS pricing models.

To our knowledge, this contribution is the first to study the Hull and White (2000) model's empirical performance based on a broad sample of liquid CDS-spread quotes. The model's attractiveness as a reduced form approach is that it can be implemented based on observable market data. We use both the standard and the approximate Hull–White pricing approach. The standard intensity-based no-arbitrage approach requires data from at least three corresponding liquid corporate bonds. The approximate no-arbitrage approach requires data from two corresponding liquid corporate bonds. Using an overall of 63,460 quotes during the period January 1, 2002 to July 7, 2006, we study 47 names in the standard pricing approach and 64 names in the approximate pricing approach, all from the Dow Jones CDX North America Investment Grade (DJ CDX.NA.IG) index universe. We evaluate the pricing performance using spread change correlations and squared relative pricing errors. Given that liquid bond data with appropriate maturities are available for the respective companies, both approaches provide satisfying result as measured by spread change correlations and squared relative pricing errors. Confirming previous results by Houweling and Vorst (2005), we find that swap rather than Treasury rates should be used as a proxy for the risk-free rate. Tests on cointegration of Hull–White model and market spreads show that a stable pricing relationship is present for about 75% of our sample companies, while the model faces seriously weak performance for the remainder cases.

The literature on CDS valuation can be divided into three major groups: Theory and models that started with the implementation of structural models, the later developed intensity-based models, and, finally, empirical evidence including commercial implementations of credit pricing models. The family of intensity-based models, also known as reduced form models, assumes that default is an unpredictable event, which is driven by some latent default intensity process. Well-known representatives are, for example, the

Jarrow and Turnbull (1995), the Duffie and Singleton (1999), and the Hull and White (2000) model, denoted as "Hull–White" hereafter. Intensity-based models show several attractive features. A great benefit lies in their flexibility and mathematical tractability. Also, Jarrow and Protter (2004) point out that while structural models assume complete information, they are in fact suffering a lack of information concerning default points and expected recovery. Furthermore, a modeler who investigates CDS spreads is unlikely to have superior information with respect to the market, so again reduced form models seem more appropriate. However, flexibility, which may be an advantage in that a model can be well fitted to some given data sample, may also prove to be a weakness. Many papers demonstrate that their approach yields decent pricing results, using one or a few sample companies. However, in-sample flexibility for some examples, of course, does not predict how a model will perform out-of-sample and for a broader cross-section of cases.

The rest of the contribution is as follows. Section 12.2 describes our data set. In Section 12.3, we sketch important ingredients and results of the Hull–White model. Section 12.4 contains our empirical pricing results for both Hull–White valuation approaches. Section 12.5 concludes the chapter.

12.2 CREDIT DEFAULT SWAP DATA SET

We pick suitable CDS names from the 125 companies universe of the sixth revision of the Dow Jones CDX North America Investment Grade Index. This provides us with a broad coverage of the U.S. market, with mostly liquid CDSs. For our sample, we downloaded CDS quotes from Open Bloomberg. The remainder data, including bond, treasury, and swap data, are from Thomson Financial Datastream.

Model implementation requires market data for at least two actively traded bonds for each obligor name. This reduces our sample to 47 names for the standard Hull–White valuation approach and to 64 names for the approximate valuation approach. After this reduction, there are still 63,460 quotes of CDSs remaining, namely 31,860 quotes for the standard and 41,576 quotes for the approximate valuation method. The data cover the period from January 1, 2002 to July 7, 2006. Given this, the study is one of the broadest empirical studies on the applicability of intensity-based models for the pricing of CDS spreads.

Our data represent U.S. investment grade issuers. The sectors Consumer Products and Retail together with Basic Industrials represent somewhat more than 55% of the market capitalization of our sample. These sectors are followed by Financials (21%), Technology, Media & Telecommunication (15%), and finally Energy (8%). This is quite in line with the overall composition of the CDX-index and demonstrates that the choice of our sample does not yield a substantial bias with respect to industry composition. Concerning the rating classes, most of our sample obligors fall in the A and BBB rating category, which is in line with the rating of the majority of the indexed companies. In detail, 30% of our obligors have an A rating and nearly 65% have a BBB rating. The 5% remainder is rated AAA or AA, while lower rating classes are negligible (as they are removed half-annually due to the fact that the index is based on investment grade companies only). Table 12.1

TABLE 12.1 Credit Default Swap Quotes' Rating Categories during the Period January 1, 2002 to July 7, 2006

Rating	2002	2003	2004	2005	2006	Avg.
AAA	208	261	262	260	135	1,126
AA	192	522	524	520	270	2,028
A	2,304	4,452	4,978	4,940	2,565	19,239
BBB	3,972	9,196	10,914	11,180	5,805	41,067
	6,676	14,431	16,678	16,900	8,775	63,460

shows the CDS quotes' rating categories in the overall sample as well as by sample year. We see that the AA-rating class somewhat loses in share while the other rating classes remain pretty stable. Quote availability raises remarkably from 2002 to 2003. Note that a lower number of quotes in the year 2006 result due to the fact that we cover less than half of that year within our sample.

Since our sample commences with relatively recent data, starting as late as January 1, 2002, the problem of missing CDS-quotes was rather negligible. Less than 0.2% of the data were missing, normally not more than 1 or 2 days in a row, so we could use linear interpolation to close these minor gaps. Weekends and holidays are not contained in the sample as only trading days are reported. Note that the CDS data represent averages between quoted bid and ask prices since actual transaction is not publicly available.* For the default-free interest rates, we choose treasury rates as well as swap rates as provided by Thomson Financial Datastream. The data were complete and hence no interpolation was needed. The same holds for the corporate bond data. We choose bonds with complete time series within our sample period, neither with any data missing nor any obvious data errors appearing.

12.3 HULL–WHITE CREDIT DEFAULT SWAP VALUATION

For the valuation of CDSs, we will use the Hull–White model. This includes the Hull–White standard intensity no-arbitrage pricing approach as well as their approximate no-arbitrage pricing approach. Skipping most of the derivation, we give a brief review of both valuation approaches.

12.3.1 Hull–White Intensity Model Valuation

In the Hull–White intensity model, the set of assumptions contains that default events, treasury rates, and recovery rates are all mutually independent and that the claim made in the event of a default is the face value plus the accrued interest. The notional principal is one currency unit. The maturity of the CDS is T, and $q(t)$ describes the risk-neutral probability of default at any given point of time t as determined in time zero. Subsequently, π describes the risk-neutral probability of no credit event occurring between time 0 and

* For some additional details on the data, see Stewart and Wagner (2008) in Chapter 11 of this volume who start from the same initial CDS universe.

time T. The expected recovery rate in a risk-neutral world $\hat{R}$ is independent of the default time and is the same for all bonds of an issuer. The present value of payments at the rate of one currency unit per year on payment dates between time 0 and time t is $u(t)$. The term $e(t)$ represents the present value of an accrual payment with a payment date immediately preceding time t. The present value of one currency unit received at time t will be $v(t)$. The total payments per year made by a CDS buyer (i.e., the insurance buyer) are denoted as w. The payments last until maturity T or a credit event, whichever occurs sooner. Finally, $A(t)$ is the time t accrued interest on the reference obligation.

Collecting all payment streams that a CDS will expectedly provide, the present value of the expected payoffs minus the present value of the expected payments is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)u(t)\mathrm{d}t - w\int_0^T q(t)[u(t) + e(t)]\mathrm{d}t - w\pi u(T)$$

The fair spread of the CDS, s, is the value of w that makes this expression equal to zero, i.e.,

$$s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)\mathrm{d}t}{\int_0^T q(t)[u(t) + e(t)]\mathrm{d}t + \pi u(T)} \tag{12.1}$$

The spread s represents the total sum of payments per year for a newly issued CDS as a relative of the swap's notional principal.

12.3.2 Hull–White Approximate Valuation

Put very simple, a portfolio consisting of a CDS and an underlying bond of the same obligor both having maturity T should be risk-free. It should, therefore, have the same payoff as a treasury bond with maturity T. Assuming for simplicity that the treasury curve is flat and that interest rates are constant, a simple no-arbitrage argument then results in the spread

$$s^\star = \text{maturity-}T \text{ corporate bond yield—maturity-}T \text{ treasury bond yield,}$$

which will typically overestimate the true spread s (Hull and White 2000). However, we are able to close most of the gap between $s^\star$ and s. Let $A^\star(t)$ represent the time-t accrued interest as a percentage of the face value on a T-year par yield bond that is issued at time zero by the reference entity with the same payment dates as the swap. We may refer to this bond as the underlying par yield corporate bond. As an approximation, use $a(t)$ as the average value for $A(t)$ under the integral in Equation 12.1 and analogously define $a^\star(t)$ as the average value for $A^\star(t)$, $0 < t < T$. This yields an approximate formula for s, where

$$s = \frac{s^\star(1 - \hat{R} - a\hat{R})}{(1 - \hat{R})(1 + a^\star)} \tag{12.2}$$

12.4 EMPIRICAL RESULTS

This section summarizes our empirical results from an analysis of the data set previously described in Section 12.2. We calculate the theoretical CDS spread for both Hull–White pricing approaches as given in Section 12.3 and then evaluate the results from various perspectives.

In their paper, Hull and White come to the conclusion that the approximate valuation (Equation 12.2) ends up with results only differing marginally from the intensity model spread (Equation 12.1). This would constitute a quick and simple method for the applied valuation of CDSs. Hence, we address both, respective model performance as compared to the market as well as relative pricing performance of the two Hull–White approaches. Additionally, we consider the swap rate as a proxy for the risk-free interest rate as an alternative to the treasury rate, which may be a bad proxy due to tax and other reasons (Elton et al. 2001). We start our exposition with the approximate approach, then turn to the standard intensity-based approach, and finally discuss some model sensitivity examples.

12.4.1 Approximate Valuation Results

We start with the calculation of CDS spreads with the Hull–White approximate valuation first using the treasury rate as a proxy for the risk-free interest rate. For our calculations, we need the 5 year CDS spread and, in theory, a 5 year corporate bond. As this is rather unlikely to find, we decided to interpolate and hence require at least one bond maturing in between 1 and 5 years and one bond maturing in 5–10 years. We decline shorter maturities as we find prices obviously not to reflect usual market spread. Also, we decline bonds that would fit the maturity range, but obviously show not to be liquid as their prices are subject to infrequent and large daily price jumps. This results in a data sample of 64 companies and 41,567 quotes.

We find that using the treasury rate misestimates the CDS spread while using the swap rate gives better results. We calculate mean squared relative pricing errors (MSRPE) via deriving model versus market spreads for each quote and setting them relative to the observed market spread before taking squares. We sum up the MSRPE results for the whole sample in Table 12.2. It becomes obvious from the relatively large MSRPE statistic of 0.874 versus 0.273 that using the treasury rate yields inferior, i.e., biased, CDS spreads.*

In a further investigation on the performance of the models, we consider whether model and market spreads tend to move together and if they do, to what extent. To this aim, the average correlation between model and market spread changes is reported in Table 12.2 as well. We see a slight but obvious advantage of the performance again under the swap rate.

Table 12.2 allows us to have a closer look at the cross-sectional variation of the spread change correlations and MSRPE statistics in our sample. It turns out that the correlation statistics end up in a large corridor between zero or even negative correlation and a nearly perfect correlation of more than 97%. The same results for the spread of the squared differences. The overall outcome with a mean correlation of nearly 60% and median in

* Unreported results show that this holds throughout all industry sectors where the treasury rate performance is between 2.3 and 5.1 times worse.

TABLE 12.2 Correlation and MSRPE Statistics for Hull–White Approximate Valuation Based on Treasury and Swap Rates

	Treasury	Swap
Correlation		
Median	0.549	0.632
Mean	0.526	0.586
Standard deviation	0.325	0.294
Minimum	−0.236	−0.062
Maximum	0.973	0.970
MSRPE		
Median	0.797	0.232
Mean	0.874	0.273
Standard deviation	0.367	0.132
Minimum	0.135	0.108
Maximum	1.878	0.706

MSRPE of 23.2% (i.e., a root-MSRPE of 48.2%) is not an excellent, but a satisfying result for the simple model (under the swap rate). Note however that the numbers have to be seen in the context of declining spreads and a sample that is completely in the investment grade sector. On July 7, 2006, 60% of our sample obligors end up with a market CDS spread of less than 40 bp, where small absolute spread changes, which may be related to market issues other than credit risk, imply large relative changes.

12.4.2 Intensity Model Valuation Results

We now implement the Hull–White intensity-based model and calculate theoretical CDS spreads. Since we discussed above that the swap rate is the appropriate proxy for the risk-free rate, we perform all calculations using only the swap rate. As it is not possible to observe the default intensity $q(t)$ of Section 3.1, we use discrete data of all available bonds and interpolate in-between. Therefore, we need liquid bonds to get an appropriate estimate for the yield and default intensity curves. The minimum number is three bonds. Again, we require at least one bond with maturity less than 5 years and one bond with maturity beyond 5 years. We always include all outstanding bonds of the respective obligor. The beneficial effect is that we avoid bias by not choosing only some bonds, while, as an adverse effect, we may also include bonds, which more or less obviously lack liquidity. As a result, 47 companies providing 31,860 quotes are included in our sample. We point out that 46 of the 47 companies were contained in our study of the approximate Hull–White approach. Hence, our samples largely overlap and we now basically study a subset of the previous names.

We start by taking a look at the correlation coefficient between our model spreads and the market spread changes. The results are presented in Table 12.3. Once the results are put in contrast to our results from the approximate valuation above, it turns out that the intensity model valuation, in fact, yields higher average spread change correlation results (65.6%) than the simple approximate valuation approach (58.7%). A median correlation of

TABLE 12.3 Correlation and MSRPE Statistics for Hull–White Approximate and Intensity-Based Valuation Using Swap Rates

	Approximate	Intensity
Correlation		
Median	0.632	0.712
Mean	0.587	0.656
Standard deviation	0.294	0.219
Minimum	−0.062	0.114
Maximum	0.970	0.948
MSRPE		
Median	0.232	0.372
Mean	0.273	0.438
Standard deviation	0.132	0.212
Minimum	0.108	0.167
Maximum	0.706	1.014

71.2% is a highly satisfactory result with a standard deviation, which is remarkably smaller (21.9% versus 29.4%).*

However, the situation changes once we consider the squared relative pricing errors in Table 12.3. In this case, a mean error of 27.3% for the approximate valuation approach clearly dominates a mis-pricing measure of 43.8% for the intensity approach. Also, the results on cross-sectional pricing stability here do not seem to support the intensity approach. As it occurs, the intensity-based approach seems to be more prone to pricing bias than the approximate approach.

Given the above results, we may conclude that there is no clear answer to the question whether intensity valuation dominates approximate valuation or vice versa. Still we have to point out that, to some extend, our results may relate to sample selection bias. With the approximate pricing, we choose the most liquid and perfect-fitting bonds only, while with intensity pricing we always use the whole set of available bonds to derive the default intensity curve. It could be that pricing bias in the intensity approach is related to the use of mis-priced low liquidity bonds. To investigate this issue, we next consider our pricing statistics for different sectors and rating classes in Tables 12.4 and 12.5.

In Table 12.4, the spread correlation results clearly favor the Basic Industrials and the Technology, Media & Telecommunication sectors which, due to intensity modeling, show absolute improvements in correlation of 15.4% and 9.2%, respectively. We point out that that these sector companies often have a wide range of bonds outstanding, giving us plenty of data for calculating default density curves. On the rating side, lower ratings improve

* Looking at single cross-sectional spread change correlations, we observe that in some cases one model approach obviously fails while the other provides excellent results. Apparently, this was a matter of one bond being seriously mis-priced due to liquidity problems or other factors we could not monitor.

TABLE 12.4 Correlation Statistics for Hull–White Approximate and Intensity-Based Valuation According to Sectors and Ratings

	Approximate	Intensity
Correlation		
Consumer Products	0.575	0.580
Energy	0.698	0.636
Financials	0.544	0.606
Basic Industrials	0.563	0.717
Technology	0.653	0.744
AAA	0.847	0.533
AA	0.550	0.576
A	0.477	0.613
BBB	0.627	0.676
Mean	0.526	0.656

their performance under intensity modeling especially with the BBB-rating now reaching 67.6% spread change correlation.*

The MSRPE results of Table 12.5 confirm the bias issue with the intensity model approach. For the sectors, best performance now is in Consumer Products and Retail as well as in Energy sectors. Basic Industrials and Technology, Media & Telecommunication sectors now perform mediocre when compared to the approximate pricing approach. Hence, this supports the hypothesis that a large number of available bonds may be to improve spread correlation while it does not reduce pricing bias. In Table 12.5, lower-rated bonds appear to be the best performers for intensity valuation.

TABLE 12.5 MSRPE Statistics for Hull–White Approximate and Intensity-Based Valuation According to Sectors and Ratings

	Approximate	Intensity
MSRPE		
Consumer Products	0.258	0.369
Energy	0.180	0.368
Financials	0.274	0.489
Basic Industrials	0.266	0.462
Technology	0.310	0.484
AAA	0.528	1.014
AA	0.195	0.763
A	0.337	0.571
BBB	0.230	0.368
Mean	0.273	0.438

* The decline in the AAA category has to be considered as random due to the fact that only one company is contained in that subsample.

12.4.3 Spread Cointegration: Are There Stable Pricing Relationships?

The results of Sections 4.1 and 4.2 show relatively satisfying pricing results based on average spread change correlations and squared pricing errors. We now examine cointegration in market versus Hull–White approximate model spreads. In case we find cointegration being present, this confirms a stable pricing relationship and supports a potential use of the model for trading and arbitrage strategies. Differences between model spread and market spread then have a stationary distribution during time, including constant mean and variance.

We test the null hypothesis that there is cointegration including a constant term on the 10%, 5%, and 1% signification level. The estimation is done using the classical Dickey–Fuller test for a unit-root on the differences between market and model spread. Once we run the test over the 64 obligors sample, we find that for 75% of our obligors we can confirm the existence of cointegration at the 95% or even 99% confidence level.* This result supports the general validity of the pricing model. However, it also points out that the Hull–White approximate model (due to various disturbing factors as given during our sample period) fundamentally fails for the remainder 25% of our 64 obligors.

We observe that spread cointegration is supported especially for companies providing sufficient bond data. In such individual cases, trading strategies could be derived given that the market spread deviates from the model spread. The spreads of AutoZone Inc. in Figure 12.1 may serve as an example, where price deviations appear stable during time and occasionally vanish over the longer run.

12.4.4 Sensitivities for Credit Default Swap Valuation Examples

In this section, we briefly present three companies that turned out to provide a decent model fit with a spread change correlation of more than 90%. We consider high, medium, and low CDS market spread levels and chose Walt Disney Co., Clear Channel Inc., and

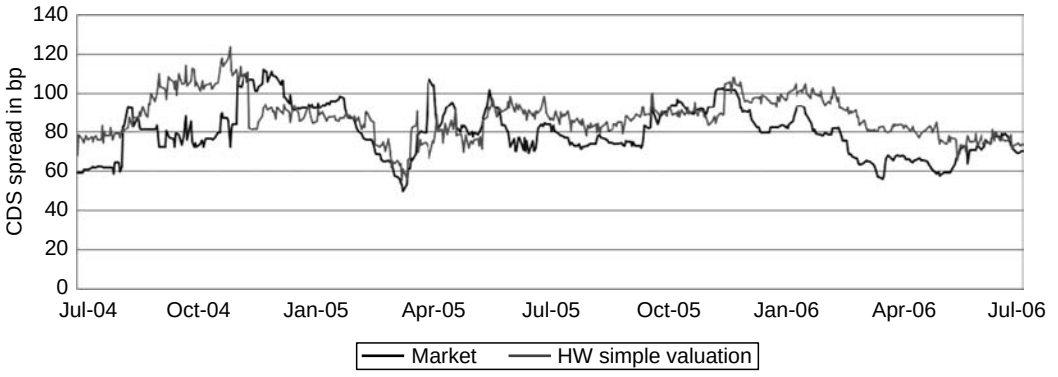


FIGURE 12.1 AutoZone Inc., Market and Hull–White approximate CDS spreads.

* These results go in hand with the previous results. Once we find individual spread change correlations reaching levels of 80% or more, we can in most cases confirm cointegration at least at the 5% level.

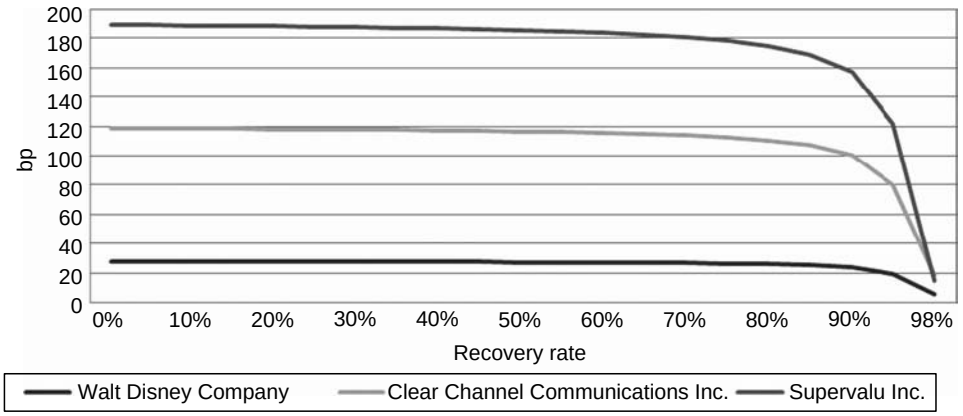


FIGURE 12.2 Sensitivity of the model spread with respect to the recovery rate: three examples.

Supervalu Inc. as example companies. On September 28, 2006, these had CDS spreads of 27.7, 116.3, and 185.3 bp, respectively.

We perform an analysis of the sensitivity of the Hull–White approximate model with respect to the main input-parameters, namely the recovery rate (Figure 12.2) and the risk-free interest rate (Figure 12.3). In Figure 12.2, the recovery rate is shown to be of subordinated importance, at least as long as it does not exceed values of 65% to 70%, a scenario, which normally seems rather unlikely. Inside this range, assuming recovery rates from 0% to 60%, the impact on the estimated CDS spreads is in the range of 1 to 5 bp, i.e., less than 5% of the respective absolute CDS spreads. In contrast to that, the level of the risk-free interest rate has a remarkable influence on the calculated CDS spreads in Figure 12.3. An interest rate reduction of 50 bp results in a spread increase of around 48 bp for Clear Channel Inc. and Supervalu Inc. This result is intuitive, since—ceteris baribus—CDS model spreads are proportional to bond minus risk-free rates. As can also be seen from Figure 12.3, a lower barrier is present in that the interest rate sensitivity shrinks to zero as the spread approaches zero.

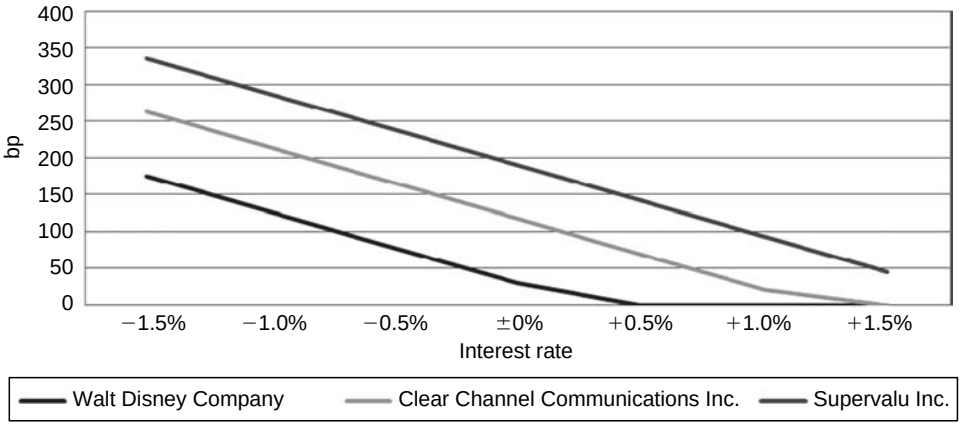


FIGURE 12.3 Sensitivity of the model spread with respect to the risk-free rate: three examples.

12.5 CONCLUSION

Although the calibration of an intensity model may provide brilliant results for several specific obligors, the present study shows that CDS pricing faces problems when applied to a broader cross-section of names. These problems start with liquidity issues and the availability of input variables and continue with a large cross-sectional dispersion of the pricing performance of a given model. We suggest that an important factor, which determines Hull–White pricing performance, is first of all the input data, while other factors such as sector specification and rating class may also play a role. This could be addressed in future research.

Despite these shortcomings, we stress that even the approximate Hull–White valuation approach can indeed produce satisfactory results. This is a remarkable result as pricing is achieved without any in-sample fitting. Especially in the BBB rating class, the approximate approach competes very well to the intensity approach, which requires a larger number of input data. The latter approach may achieve superior pricing results especially for the high-quality AAA to A rating classes. All in all, both Hull–White approaches represent powerful pricing tools, ready to calculate fair CDS spreads.

Given partly overlapping CDX samples, we may compare the results of this study to those of Stewart and Wagner (2008) in Chapter 11. Using percentage median spread change correlations as the aggregate pricing performance measure indicates that a tentative performance ranking would imply the following order: Hull–White intensity (71%) $\geq$ Hull–White approximate (63%) $\geq$ trinomial trees (60%) $\geq$ CreditGrades (42%). This suggests that Hull–White pricing may on average well compete with structural pricing approaches such as CreditGrades or with hybrid valuation via trinomial trees. Future research may address CDS pricing models and their performance in more detail.

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Part IV

Default Risk, Recovery Risk, and Rating

CHAPTER 13

The Link between Default and Recovery Rates: Theory, Empirical Evidence, and Implications

Edward I. Altman, Brooks Brady, Andrea Resti, and Andrea Sironi

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13.1 INTRODUCTION

Credit risk affects virtually every financial contract. Therefore, the measurement, pricing, and management of credit risk have received much attention from financial economists, bank supervisors and regulators, and from financial market practitioners. Following the recent attempts of the Basel Committee on Banking Supervision (1999, 2001) to reform the capital adequacy framework by introducing risk-sensitive capital requirements, significant additional attention has been devoted to the subject of credit-risk measurement by the international regulatory, academic, and banking communities.

This chapter analyzes and measures the association between aggregate default and recovery rates on corporate bonds, and seeks to empirically explain this critical relationship. After a brief review of the way credit-risk models explicitly or implicitly treat the recovery rate variable, Section 13.2 examines the recovery rates on corporate bond defaults over the period 1982–2002. We attempt to explain recovery rates by specifying rather straightforward linear, logarithmic, and logistic regression models. The central thesis is that aggregate recovery rates are basically a function of supply and demand for the securities. Our econometric univariate and multivariate time-series models explain a significant portion of the variance in bond recovery rates aggregated across all seniority and collateral levels. In Sections 13.3 and 13.4 we briefly examine the effects of the relationship between defaults and recoveries on credit VaR (value at risk) models as well as on the procyclicality effects of the new capital requirements proposed by the Basel Committee, and then conclude with some remarks on the general relevance of our results.

13.2 RELATIONSHIP BETWEEN DEFAULT RATES AND RECOVERY RATES IN CREDIT-RISK MODELING: A REVIEW OF THE LITERATURE

Credit-risk models can be divided into three main categories: (1) first generation structural-form models, (2) second generation structural-form models, and (3) reduced-form models. Rather than going through a discussion of each of these well-known approaches

and their advocates in the literature, we refer the reader to our earlier report for ISDA (2001) which carefully reviews the literature on the conceptual relationship between the firm's probability of default (PD) and the recovery rate (RR) after default to creditors.*

During the last few years, new approaches explicitly modeling and empirically investigating the relationship between PD and RR have been developed. These include Frye (2000a,b), Jokivuolle and Peura (2003), and Jarrow (2001). The model proposed by Frye draws from the conditional approach suggested by Finger (1999) and Gordy (2000). In these models, defaults are driven by a single systematic factor—the state of the economy—rather than by a multitude of correlation parameters. The same economic conditions are assumed to cause default to rise, for example, and RRs to decline. The correlation between these two variables therefore derives from their common dependence on the systematic factor. The intuition behind Frye's theoretical model is relatively simple: If a borrower defaults on a loan, a bank's recovery may depend on the value of the loan collateral. The value of the collateral, like the value of other assets, depends on economic conditions. If the economy experiences a recession, RRs may decrease just as default rates tend to increase. This gives rise to a negative correlation between default rates and RRs. The model originally developed by Frye (2000a) implied recovery from an equation that determines collateral. His evidence is consistent with the most recent U.S. bond market data, indicating a simultaneous increase in default rates and losses given default (LGDs)† in 1999–2001.‡ Frye's (2000b,c) empirical analysis allows him to conclude that in a severe economic downturn, bond recoveries might decline 20%–25% points from their normal-year average. Loan recoveries may decline by a similar amount, but from a higher level.

Jarrow (2001) presents a new methodology for estimating RRs and PDs implicit in both debt and equity prices. As in Frye (2000a,b), RRs and PDs are correlated and depend on the state of the economy. However, Jarrow's methodology explicitly incorporates equity prices in the estimation procedure, allowing the separate identification of RRs and PDs and the use of an expanded and relevant dataset. In addition, the methodology explicitly incorporates a liquidity premium in the estimation procedure, which is considered essential in the light of the high variability in the yield spreads between risky debt and U.S. Treasury securities.

A rather different approach is the one proposed by Jokivuolle and Peura (2003). The authors present a model for bank loans in which collateral value is correlated with the PD. They use the option pricing framework for modeling risky debt: The borrowing firm's total asset value determines the event of default. However, the firm's asset value does not determine the RR. Rather, the collateral value is in turn assumed to be the only stochastic element determining recovery. Because of this assumption, the model can be implemented using an exogenous PD, so that the firm's asset value parameters need not be estimated. In this respect, the model combines features of both structural- and reduced-form models. Assuming a positive correlation between a firm's asset value and collateral value, the

* See Altman et al. (2001) for a formal discussion of this relationship.

† LGD indicates the amount actually lost (by an investor or a bank) for each dollar lent to a defaulted borrower. Accordingly, LGD and RR always add to one. One may also factor into the loss calculation of the last coupon payment that is usually not realized when a default occurs (Altman 1989).

‡ Gupton et al. (2001) provide clear empirical evidence of this phenomenon.

authors obtain a similar result as that of Frye's (2000a): realized default rates and recovery rates have an inverse relationship.

Using Moody's historical bond market data, Hu and Perraudin (2002) examine the dependence between recovery rates and default rates. They first standardize the quarterly recovery data in order to filter out the volatility of recovery rates given by the variation over time in the pool of borrowers rated by Moody's. They find that correlations between quarterly recovery rates and default rates for bonds issued by U.S.-domiciled obligors are 0.22 for post 1982 data (1983–2000) and 0.19 for the 1971–2000 period. Using extreme value theory and other nonparametric techniques, they also examine the impact of this negative correlation on credit VaR measures and find that the increase is statistically significant when confidence levels exceed 99%.

Bakshi et al. (2001) enhance the reduced-form models briefly presented above to allow for a flexible correlation between the risk-free rate, the default probability, and the recovery rate. On the basis of some preliminary evidence published by rating agencies, they force recovery rates to be negatively associated with default probability. They find some strong support for this hypothesis through the analysis of a sample of BBB-rated corporate bonds: More precisely, their empirical results show that, on average, a 4% worsening in the (risk-neutral) hazard rate is associated with a 1% decline in (risk-neutral) recovery rates.

Compared to the above mentioned contributions, this study extends the existing literature in three main directions. First, the determinants of defaulted bonds' recovery rates are empirically investigated. While most of the above mentioned recent studies concluded in favor of an inverse relationship between these two variables, based on the common dependence on the state of the economy, none of them empirically analyzed the specific determinants of recovery rates. While our analysis shows empirical results that appear consistent with the intuition of a negative correlation between default rates and RRs, we find that a single systematic risk factor—i.e., the performance of the economy—is less predictive than the above mentioned theoretical models would suggest.

Second, our study is the first one to examine, both theoretically and empirically, the role played by supply and demand of defaulted bonds in determining aggregate recovery rates. Our econometric univariate and multivariate models assign a key role to the supply of defaulted bonds and show that these variables together with variables that proxy the size of the high yield bond market explain a substantial proportion of the variance in bond recovery rates aggregated across all seniority and collateral levels.

Third, our simulations show the consequences that the negative correlation between default and recovery would have on VaR models and on the procyclicality effect of the capital requirements recently proposed by the Basel Committee. Indeed, while our results on the impact of this correlation on credit-risk measures (such as unexpected loss and value at risk) are in line with the ones obtained by Hu and Perraudin (2002), they show that, if a positive correlation highlighted by bond data were to be confirmed by bank data, the procyclicality effects of "Basel 2" might be even more severe than expected if banks use their own estimates of LGD (as in the "advanced" internal ratings-based approach, also known as the "IRB approach").

As concerns specifically the Hu and Perraudin paper, it should be pointed out that they correlate recovery rates (or percentage of par which is the same thing) with issuer-based

default rates. Our models assess the relationship between dollar-denominated default and recovery rates and, as such, can assess directly the supply/demand aspects of the defaulted debt market. Moreover, besides assessing the relationship between default and recovery using *ex post* default rates, we explore the effect of using *ex ante* estimates of the future default rates (i.e., default probabilities) instead of actual, realized defaults. As will be shown, however, while the negative relationship between RR and both *ex post* and *ex ante* default rates is empirically confirmed, probabilities of default show a considerably lower explanatory power. Finally, it should be emphasized that while our chapter and the one by Hu and Perraudin reach similar conclusions, albeit from very different approaches and tests, it is important that these results become accepted and are subsequently reflected in future credit-risk models and public-policy debates and regulations. For these reasons, concurrent confirming evidence from several sources are beneficial, especially if they are helpful in specifying fairly precisely the default rate/recovery rate nexus.

13.3 EXPLAINING AGGREGATE RECOVERY RATES ON CORPORATE BOND DEFAULTS: EMPIRICAL RESULTS

The average loss experience on credit assets is well documented in studies by the various rating agencies (Moody's, S&P, and Fitch) as well as by academics.* Recovery rates have been observed for bonds, stratified by seniority, as well as for bank loans. The latter asset class can be further stratified by capital structure and collateral type (Van de Castle and Keisman 2000). While quite informative, these studies say nothing about the recovery versus default correlation. The purpose of this section is to empirically test this relationship with actual default data from the U.S. corporate bond market over the last two decades. As pointed out in Section 13.1, there is strong intuition suggesting that default and recovery rates might be correlated. Accordingly, this section of our study attempts to explain the link between the two variables, by specifying rather straightforward statistical models.†

We measure aggregate annual bond recovery rates (henceforth: BRR) by the weighted average recovery of all corporate bond defaults, primarily in the United States, over the period 1982–2001. The weights are based on the market value of defaulting debt issues of publicly traded corporate bonds.‡ The logarithm of BRR (BLRR) is also analyzed.

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* See Altman and Kishore (1996), Altman and Arman (2002), FITCH (1997, 2001), Standard and Poor's (2000).

† We will concentrate on average annual recovery rates but not on the factors that contribute to understanding and explaining recovery rates on individual firm and issue defaults. Madan and Unal (1998) propose a model for estimating risk-neutral expected recovery rate distributions, not empirically observable rates. The latter can be particularly useful in determining prices on credit derivative instruments, such as credit default swaps.

‡ Prices of defaulted bonds are based on the closing "bid" levels on or as close to the default date as possible. Precise-date pricing was only possible in the last 10 years, or so, since market-maker quotes were not available from the NYU Salomon Center database prior to 1990 and all prior date prices were acquired from secondary sources, primarily the *S&P Bond Guides*. Those latter prices were based on end-of-month closing bid prices only. We feel that more exact pricing is a virtue since we are trying to capture supply and demand dynamics which may impact prices negatively if some bondholders decide to sell their defaulted securities as fast as possible. In reality, we do not believe this is an important factor since many investors will have sold their holdings prior to default or are more deliberate in their "dumping" of defaulting issues.

The sample includes annual and quarterly averages from about 1300 defaulted bonds for which we were able to get reliable quotes on the price of these securities just after default. We utilize the database constructed and maintained by the NYU Salomon Center, under the direction of one of the authors. Our models are both univariate and multivariate least square regressions. The univariate structures can explain up to 60% of the variation of average annual recovery rates, while the multivariate models explain as much as 90%.

The rest of this section will proceed as follows. We begin our analysis by describing the independent variables used to explain the annual variation in recovery rates. These include supply-side aggregate variables that are specific to the market for corporate bonds, as well as macroeconomic factors (some demand side factors, like the return on distressed bonds and the size of the "vulture" funds market, are discussed later). Next, we describe the results of the univariate analysis. We then present our multivariate models, discussing the main results and some robustness checks.

13.3.1 Explanatory Variables

We proceed by listing several variables we reasoned could be correlated with aggregate recovery rates. The expected effects of these variables on recovery rates will be indicated by a $+/-$ sign in parentheses. The exact definitions of the variables we use are:

BDR (−): The weighted average default rate on bonds in the high yield bond market and its logarithm (BLDR, −). Weights are based on the face value of all high yield bonds outstanding each year and the size of each defaulting issue within a particular year.*

BDRC (−): One year change in BDR.

BOA (−): The total amount of high yield bonds outstanding for a particular year (measured at midyear in trillions of dollars) and represents the potential supply of defaulted securities. Since the size of the high yield market has grown in most years over the sample period, the BOA variable is picking up a time-series trend as well as representing a potential supply factor.

BDA (−): We also examined the more directly related bond defaulted amount as an alternative for BOA (also measured in trillions of dollars).

GDP (+): The annual GDP growth rate.

GDPC (+): The change in the annual GDP growth rate from the previous year.

* We did not include a variable that measures the distressed, but not defaulted, proportion of the high yield market since we do not know of a time-series measure that goes back to 1987. We define distressed issues as yielding more than 1000 basis points over the risk-free 10 year Treasury bond rate. We did utilize the average yield spread in the market and found it was highly correlated (0.67) to the subsequent one year's default rate and hence did not add value (see discussion below). The high yield bond yield spread, however, can be quite helpful in forecasting the following year's BDR, a critical variable in our model (see our discussion of a default probability prediction model in Section 13.2.6).

GDPI (−): Takes the value of 1 when GDP growth was less than 1.5% and 0 when GDP growth was greater than 1.5%.

SR (+): The annual return on the S&P 500 stock index.

SRC (+): The change in the annual return on the S&P 500 stock index from the previous year.

13.3.2 Basic Explanatory Variable: Default Rates

It is clear that the supply of defaulted bonds is most vividly depicted by the aggregate amount of defaults and the rate of default. Since virtually all public defaults most immediately migrate to default from the noninvestment grade or "junk" bond segment of the market, we use that market as our population base. The default rate is the par value of defaulting bonds divided by the total amount outstanding, measured at face values. Table 13.1 shows the default rate data from 1982–2001, as well as the weighted average

TABLE 13.1 Default Rates, Recovery Rates, and Losses

Year	Par Value Outstanding[a] ($ MMs)	Par Value of Defaults[b] ($ MMs)	Default Rate (%)	Weighted Price after Default (Recovery Rate)	Weighted Coupon (%)	Default Loss[c] (%)
2001	649,000	63,609	9.80	25.5	9.18	7.76
2000	597,200	30,295	5.07	26.4	8.54	3.95
1999	567,400	23,532	4.15	27.9	10.55	3.21
1998	465,500	7464	1.60	35.9	9.46	1.10
1997	335,400	4200	1.25	54.2	11.87	0.65
1996	271,000	3336	1.23	51.9	8.92	0.65
1995	240,000	4551	1.90	40.6	11.83	1.24
1994	235,000	3418	1.45	39.4	10.25	0.96
1993	206,907	2287	1.11	56.6	12.98	0.56
1992	163,000	5545	3.40	50.1	12.32	1.91
1991	183,600	18,862	10.27	36.0	11.59	7.16
1990	181,000	18,354	10.14	23.4	12.94	8.42
1989	189,258	8110	4.29	38.3	13.40	2.93
1988	148,187	3944	2.66	43.6	11.91	1.66
1987	129,557	1736	1.34	62.0	12.07	0.59
1986	90,243	3156	3.50	34.5	10.61	2.48
1985	58,088	992	1.71	45.9	13.69	1.04
1984	40,939	344	0.84	48.6	12.23	0.48
1983	27,492	301	1.09	55.7	10.11	0.54
1982	18,109	577	3.19	38.6	9.61	2.11
Weighted average			4.19	37.2	10.60	3.16

Note: Default rate data from 1982–2001 are shown, as well as the weighted average annual recovery rates and the loss rates. The time series show a high correlation (75%) between default and recovery rates.

[a] Measured at midyear, excludes defaulted issues.

[b] Does not include Texaco's bankruptcy in 1987.

[c] Includes lost coupon as well as principal loss.

annual recovery rates (our dependent variable) and the default loss rate (last column). Note that the average annual recovery is 41.8% (weighted average 37.2%) and the weighted average annual loss rate to investors is 3.16%.* The correlation between the default rate and the weighted price after default amounts to .75.

13.3.3 Demand and Supply of Distressed Securities

The logic behind our demand/supply analysis is both intuitive and important, especially, since as we have seen, most credit-risk models do not formally and statistically consider this relationship. On a macroeconomic level, forces that cause default rates to increase during periods of economic stress also cause the value of assets of distressed companies to decrease. Hence the securities' values of these companies will likely be lower. While the economic logic is clear, the statistical relationship between GDP variables and recovery rates is less significant than what one might expect. We hypothesized that if one drills down to the distressed firm market and its particular securities, we can expect a more significant and robust negative relationship between default and recovery rates.†

The principal purchasers of defaulted securities, primarily bonds and bank loans, are niche investors called distressed asset or alternative investment managers—also called vultures. Prior to 1990, there was little or no analytic interest in these investors, indeed in the distressed debt market, except for the occasional anecdotal evidence of performance in such securities. Altman (1991) was the first to attempt an analysis of the size and performance of the distressed debt market and estimated, based on a fairly inclusive survey, that the amount of funds under management by these so-called vultures was at least $7.0 billion in 1990 and if you include those investors who did not respond to the survey and nondedicated investors, the total was probably in the $10–$12 billion range. Cambridge Associates (2001) estimated that the amount of distressed assets under management in 1991 was $6.3 billion. Estimates since 1990 indicate that the demand did not rise materially until 2000–2001, when the estimate of total demand for distressed securities was about $40–$45 billion as of December 31, 2001 and $60–$65 billion in 1 year later (Altman 2003). So, while the demand for distressed securities grew slowly in the 1990s and early in the next decade, the supply (as we will show) grew enormously.

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* The loss rate is impacted by the lost coupon at default as well as the more important lost principal. The 1987 default rate and recovery rate statistics do not include the massive Texaco default since it was motivated by a lawsuit which was considered frivolous resulting in a strategic bankruptcy filing and a recovery rate (price at default) of over 80%. Including Texaco would have increased the default rate by over 4% and the recovery rate to 82% (reflecting the huge difference between the market's assessment of asset values versus liabilities, not typical of bankrupt companies). The results of our models would be less impressive, although still quite significant, with Texaco included.

† Consider the latest highly stressful period of corporate bond defaults in 2000–2002. The huge supply of bankrupt firms' assets in sectors like telecommunications, airlines, and steel, to name a few, has had a dramatic negative impact on the value of the firms in these sectors as they filed for bankruptcy and attempted a reorganization under Chapter 11. Altman (2003) estimated that the size of the U.S. distressed and defaulted public and private debt markets swelled from about $300 billion (face value) at the end of 1999 to about $940 billion by year end 2002. And, only 1 year in that 3 year period was officially a recession year (2001). As we will show, the recovery rate on bonds defaulting in this period was unusually low as telecom equipment, large body aircraft, and steel assets of distressed firms piled up.

On the supply side, the last decade has seen the amounts of distressed and defaulted public and private bonds and bank loans grow dramatically in 1990–1991 to as much as $300 billion (face value) and $200 billion (market value), then recede to much lower levels in the 1993–1998 period and grow enormously again in 2000–2002 to the unprecedented levels of $940 billion (face value) and almost $500 billion market value as of December 2002. These estimates are based on calculations in Altman (2003) from periodic market calculations and estimates.*

On a relative scale, the ratio of supply to demand of distressed and defaulted securities was something like ten to one in both 1990–1991 and also in 2000–2001. Dollarwise, of course, the amount of supply-side money dwarfed the demand in both periods. And, as we will show, the price levels of new defaulting securities was relatively very low in both periods—at the start of the 1990s and again at the start of the 2000 decade.

13.3.4 Univariate Models

We begin the discussion of our results with the univariate relationships between recovery rates and the explanatory variables described in the previous section. Table 13.2 displays the results of the univariate regressions carried out using these variables. These univariate regressions, and the multivariate regressions discussed in the following section, were calculated using both the recovery rate (BRR) and its natural log (BLRR) as the dependent variables. Both results are displayed in Table 13.2, as signified by an "x" in the corresponding row.

We examine the simple relationship between bond recovery rates and bond default rates for the period 1982–2001. Table 13.2 and Figure 13.1 show several regressions between the two fundamental variables. We find that one can explain about 51% of the variation in the annual recovery rate with the level of default rates (this is the linear model, regression 1) and as much as 60%, or more, with the logarithmic and power† relationships (regressions 3 and 4). Hence, our basic thesis that the rate of default is a massive indicator of the likely average recovery rate amongst corporate bonds appears to be substantiated.‡

The other univariate results show the correct sign for each coefficient, but not all of the relationships are significant. BDRC is highly negatively correlated with recovery rates, as shown by the very significant *t*-ratios, although the *t*-ratios and *R*-squared values are not as significant as those for BLDR. BOA and BDA are, as expected, both negatively correlated with recovery rates with BDA being more significant on a univariate basis. Macroeconomic

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* Defaulted bonds and bank loans are relatively easy to define and are carefully documented by the rating agencies and others. Distressed securities are defined here as bonds selling at least 1000 basis points over comparable maturity Treasury bonds (we use the 10 year T-bond rate as our benchmark). Privately owned securities, primarily bank loans, are estimated as 1.4–1.8× the level of publicly owned distressed and defaulted securities based on studies of a large sample of bankrupt companies (Altman 2003).

† The power relationship ($BRR = e^{b0} \times BDR^{b1}$) can be estimated using the following equivalent equation: $BLRR = b_0 + b_1 \times BLDR$ (power model).

‡ Such an impression is strongly supported by a −80% rank correlation coefficient between BDR and BRR (computed over the 1982–2001 period). Note that rank correlations represent quite a robust indicator, since they do not depend upon any specific functional form (e.g., log, quadratic, power, etc.).

TABLE 13.2 Univariate Regressions, 1982–2001

	Market Variables									
Regression #	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Dependent variable										
BRR	×		×		×		×		×	
BLRR		×		×		×		×		×
Explanatory variables: coefficients and *t*-ratios										
Constant	0.509	−0.668	0.002	−1.983	0.432	−0.872	0.493	−0.706	0.468	−0.772
	(18.43)	(−10.1)	(0.03)	(−10.6)	(24.9)	(−20.1)	(13.8)	(−8.05)	(19.10)	(−13.2)
BDR	−2.610	−6.919								
	(−4.36)	(−4.82)								
BLDR			−0.113	−0.293						
			(−5.53)	(−5.84)						
BDRC					−3.104	−7.958				
					(−4.79)	(−4.92)				
BOA							−0.315	−0.853		
							(−2.68)	(−2.95)		
BDA									−4.761	−13.122
									(−3.51)	(−4.08)
Goodness of fit measures										
R-square	0.514	0.563	0.630	0.654	0.560	0.574	0.286	0.326	0.406	0.481
Adjusted *R*-square	0.487	0.539	0.609	0.635	0.536	0.550	0.246	0.288	0.373	0.452
F-Stat	19.03	23.19	30.61	34.06	22.92	24.22	7.21	8.69	12.31	16.67
(*P*-value)	0.000	0.000	0.000	0.000	0.000	0.000	0.015	0.009	0.003	0.001
Residual tests										
Serial correlation LM, 2 lags (Breusch–Godfrey)	1.021	1.836	1.522	2.295	1.366	2.981	1.559	1.855	3.443	2.994
(*P*-value)	0.600	0.399	0.467	0.317	0.505	0.225	0.459	0.396	0.179	0.224
Heteroscedasticity (White, Chi square)	0.089	1.585	0.118	1.342	8.011	5.526	2.389	1.827	0.282	1.506
(*P*-value)	0.956	0.453	0.943	0.511	0.018	0.063	0.303	0.401	0.868	0.471
Number of observations	20	20	20	20	20	20	20	20	20	20

	Macrovariables									
Regression #	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Dependent variable										
BRR	×		×		×		×		×	
BLRR		×		×		×		×		×
Explanatory variables										
Constant	0.364	−1.044	0.419	−0.907	0.458	−0.804	0.387	−1.009	0.418	−0.910
	(7.59)	(−8.58)	(18.47)	(−15.65)	(15.42)	(−10.8)	(10.71)	(−11.3)	(16.42)	(−14.4)
GDP	1.688	4.218								
	(1.30)	(1.28)								
GDPC			2.167	5.323						
			(2.31)	(2.22)						
GDPI					−0.101	−0.265				
					(−2.16)	(−2.25)				
SR							0.205	0.666		
							(1.16)	(1.53)		
SRC									0.095	0.346
									(0.73)	(1.07)
Goodness of fit measures										
R-square	0.086	0.083	0.228	0.215	0.206	0.220	0.070	0.115	0.029	0.060
Adjusted *R*-square	0.035	0.032	0.186	0.171	0.162	0.176	0.018	0.066	−0.025	0.007
F-Stat	1.69	1.64	5.33	4.93	4.66	5.07	1.36	2.35	0.53	1.14
(*P*-value)	0.211	0.217	0.033	0.040	0.045	0.037	0.259	0.143	0.475	0.299
Residual tests										
Serial correlation LM, 2 lags (Breusch–Godfrey)	2.641	4.059	0.663	1.418	0.352	1.153	3.980	5.222	3.479	4.615
(*P*-value)	0.267	0.131	0.718	0.492	0.839	0.562	0.137	0.073	0.176	0.100
Heteroscedasticity (White, Chi square)	2.305	2.077	2.254	2.494	0.050	0.726	2.515	3.563	3.511	4.979
(*P*-value)	0.316	0.354	0.324	0.287	0.823	0.394	0.284	0.168	0.173	0.083
Number of observations	20	20	20	20	20	20	20	20	20	20

Note: Variables explaining annual recovery rates on defaulted corporate bonds. Results of a set of univariate regressions carried out between the recovery rate (BRR) or its natural log (BLRR) and an array of explanatory variables: The default rate (BDR), its log (BLDR), and its change (BDRC); the outstanding amount of bonds (BOA) and the outstanding amount of defaulted bonds (BDA); the GDP growth rate (GDP), its change (GDPC), and a dummy (GDPI) taking the value of 1 when the GDP growth is less than 1.5%; the S&P 500 stock market index (SR) and its change (SRC).

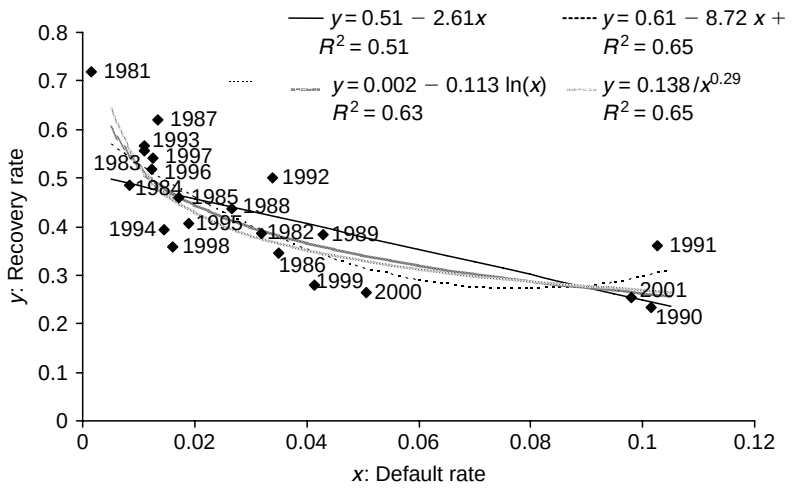


FIGURE 13.1 Univariate Models: Results of a set of univariate regressions carried out between the recovery rate (BRR) or its natural log (BLRR) and the default rate (BDR) or its natural log (BLDR). See Table 13.2 for more details.

variables did not explain as much of the variation in recovery rates as the corporate bond market variables explained; their poorer performance is also confirmed by the presence of some heteroscedasticity and serial correlation in the regression's residuals, hinting at one or more omitted variables. We will come back to these relationships in the next paragraphs.

13.3.5 Multivariate Models

We now specify some more complex models to explain recovery rates, by adding several variables to the default rate. The basic structure of our most successful models is

$$\text{BRR} = f(\text{BDR, BDRC, BOA, or BDA})$$

Some macroeconomic variables will be added to this basic structure, to test their effect on recovery rates.

Before we move on to the multivariate results, Table 13.3 reports the cross-correlations among our regressors (and between each of them and the recovery rate BRR); values greater than 0.5 are highlighted. A rather strong link between GDP and BDR emerges, suggesting that default rates are, as expected, positively correlated with macrogrowth measures. Hence, adding GDP to the BDR/BRR relationship is expected to blur the significance of the results. We also observe a high positive correlation between BDA (absolute amount of all defaulted bonds) and the default rate.

We estimate our regressions using 1982–2001 data in order to explain recovery rate results and to predict 2002 rates. This involves linear and log–linear structures for the two

TABLE 13.3 Correlation Coefficients among the Main Variables

	BDR	BOA	BDA	GDP	SR	BRR
BDR	1.00	.33	**.73**	**−.56**	−.30	**−.72**
BOA		1.00	**.76**	.05	−.21	**−.53**
BDA			1.00	−.26	−.49	**−.64**
GDP				1.00	−.02	.29
SR					1.00	.26
BRR						1.00

Note: Cross-correlations among our regressors (and between each of them and the recovery rate BRR); values greater than .5 are in bold. A strong link between GDP and BDR emerges, suggesting that default rates are, as expected, positively correlated with macrogrowth measures.

key variables—recovery rates (dependent) and default rates (explanatory)—with the log–linear relationships somewhat more significant. These results appear in Table 13.4.

Regressions 1 through 6 build the "basic models": Most variables are quite significant based on their *t*-ratios. The overall accuracy of the fit goes from 71% (65% adjusted *R*-square) to 87% (84% adjusted).

The actual model with the highest explanatory power and lowest "error" rates is the power model* in regression 4 of Table 13.4. We see that all of the four explanatory variables have the expected negative sign and are significant at the 5% or 1% level. BLDR and BDRC are extremely significant, showing that the level and change in the default rate are highly important explanatory variables for recovery rates. Indeed the variables BDR and BDRC explain up to 80% (unadjusted) and 78% (adjusted) of the variation in BRR simply based on a linear or log–linear association. The size of the high yield market also performs very well and adds about 6%/7% to the explanatory power of the model. When we substitute BDA for BOA (regressions 5 and 6), the latter does not look statistically significant, and the *R*-squared of the multivariate model drops slightly to 0.82 (unadjusted) and 0.78 (adjusted). Still, the sign of BDA is correct (+). Recall that BDA was more significant than BOA on a univariate basis (Table 13.2).

Macrovariables are added in columns 7–10: We are somewhat surprised by the low contributions of these variables since there are several models that have been constructed that utilize macrovariables, apparently significantly, in explaining annual default rates.†

As concerns the growth rate in annual GDP, the univariate analyses presented in Tables 13.2 and 13.3 had shown it to be significantly negatively correlated with the bond default rate (−0.78, see Table 13.3); however, the univariate correlation between recovery rates (both BRR and BLRR) and GDP growth is relatively low (see Table 13.2), although with the

* Like its univariate cousin, the multivariate power model can be written using logs. For example, $BLRR = b_0 + b_1 \times BLDR + b_2 \times BDRC + b_3 \times BOA$ becomes $BRR = exp[b_0] \times BDR^{b1} \times exp[b_2 \times BDRC + b_3 \times BOA]$ and takes its name from BDR being raised to the power of its coefficient.

† See Jonsson and Fridson (1996), Keenan et al. (1999), Fridson et al. (1997), Helwege and Kleiman (1997), and Chacko and Mercier (2001).

TABLE 13.4 Multivariate Regressions, 1982–2001

	Linear and Logarithmic Models										Logistic Models				
Regression #	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Dependent variable															
BRR	×		×		×		×		×		×	×	×	×	×
BLRR		×		×		×		×		×					
Explanatory variables: coefficients and *t*-ratios															
Constant	0.514	−0.646	0.207	−1.436	0.482	−1.467	0.529	−1.538	0.509	−1.447	−0.074	−0.097	0.042	0.000	0.000
	(19.96)	(−11.34)	(2.78)	(−8.70)	(20.02)	(−6.35)	(11.86)	(−9.07)	(14.65)	(−8.85)	(−0.64)	(−0.92)	(0.44)	(0.00)	(0.00)
BDR	−1.358	−3.745			−1.209		−1.513		−1.332		12.200	6.713	5.346	7.421	6.487
	(−2.52)	(−3.13)			(−1.59)		(−2.28)		(−2.33)		(4.14)	(2.82)	(1.55)	(2.59)	(2.64)
BLDR			−0.069	−0.176		−0.167		−0.222		−0.169					
			(−3.78)	(−4.36)		(−2.94)		(−4.64)		(−4.17)					
BDRC	−1.930	−4.702	−1.748	−4.389	−2.039	−4.522	−1.937	−4.415	−1.935	−4.378		8.231	8.637	8.304	8.394
	(−3.18)	(−3.50)	(−3.39)	(−3.84)	(−3.03)	(−3.35)	(−3.11)	(−4.05)	(−3.09)	(−3.87)		(3.339)	(3.147)	(3.282)	(3.315)
BOA	−0.164	−0.459	−0.141	−0.410			−0.153	−0.328	−0.162	−0.387		0.742		0.691	0.736
	(−2.13)	(−2.71)	(−2.12)	(−2.78)			(−1.86)	(−2.20)	(−2.03)	(−2.63)		(2.214)		(1.927)	(2.136)
BDA					−1.203	−3.199							8.196		
					(−0.81)	(−1.12)							(1.064)		
GDP							−0.387	−2.690						1.709	
							(−0.43)	(−1.62)						(0.473)	

SR									0.020	0.213					−0.242
									(0.192)	(1.156)					(−0.56)
Goodness of fit measures															
R-square	0.764	0.819	0.826	0.867	0.708	0.817	0.767	0.886	0.764	0.878	0.534	0.783	0.732	0.786	0.787
Adjusted *R*-square	0.720	0.785	0.793	0.842	0.654	0.782	0.704	0.856	0.702	0.845	0.508	0.742	0.682	0.729	0.731
F-Stat	17.250	24.166	25.275	34.666	12.960	23.752	12.320	29.245	12.168	26.881	20.635	19.220	14.559	13.773	13.876
(*P*-value)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Residual tests															
Serial correlation LM, 2 lags (Breusch–Godfrey)	3.291	2.007	1.136	0.718	1.235	0.217	3.344	0.028	5.606	1.897	1.042	2.673	1.954	2.648	5.899
(*P*-value)	0.193	0.367	0.567	0.698	0.539	0.897	0.188	0.986	0.061	0.387	0.594	0.263	0.376	0.266	0.052
Heteroscedasticity (White, Chi square)	5.221	5.761	5.049	5.288	12.317	12.795	5.563	4.853	6.101	6.886	0.008	5.566	9.963	5.735	5.948
(*P*-value)	0.516	0.451	0.538	0.507	0.055	0.046	0.696	0.773	0.636	0.549	0.996	0.474	0.126	0.677	0.653
Number of observation	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20

Note: Results of a set of multivariate regressions based on 1982–2001 data. Regressions 1 through 6 build the "basic models": most variables are quite significant based on their *t*-ratios. The overall accuracy of the fit goes from 71% (65% adjusted *R*-square) to 87% (84% adjusted); the model with the highest explanatory power and lowest "error" rates is the power model in regression 4. Macrovariables are added in columns 7–10, showing a poor explanatory power. A set of logistic estimates (cols. 11 to 15) are provided to account for the fact that the dependent variable (recovery rates) is bounded between 0 and 1.

appropriate sign (+). Note that, when we utilize the change in GDP growth (GDPC, Table 13.2, regressions 5 and 6), the significance improves markedly.

When we introduce GDP to our existing multivariate structures (Table 13.4, regressions 7 and 8), not only is it insignificant, but it also has a counterintuitive sign (negative). The GDPC variable leads to similar results (not reported). No doubt, the high negative correlation between GDP and BDR reduces the possibility of using both in the same multivariate structure.

We also postulated that the return of the stock market could impact prices of defaulting bonds in that the stock market represented investor expectations about the future. A positive stock market outlook could imply lower default rates and higher recovery rates. For example, earnings of all companies, including distressed ones, could be reflected in higher stock prices. Table 13.4, regressions 9–10, show the association between the annual S&P 500 Index stock return (*SR*) and recovery rates. Note the insignificant *t*-ratios in the multivariate model, despite the appropriate signs. Similar results (together with low R^2s) emerge from our univariate analysis (Table 13.2), where the change in the S&P return (SRC) was also tested.

Since the dependent variable (BRR) in most of our regressions is bounded by 0 and 1, we have also run the same models using a logistic function (Table 13.4, columns 11–15). As can be seen, *R*-squares and *t*-ratios are broadly similar to those already shown above. The model in column 12, including BDR, BDRC, and BOA explains as much as 74% (adjusted *R*-square) of the recovery rate's total variability. Macroeconomic variables, as before, tend to have no evident effect on BDR.

13.3.6 Robustness Checks

This section hosts some robustness checks carried out to verify how our results would change when taking into account several important modifications to our approach.

13.3.6.1 Default Probabilities

The models shown above are based on the actual default rate experienced in the high yield, speculative grade market, (BDR) and reflect a coincident supply/demand dynamic in that market. One might argue that this *ex post* analysis is conceptually different from the specification of an *ex ante* estimate of the default rate.

We believe both specifications are important. Our previous *ex post* models and tests are critical in understanding the actual experience of credit losses and, as such, impact credit management regulation and supervision, capital allocations, and credit policy and planning of financial institutions. On the other hand, *ex ante* probabilities (PDs) are customarily used in VaR models in particular and risk management purposes in general; however, their use in a regression analysis of recovery rates might lead to empirical tests which are inevitably limited by the models used to estimate PDs and their own biases. The results of these tests might therefore not be indicative of the true relationship between default and recovery rates.

In order to assess the relationship between *ex ante* PDs and BRRs, we used PDs generated through a well-established default rate forecasting model from Moody's

(Keenan et al. 1999). This econometric model is used to forecast the *global* speculative grade *issuer* default rate and was fairly accurate ($R^2 = 0.8$) in its explanatory model tests.*

The results of using Moody's model to explain our recovery rates did demonstrate a significant negative relationship but the explanatory power of the multivariate models was considerably lower (adjusted $R^2 = 0.39$), although still impressive with significant *t*-tests for the change in PD and the amount of bonds outstanding (all variables had the expected sign). Note that, since the Moody's model is for global issuers and our earlier tests are for U.S. dollar-denominated high yield bonds, we did not expect that their PD model would be nearly as accurate in explaining U.S. recovery rates.

13.3.6.2 Quarterly Data

Our results are based on yearly values, so we wanted to make sure that higher-frequency data would confirm the existence of a link between default rates and recoveries. On the basis of quarterly data,† a simple, univariate estimate (Table 13.5) shows that: (1) BDR is still strongly significant, and shows the expected sign; (2) the *R*-square looks relatively modest (23.9%) versus 51.4 for the annual data, because quarterly default rates and recovery rates tend to be very volatile (due to some "poor" quarters with only very few defaults).

Using a moving average of four quarters (BRR4W, weighted by the number of defaulted issues) we estimated another model (using BDR, its lagged value and its square, see last column in Table 13.5), obtaining a much better *R*-square (72.4%). This suggests that the link between default rates and recovery is somewhat "sticky" and—although confirmed by quarterly data—is better appreciated over a longer time interval.‡ Note that the signs of the coefficients behave as expected: for example, an increase in quarterly BDRs from 1% to 3% reduces the expected recovery rate from 39% to 31% within the same quarter, while a further decrease to 29% takes place in the following three months.

13.3.6.3 Risk-Free Rates

We considered the role of risk-free rates in explaining recovery rates, since these, in turn, depend on the discounted cash flows expected from the defaulted bonds. We therefore added to our best models (e.g., columns 3–4 in Table 13.4) some rate variables (namely, the 1 and 10 year U.S. dollar Treasury rates taken from the Federal Reserve Board of Governors, the corresponding discount rates or, alternatively, the "steepness" of the yield curve, as measured by the difference between 10 and 1 year rates). The results are disappointing, since none of these variables ever is statistically significant at the 10% level.§

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* Thus, far, Moody's has tested their forecasts for the 36 month period 1999–2001 and found that the correlation between estimated (PD) and actual default rates was greater than 0.90 (Hamilton et al. 2003). So, it appears that there can be a highly correlated link between estimated PDs and actual BDRs. By association, therefore, one can infer that accurate PD models can be used to estimate recovery rates and LGD.

† We had to refrain from using monthly data simply because of missing values (several months show no defaults, so it is impossible to compute recovery rates when defaulted bonds amount to zero).

‡ This is confirmed by the equation residuals, which look substantially autocorrelated.

§ This might also be due to the fact that one of our regressors (BOA, the amount of outstanding bonds) indirectly accounts for the level of risk-free rates, since lower rates imply higher market values and vice versa. Even removing BOA, however, risk-free rates cannot be found to be significant inside our model.

TABLE 13.5 Quarterly Regressions, 1990–2002

Dependent variable		
	BRR	**BRR4W**
Explanatory variables: coefficients and *t*-ratios		
Constant	45.48	48.12
	(17.61)	(39.3)
BDR	−5.77	−8.07
	(−3.92)	(−4.24)
BDR(−1)		−2.62
		(−2.87)
BDRSQ		1.10
		(2.87)
Goodness of fit measures		
R-square	0.239	0.724
Adjusted *R*-square	0.223	0.705
F-Stat	15.36	38.47
(*P*-value)	0.000	0.000
Residual tests		
Serial correlation LM, 2 lags (Breusch–Godfrey)	1.129	10.456
(*P*-value)	0.332	0.000
Heteroscedasticity (White, Chi square)	4.857	1.161
(*P*-value)	0.012	0.344
Number of observations	51	48

Note: Univariate estimates based on quarterly data. It appears that: (1) BDR is strongly significant, and has the expected sign; (2) the *R*-square looks relatively modest, because quarterly default rates and recovery rates tend to be very volatile; (3) expressing recovery rates as the moving average of four quarterly data (BRR4W) improves the *R*-square, suggesting that the link between default rates and recovery is somewhat "sticky."

13.3.6.4 Returns on Defaulted Bonds

We examined whether the return experienced by the defaulted bond market affects the demand for distressed securities, thereby influencing the "equilibrium price" of defaulted bonds. To do so, we considered the 1-year return on the Altman-NYU Salomon Center Index of Defaulted Bonds (BIR), a monthly indicator of the market weighted average performance of a sample of defaulted publicly traded bonds.* This is a measure of the price changes of existing defaulted issues as well as the "entry value" of new defaults and, as such, is impacted by supply and demand conditions in this "niche" market.† On a

* More details can be found in Altman (1991) and Altman (2003). Note that we use a different time frame in our analysis (1987–2001), because the defaulted bond index return (BIR) has only been calculated since 1987.

† We are aware of the fact that the average recovery rate on newly defaulted bond issues could influence the level of the defaulted bond index and vice versa. The vast majority of issues in the index, however, are usually comprised of bonds that have defaulted in prior periods. And, as we will see, while this variable is significant on an univariate basis and does improve the overall explanatory power of the model, it is not an important contributor.

univariate basis, the BIR shows the expected sign (+) with a *t*-ratio of 2.67 and explains 35% of the variation in BRR. However, when BIR is included in multivariate models, its sign remains correct, but the significance is usually below 10%.

13.3.6.5 Outliers

The limited width of the time series on which our coefficient estimates are based suggests that they might be affected by a small number of outliers. We checked for this by eliminating 10% of the observations, choosing those associated with the highest residuals,* and running our regressions again. The results (not reported to save room) totally confirm the estimates shown in Table 13.4, e.g., for model 3, the coefficients associated with BLDR (−0.15), BDRC (−4.26), and BOA (−0.45) are virtually unchanged, and remain significant at the 1% level; the same happens for model 4 (the coefficients being, respectively, −0.15, −4.26, and −0.45).

13.3.6.6 GDP Dummy and Regime Effects

We saw, in our multivariate results, that the GDP variable lacks statistical significance and tends to have a counterintuitive sign when added to multivariate models. The fact that GDP growth is highly correlated with default rates, our primary explanatory variable, looks like a sensible explanation for this phenomenon. To try and circumvent this problem, we used a technique similar to Helwege and Kleiman (1997): they postulate that, while a change in GDP of say 1% or 2% was not very meaningful in explaining default rates when the base year was in a strong economic growth period, the same change was meaningful when the new level was in a weak economy. Following their approach, we built a dummy variable (GDPI) which takes the value of 1 when GDP grows at less than 1.5% and 0 otherwise. The univariate GDPI results show a somewhat significant relationship with the appropriate negative sign (Table 13.2); however, when one adds the "dummy" variable GDPI to the multivariate models discussed above, the results (not reported) show no statistically significant effect, although the sign remains appropriate.

We also checked whether the relationship between default rates and recoveries outlined in Table 13.4 experiences a structural change depending on the economy being in a "good" or "bad" regime. To do so, we re-estimated our multivariate regressions and reported the results in the table after removing recession years (simply defined as years showing a negative real GDP growth rate); the results (not reported) confirmed the results shown in Table 13.4. This suggests that our original estimates are not affected by recession periods.

13.3.6.7 Seniority and Original Rating

Our study considers default rates and recoveries at an aggregate level. However, to strengthen our analysis and get some further insights on the PD/LGD relationship, we also considered recovery rates broken down by seniority status and by original rating.

Table 13.6 shows the results obtained, on such data, for our basic univariate, translog model. One can see that the link between recovery rates and default frequencies remains statistically significant for all seniority and rating groups. However, such a link tends to be somewhat weaker for subordinated bonds and for junk issues. Moreover, while the

* This amounts to 2 years out of 20, namely 1987 and 1997.

TABLE 13.6 Univariate Regressions on Data Broken by Seniority Status and Original Rating

Dependent variable	**Data Broken by Seniority Status**		**Data Broken by Original Rating[c]**	
	RR on Senior[a] Bonds (log)	**RR on Subordinated[b] Bonds (log)**	**RR on Investment-Grade Bonds (log)**	**RR on Junk[d] Bonds (log)**
Explanatory variables: coefficients and *t*-ratios				
Constant	2.97	2.58	2.70	2.70
	(13.91)	(8.79)	(9.77)	(10.99)
BLDR	−0.24	−0.23	−0.31	−0.23
	(−4.30)	(−2.90)	(−4.16)	(−3.41)
Goodness of fit measures				
R-square	0.507	0.319	0.519	0.421
Adjusted *R*-square	0.479	0.281	0.489	0.385
F-Stat	18.487	8.431	17.291	11.652
(*P*-value)	0.000	0.009	0.001	0.004
Residual tests				
Serial correlation LM, 2 lags (Breusch–Godfrey)	0.6674	1.66	4.0302	0.162
(*P*-value)	0.5268	0.2213	0.0415	0.852
Heteroscedasticity (White, Chi square)	0.719	0.2001	0.4858	0.4811
(*P*-value)	0.5015	0.8205	0.6245	0.6273
Number of observations	20	20	18	18

Note: A set of univariate regressions is shown, based on recovery rates broken down by seniority status and by original rating. The link between recovery rates and default frequencies remains statistically significant for all seniority and rating groups. However, it grows weaker for subordinated bonds and for junk issues. Moreover, recovery rates on investment-grade bonds seem to react more steeply to changes in the default rate.

[a] Senior-secured and senior-unsecured bonds.

[b] Senior-subordinated, subordinated, and discount bonds.

[c] Years 1993 ad 1994 not included because no default took place on investment-grade issues.

sensitivity of BRR to the default rate looks similar for both seniority classes, recovery rates on investment-grade bonds seem to react more steeply to changes in the default rate. In other words, the price of defaulted bonds with an original rating between AAA and BBB decreases more sharply as defaults become relatively more frequent. Perhaps the reason for this is that original issue investment-grade defaults tend to be larger than noninvestment-grade failures and the larger amounts of distressed assets depresses the recovery rates even greater in difficult periods.

13.4 IMPLICATIONS FOR CREDIT VaR MODELS, CAPITAL RATIOS, AND PROCYCLICALITY

The results of our empirical tests have important implications for a number of credit risk related conceptual and practical areas. This section reviews two key areas that can be

significantly affected when one of the factors in that default rates is, in fact, negatively correlated with recovery rates. These are (1) credit VaR models and (2) the potential impact of our findings on the procyclicality of capital requirements debated by the Basel Committee.*

13.4.1 VaR Models

Most credit VaR models treat recovery rates as deterministic (like in the credit risk+ model proposed by Credit Suisse Financial Products 1997) or stochastic but independent from default probabilities (like in the Creditmetrics framework: Finger et al. 1997). The impact of a negative correlation between recovery rates and default rates is generally overlooked. In order to assess this impact, we ran Monte Carlo simulations on a sample portfolio of bank loans and compared the key risk measures (expected and unexpected losses) obtained by the two above mentioned models to those generated when recovery rates are treated as stochastic and negatively correlated with PDs.

The results of our simulations are revealing, indicating that both the expected loss and the unexpected loss are vastly understated if one assumes that PDs and RRs are uncorrelated.† As long as the PDs used in VaR models can be thought of as an *ex ante* estimate of actual DRs, this implies that the risk measures generated by such models are biased.

Summing up, if default rates (and PDs, which can be thought of as *ex ante* estimates of actual DRs) are found to be correlated with RRs, then not only the risk measures based on standard errors and percentiles (i.e., the unexpected losses) could be seriously underestimated, but the amount of expected losses on a given credit portfolio (on which banks' provisioning policies should be based) could also be misjudged. Therefore, credit models that do not carefully factor in the negative correlation between PDs and RRs might lead to insufficient bank reserves and cause unnecessary shocks to financial markets.

13.4.2 RR/PD Link and Procyclicality Effects

Procyclicality involves the sensitivity of regulatory capital requirements to economic and financial market cycles. Since ratings and default rates respond to the cycle, the new internal ratings-based (IRB) approach proposed by the Basel Committee risks increasing capital charges, and limiting credit supply, when the economy is slowing (the reverse being true when the economy is growing at a fast rate).

Such procyclicality effects might be thought to be exacerbated by the correlation between DRs and RRs found in our study (and in some of the contributions quoted in Section 13.1); in other words, low recovery rates when defaults are high would amplify

* We will simply summarize here our conclusions based on several simulation analyses, discussed in greater detail in Altman et al. (2001).

† Both expected losses and VaR measures associated with different confidence levels tend to be underestimated by approximately 30%.

cyclical effects. This would result from the fact that a negative correlation between default rates and recovery rates would lead to more sensitive capital requirements. For example, in a recession period with increasing default rates, recovery rates would decrease leading to higher credit losses. This would in turn lead to higher capital requirements and, correspondingly, possibly to a decrease in the supply of bank credit to the economy, thereby exacerbating the recession. On the other side, in a strong economic growth period with decreasing default rates, recovery rates would increase leading to lower credit losses and lower bank capital requirements. This would in turn allow an expansion of bank credit, thereby favoring economic growth.

This procyclicality effect would especially be true under the so-called advanced IRB approach, where banks are free to estimate their own recovery rates and might tend to revise them downwards when defaults increase and ratings worsen.

The impact of such a mechanism was assessed, for example, in Resti (2002), based on simulations over a 20 year period, using a standard portfolio of bank loans (the composition of which is adjusted through time according to S&P transition matrices). Two results of these simulations are worth mentioning. First, the procyclicality effect is driven more by up- and downgrades, rather than by default rates; in other words, adjustments in credit supply needed to comply with capital requirements respond mainly to changes in the structure of weighted assets, and only to a lesser extent to actual credit losses (except in extremely high default years). Second, when RRs are permitted to fluctuate with default rates, the procyclicality effect increases significantly. Moreover, bank spreads, too, become more volatile, since revisions in short-term RR estimates are factored into loan prices.

One might object that in these simulations banks basically react to short-term results, and that regulation should encourage advanced IRB systems to use long-term average recovery rates. However, while the use of long-term RRs would make procyclicality effects less marked, it would also force banks to maintain a less updated picture of their risks, thereby trading stability for precision.

13.5 CONCLUDING REMARKS

This chapter analyzed the link between aggregate default rates/probabilities and the loss given default on corporate bonds, both from a theoretical and an empirical standpoint. As far as the theoretical aspects are concerned, most of the literature on credit-risk management models and tools treats the recovery rate variable as a function of historic average default recovery rates (conditioned perhaps on seniority and collateral factors), but in almost all cases as independent of expected or actual default rates. This appears rather simplistic and unrealistic in the light of our empirical evidence.

We examined the recovery rates on corporate bond defaults, over the period 1982–2002, by means of rather straightforward statistical models. These models assign a key role to the supply of defaulted paper (default rates) and explain a substantial proportion of the variance in bond recovery rates aggregated across all seniority and collateral levels.

These results have important implications for portfolio credit-risk models, for markets which depend on recovery rates as a key variable (e.g., securitizations, credit derivatives, etc.), and for the current debate on the revised BIS guidelines for capital requirements on bank assets.

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CHAPTER 14

Business and Financial Indicators: What Are the Determinants of Default Probability Changes?

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14.1 INTRODUCTION

Understanding and modeling the common dynamics of default probabilities is a key concern for practitioners and academics. Probability changes have a significant impact on the prices of credit risky instruments, and also on the way credit risk should be managed. Regulation issues especially through economic capital computations, as well as large variations in default rates over the last 5 years have renewed interest in this topic. Common dynamic changes in probabilities have mainly been investigated using historical rating migration matrices or, from pricing perspectives, by modeling credit quantities (in particular credit spreads). It turns out that significant differences in probability levels have been observed through expansion and recession periods (Nickell et al. 2000).

Researchers have concentrated on modeling and forecasting these changes either by filtering out unobservable common drivers (Duffee 1999 or Driessen 2005 under the pricing measure, Koopman et al. 2008 or Wendin and McNeil 2006 under the historical measure) or using stock and bond markets factors (Janosi et al. 2002). These approaches find their origins in structural models (firm-value based) and reduced-form models (intensity based). The former are closely linked to financial markets as they model default as the first time at which a firm asset value falls below its liabilities. But they regularly fail to provide accurate probability estimates. This is essentially due to structural models heavily relying on stock market components which are noisy and contain bubbles, as pointed out by Janosi et al. (2002). Another reason for this failure is that some parameters, such as the asset volatility, are difficult to observe and calibrate. The intensity-based methodology is

more convenient for pricing and provides simple extensions to multiple assets (Duffie and Singleton 2003).* However, since dynamics are exogenously specified, this approach ignores what the default mechanisms are and leads to disappointing predictive power as well. It is worth noticing that firm-specific variations have also been documented. This last stream of studies relies on different modeling techniques, namely credit scoring or hazard rate models (Duffie et al. 2007) which will be used in this chapter.

A key issue lies in the identification of the systematic drivers of default probabilities. Some studies observed surprisingly low sensitivities of spreads to interest rate factors or to stock market variables (Collin-Dufresne et al. 2001), but explanations are lacking. This chapter contributes to the existing literature by considering a new set of potential determinants that explicitly link the default process to economic fundamentals. This study shows that including all economic components provides more accurate measures than those obtained from financial market factors. For the first time, the critical roles of business cycle related indicators as well as aggregate credit market factors are identified and quantified. An additional contribution lies in the use of past information, jointly with contemporaneous one. We show that large benefits can be obtained by taking into account lagged effects between the economy and the default cycle. The empirical part of the chapter relies on a unique rating event database that captures more precisely the default probabilities, whereas the previous body of evidence is based on a noisy measure of these probabilities, namely the credit spreads.

We believe there are four main contributions to this chapter. First, we analyze and quantify the impact of financial markets, business cycle indicators, and credit market indicators on default probabilities of various rating classes.

Second, we show that although all economic variables help explaining default likelihoods, their explanatory power differs greatly between investment-grade (IG) and non-investment-grade (NIG) classes. As issuer quality decreases, the dominant systematic factors change from financial markets to business cycle and endogenous default indicators. Relying solely on financial information to explain default probabilities leads to poor results for NIG firms. Default peaks are underestimated, whereas default probabilities predicted by the model overshoot realized probabilities during stable and low-default periods.

Third, we establish the critical importance of combining past information with contemporaneous factors. We show that default probability changes are the joint effects of past shocks and subsequent economic trends. Resulting lead–lag effects between the economy and the default cycle partially explain the lower speed and higher persistence of the default cycle.

Fourth, we introduce a semi-parametric framework which explicitly takes into account the term structure of hazard rates. Firm ageing effects as well as calendar-time effects are explicitly taken into account. Models are set up in continuous time using the natural time-to-default framework and taking care of censoring. In particular, default probabilities can consistently be computed at all horizons. A specification test is also introduced.

* We abusively use indifferently the terms of hazard rates and intensities in this chapter. We only point out the differences when outlining our framework.

The chapter is organized as follows. Section 14.2 briefly presents the default data. Section 14.3 presents our modeling framework. We then propose a list of potential default determinants that will be investigated in Section 14.4. We extensively study sensitivities of the default cycle with respect to financial markets, business, and endogenous indicators through conditional single-factor hazard models in Section 14.5. Section 14.6 gathers main results and their implications looking at multifactor models. We check mis-specifications of financial market-based models, and assess the benefits of considering other economic components. Our results deliver clear empirical guidelines for modeling. They rely on analyses of predictive variables of default probabilities exploiting the history of companies in the rating process. This study opens new avenues to investigate the intensity-based pricing and the management of default risk. It more particularly sheds light on relevant systematic factors for the construction of realistic stress tests of credit portfolios in the Basel II framework.

14.2 DEFAULT DATASET

We extract information on times-to-default from the Standard & Poor's (S&P's) CreditPro 6.6 rating database. As the first goal of ratings consists of providing a cross-sectional ranking of firms with respect to their default likelihood, they allow to classify firms into homogenous classes of default risk. The database contains S&P's rating histories for 10,439 companies over the period January 1981 to December 2003. The CreditPro database has already been used and extensively described by Bangia et al. (2002) and Couderc (2005). Overall 33,044 rating migrations are recorded in CreditPro as well as 1386 defaults and default rate ranges from 3% to 29% across industries. Within our sample, firms are classified by industrial groups distributed among 93 countries; 6897 firms or 66% are U.S. ones. Moreover, S&P attributes 25 distinct ratings plus the not-rated (NR) one, but we aggregate the data coming from a grade and its plus/minus modifiers because of minimal population requirements. Besides, all grades below B have been put in the CCC class.

Rating events require careful treatment as three sources of censoring are present in the database. The first type of right censoring is an inherent feature of any rating database as most companies survive after the end of the recordings. Another type of right censoring requires specific consideration. Some companies leave the rating process and fall into the NR category. Several reasons may explain this fact: the rated company may be acquired by another firm or may simply decide no longer to be rated by S&P. In the database, we can identify firms that migrated to NR and subsequently defaulted. Therefore, the NR class is not a complete loss of information: although there is no longer any indication of credit quality, an NR firm is a nondefaulter. Finally, left censoring arises from 1371 issuers having already received a rating before they were included in the database (i.e., before January 1981). We do not have information about the attribution date of their first rating and therefore, for robustness checks, we run all estimations both on the full sample and on the reduced sample excluding left-censored data (the reduced sample contains 9068 companies and 25,993 rating migrations). It is worthwhile to notice that these left-censored issuers faced the 1991 and 2001 U.S. recessions. The number of issuers tracked by the database

increases linearly. Consequently, one third of the total number of issuers were active on average for 8 years in 1991 and 18 years in 2001, and another one third only faced the 2001 recession after an average of 4 years in the rating process.

The database allows to consider two types of durations, implying two different approaches to the behavior of default probabilities. On the one hand, we can look at times-to-default from entry in a risk class up to the last available observation. Such durations provide a picture of default riskiness over the whole life of the firms without any assumption on the rating process behavior. On the other hand, we can examine times-to-default conditional on staying in a given risk class up to the default time. In this case, it would require assumptions on rating migration dynamics so as to design the default riskiness of a firm over the long term.

14.3 MODELS OF DEFAULT PROBABILITIES

Hereafter, we develop a powerful framework to analyze impacts of systematic factors. A straightforward practice to study the effects of structural factors on a given variable consists in performing regressions. Investigations of spreads follow this approach but it cannot be used for probability changes because hazard rates or equivalently instantaneous default probabilities are not directly observable. Since hazard rates fully characterize a default time distribution, modeling hazard rates leads to specifying the conditional distribution of times-to-default, and all parameters can be estimated in a single stage. We rely on the standard way to construct hazard models, with log-intensities that are linear in the factors. We briefly recall the basic parametric framework before turning to our more complete semi-parametric framework.

14.3.1 Factor Models of Intensities

For a firm i in a given risk class, let D_i denote the uncensored duration up to default, or time-to-default, and C_i the censored duration. $U_i = \min(C_i, D_i)$ is the time at which the firm leaves the class either because of censoring (C_i) or default (D_i). The U_i are therefore the true observations. We also let $\mathbf{Z}$ and u denote respectively a vector of explanatory variables and the time-to-default or ageing time, while t_i denotes the date at which a firm i enters into the class. Hence, $u + t_i$ represents calendar time. We consider intensities λ^i as exponential affine functions which remain constant between two observations of the factors. Conditional on the realization of the factors, durations are exponentially distributed between factor updates:

$$\lambda^i(u) = \lambda(u, t_i) = \exp\left[\gamma + \beta' \mathbf{Z}(t_i, u + t_i)\right] \quad \forall i \tag{14.1}$$

where $\mathbf{Z}$ can include a combination of time-dependent and time-independent covariates. The exponential assumption could be relaxed by replacing the constant e^{γ} by another formulation. For instance, one could impose a conditional Weibull hazard model. The survival probability for a firm i beyond the time $u + t_i$ can then be retrieved as $P(U > u|t_i) = \exp\left[-\int_0^u \lambda(u, t_i)\mathrm{d}u\right]$.

This parametric framework allows us to use maximum likelihood to efficiently estimate $\widehat{\beta}$. Details of maximum likelihood estimation in this context are provided in Appendix. The estimation of such models is therefore straightforward and both censored and uncensored durations contribute to the likelihood.

The accuracy of the above estimator depends on the proper specification of the model, on the selection of explanatory variables, and on time-to-default being conditionally exponentially distributed. However, some empirical studies have demonstrated that these simple models are mis-specified even if they successfully take into account all systematic factors. Fledelius et al. (2004) and Couderc (2005) have shown that economic changes create bumps in hazard rates and have established that the distribution of times-to-default is not exponential (i.e., the hazard is not constant with u). In the remainder of this chapter, we propose to use a semi-parametric framework for hazard models which relaxes this exponential assumption. The nonparametric component of these models allows us to extract the baseline hazard, which is the true conditional distribution of times-to-default which characterizes the whole term structure of hazard rates. The parametric part of the models reflects shocks to the hazard rates because of changes in economic and financial factors. Once these shocks have been accounted for, the remaining bumps in the baseline hazard indicate the proportion of the default cycle which is not captured by the factors. If the factors were able to fully describe the patterns of historical times-to-defaults, then the baseline hazard would be constant and we would fall back on the class of simple models described above.

14.3.2 Semi-Parametric Framework for Intensities

In our semi-parametric setting, calendar-time effects, or factor impacts, can easily be separated from pure duration or life cycle effects on hazard rate deformations. To build our setup, we start from a fully nonparametric estimator of hazard rates and add a multiplicative parametric component (Cox 1972, 1975 proportional hazard methodology). The nonparametric baseline hazard is estimated using the GRHE (gamma Ramlau-Hansen estimator) as the solution of the maximum likelihood objective function, while the parametric part is estimated by partial likelihood.

14.3.2.1 Basic Estimator

The GRHE introduced by Couderc (2005) is based on a convenient gamma kernel smoother of the hazard rate and allows to recover the term structure of hazard rates. Standard smooth estimators suffer from large bias and oversmoothing, which lead to incorrect inferences on hazard rates. The GRHE however is free of boundary bias and is able to capture changes in intensities in the short run (which may cover up to 5 years) as well as subsequent deformations.* Not using an unbiased estimator would lead to strongly inaccurate estimations and assessments of hazard models.

* This feature has already been widely documented in the case of density function estimation. Standard smoothers are biased because of the inadequacy between the domain of definition of the kernel and one of the data. For instance, see Chen (2000).

The necessary conditions ensuring the consistency of the estimator are assumed to be met. Accordingly, all firms in a given risk class are assumed to be homogenous and conditionally independent. Censoring mechanisms which may prevent from observing firms up to their default time are random and independent from the default process. For a given firm, these mechanisms are reported through a process $Y^i(u)$, which equals one only if the firm was observed at least up to the time-to-default u. The most important building block of the GRHE then lies in the following assumption.

Assumption 14.1. *The intensity of individual firms satisfies the multiplicative intensity model:*

$$\lambda^i(u) = \alpha(u)Y^i(u) \tag{14.2}$$

where $\alpha(u)$ is deterministic and called the hazard rate, whereas $Y^i(u)$ is a predictable and observable process.

The difference between the intensity and the hazard rate resides in their observability The estimator of the hazard rate is specified as follows.

Definition 14.1. *The gamma kernel estimator $\widehat{\alpha}(u)$ of the hazard rate (GRHE) is defined by*

$$\widehat{\alpha}(u) = \int_0^\infty \frac{1}{Y(s)} \frac{s^{u/b}\mathrm{e}^{-s/b}}{b^{u/b+1}\Gamma\left(\frac{u}{b}+1\right)} \mathrm{d}N_s \tag{14.3}$$

where

$\mathrm{d}N_s$ *counts the number of defaults occurring at time s*

$Y(u)$ *is computed as the number of firms for which the last time of observation is greater than u*

$Y(u)$ handles censoring and is the sum of the $Y^i(u)$ over i. It is usually called the risk set. b is a smoothing parameter, the so-called bandwidth. The intuition behind the above nonparametric estimator is the following: probabilities of default in the very short run $[\alpha(u)\,\mathrm{d}u]$, knowing survival up to time u, are estimated as a weighted average of past (if any), current, and subsequent observed instantaneous default frequencies $\left[\frac{\mathrm{d}N_s}{Y(s)}\right]$. The weights are determined by the kernel, the bandwidth, as well as by the durations between u and observed default events.

The restrictions imposed on the process $Y(u)$ are sufficiently weak to permit more complex specifications of this process. In what follows, we rely on this multiplicative intensity model to characterize economic shocks on time-to-default distributions.

14.3.2.2 Factor Models for the Term Structure of Hazard Rates

Let **Z** denote a vector of factors. We enrich standard factor models of hazard rates from Section 14.3.1 by relaxing the conditional exponential assumption, or equivalently by not constraining the term structure of hazard rates. This is done by using a semi-parametric framework. As discussed previously, the parametric part reflects the impact of economic

and financial factors on default intensities, whereas the baseline hazard or conditional distribution of times-to-default is estimated through a slight modification of the GRHE.

Assumption 14.2. *The intensities conditional on structural factors are proportional to a class baseline intensity* $\lambda^{\circ}(u)$ *representing the common intensity shape*:

$$\begin{aligned}\lambda(u,\, t_i) &= \lambda^{\circ}(u)\exp\left[\beta'\mathbf{Z}(t_i,\, u+t_i)\right] \quad \forall i \\ &= \alpha^{\circ}(u)Y^i(u)\exp\left[\beta'\mathbf{Z}(t_i,\, u+t_i)\right] \quad \forall i \end{aligned} \tag{14.4}$$

where $\mathbf{Z}(t_i, u+t_i)$ *is the set of structural variables taken at the date of entry* t_i *of the firm i in the class or at calendar time* $u+t_i$, *and* β *is the vector of sensitivities associated with a given risk class.*

Provided that structural variable dynamics are not explosive, the gamma kernel estimator of the baseline hazard $\alpha^{\circ}(u)$ is as follows.

Corollary 14.1. *Under Assumptions 14.1 and 14.2, a semi-parametric estimator of the baseline hazard function is given by*

$$\widehat{\widehat{\alpha}}^{\circ}(u) = \int_0^{\infty} \frac{1}{\widehat{\widehat{Y}}(s)} \frac{s^{u/b}\mathrm{e}^{-s/b}}{b^{u/b+1}\Gamma\left(\frac{t}{b}+1\right)} \mathrm{d}N_s \tag{14.5}$$

$$\widehat{\widehat{Y}}(s) = \sum_i Y^i(s)\exp\left[\widehat{\widehat{\beta}}'\mathbf{Z}(t_i,\, u+t_i)\right] \tag{14.6}$$

A convenient feature of this model is that an estimate $\widehat{\widehat{\beta}}$ of the sensitivities β can be derived separately from the baseline intensity through Cox partial likelihood:

$$\widehat{\widehat{L}} = \prod_{k=1}^{n} \frac{Y^k(u_k)\exp\left[\beta'\mathbf{Z}(t_k,\, u_k+t_k)\right]}{\sum_i Y^i(u_k)\exp\left[\beta'\mathbf{Z}(t_i,\, u_k+t_i)\right]} \tag{14.7}$$

where

u_k is the observed duration of firm k

t_k is its date of entry in the class

A complete survey of related models, estimation techniques, and asymptotic can be found in Andersen and Gill (1982) or Andersen et al. (1997).

This two-stage estimation technique does not affect the nonparametric estimation of the baseline intensity as the speed of convergence of the partial likelihood estimator is of order $\frac{1}{\sqrt{n}}$ and therefore higher than that of the kernel estimator. The parametric factor model from Equation 14.1 corresponds to the case where the constraint $\alpha^{\circ}(u) = \mathrm{e}^{\gamma}$ has been imposed. The estimators $\widehat{\beta}$ and $\widehat{\widehat{\beta}}$ converge to the same parameter β, but $\widehat{\widehat{\beta}}$ has a slower rate of convergence to its asymptotic distribution since it does not use the full

likelihood.* Corollary 14.1 is a direct consequences of Andersen et al. (1997) and Couderc (2005), $\widehat{\widehat{\alpha}}^{\circ}$ being the maximum likelihood estimator of the baseline hazard.

The relative performance of the various factors can be tested by comparing the hazard rate $\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}$ of the fully nonparametric specification ($\beta = 0$) and the baseline hazards $\widehat{\widehat{\alpha}}^{\circ}$ of semi-parametric models. The lower $\widehat{\widehat{\alpha}}^{\circ}$ is with respect to $\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}$, the higher the proportion of changes in default hazard rates captured by the factors. Moreover, the comparison between $\widehat{\widehat{\alpha}}^{\circ}$ and the constant hazard $e^{\widehat{\gamma}}$ of equivalent parametric models allows us to check mis-specifications of standard hazard rate models. More precisely, we can observe deviations from the conditional exponential hypothesis and their distributions through the term structure of times-to-default.

14.4 POTENTIAL DETERMINANTS OF DEFAULT

In this section, we discuss some potential common determinants of default intensities that will be tested in our model. Mainstream models that attempt to explain risk-neutral default probabilities are usually calibrated on financial variables: interest rates and equity information. Another stream of literature focuses on long-term economic and credit cycles, and relies mainly on macroeconomic variables to explain default rates. The business cycle has for example been factored in time-series analyses of procyclicality with the bankruptcy cycle (Koopman and Lucas 2004). To our knowledge, there exists no systematic study of the common determinants of default including both financial and nonfinancial variables, and we will attempt to fill that gap here. Given that our sample is primarily American, we use U.S. explanatory variables. Our economic data were extracted from the Federal Reserve of St. Louis Web site and Bloomberg. All factors are annualized, deseasonalized, and updated monthly or quarterly.

14.4.1 Financial Market Information

The stock and bond markets are sources of information used both by structural (i.e., firmvalue based, la Merton) and reduced-form models. Reduced-form models typically require to design a stochastic process of interest rates. For instance, Duffee (1999) relies on a two-factor interest rate model to extract default intensities. In a structural approach, the volatility and the return of the firm asset determine how close a firm is to its liability barrier which represents the default threshold. Cremers et al. (2006) show that individual stock return and volatility are significant determinants of spreads. Using an aggregate measure, Janosi et al. (2002) and Collin-Dufresne et al. (2001) also establish the impact of the S&P500 index and the VIX (volatility index) on credit spreads, respectively. The latter further claim that using individual stock volatility rather than index volatility does not modify their results. We consequently test the following factors on hazard rates:

1. Annual return on S&P500: As a measure of asset levels, the higher the stock return, the higher the distance to default should be. An increase in equity prices tends to

* We checked this last point on the subsequent multifactor models. We first found that all sensitivities keep the same signs at the same horizons when switching from a parametric factor model to a semi-parametric specification. Small variations in magnitude can be observed between $\widehat{\beta}$ and $\widehat{\widehat{\beta}}$. However, using bootstrap technique on the distance $\left\|\widehat{\widehat{\beta}} - \widehat{\beta}\right\|_2$, we could never reject the null hypothesis ($\widehat{\widehat{\beta}} - \widehat{\beta}$) at a 95% confidence level.

decrease firm leverage and therefore pushes down default probabilities. Moreover, from an economic standpoint, short- and mid-term economic performances should be positively correlated with S&P500's returns. We expect a negative impact on default intensities (i.e., intensities should be a decreasing function of the factor).

2. Volatility of S&P500 returns: In a traditional Merton (1974)-type model, the two drivers of default probabilities are leverage and the volatility of firms assets. The implied volatility of equity returns is often used as a proxy for the latter. We use the realized annualized volatility computed over 60 trading days. We expect it to have a positive impact on default intensities.
3. Ten year treasury yield: Higher interest rate levels imply higher cost of borrowing. Hence, this variable could impact positively on default probabilities. However, interest rates tend to be lower in contraction periods and higher in expansions. The ultimate impact on intensities is therefore uncertain and may depend on issuer quality.
4. Slope of term structure (10 year rate minus 1 year rate): Steep-term structures of interest rates are usually associated with strong growth prospects. It can also reflect expectations of higher future spot rates. We expect this variable to impact negatively on mid- to long-term intensities.

14.4.2 Business Cycle

We believe that it is crucial to extract information from the business cycle. If stocks were available for all firms and markets were fully efficient, financial market and business cycle variables might be redundant. As mentioned, the return on the market index does not constitute a perfect proxy for the state of the economy. To complement financial variables, Fons (1991) regresses default rates on the GDP (gross domestic product) growth, and Helwege and Kleiman (1997) add the NBER (National Bureau of Economic Research) economic indicator. These business cycle indicators explain 30% of the annual default probabilities. We will use these variables in our estimations and will also include the personal income growth as in Duffie et al. (2007).

1. Real GDP growth: As a signal of current macroeconomic conditions, this variable should be negatively correlated with short-term probabilities.
2. Industrial production growth: This is an alternative growth measure which should have a similar impact as that of GDP growth. Its advantage over GDP growth is that it offers more frequent updates (monthly versus quarterly).
3. Personal income growth: It has the same expected impact as the previous two variables. This business factor is more volatile and should consequently be less persistent. This indicator could also convey some slightly lagged information on past business conditions.
4. CPI (consumer price index) growth: Inflation is again a general indicator of economic conditions. We expect to observe a negative correlation with short-term default probabilities, as high inflation has often been associated with growth.

14.4.3 Credit Market Information

In addition to general economic variables and financial information, more specific credit factors should prove valuable in explaining default intensities. Although corporate bond spreads do not only reflect changes in underlying default probabilities (Ericsson and Renault 2006), they should still contain some forward-looking information on default probabilities. Moreover, when spread variations are due to changes in the default risk premium, they involve changes in expectations of future economic conditions. Spread factors may therefore be more persistent than other market factors.

1. Spread of long-term BBB bonds over treasuries: They reflect future default probabilities, expected recoveries, as well as default and liquidity premia. It should therefore be positively correlated with default intensities.
2. Spread of long-term BBB bonds over AAA bonds: This variable factors in the risk aversion of investors and may be a measure of their risk forecast. It filters out mixed effects contained into the BBB spread. Furthermore, an increase in the relative spread may reflect an increase in firms asset volatilities (Prigent et al. 2001). We therefore expect default intensities to increase with this variable.
3. Net issues of treasury securities: This indicator should positively impact short-term probabilities of default as higher deficit and borrowing is an indicator of economic difficulties (it is at least negatively correlated with the business cycle). Furthermore, high public sector borrowing may crowd out private borrowers and lead to increased financial difficulties for firms. However, if borrowing is used for investments, an increase in treasury issuance may be linked to stronger growth in the long term and decreasing probabilities of default.
4. Money lending (M2 − M1) and bank credit growth: These factors measure credit liquidity and should be associated with default intensities. It is well known that the information content of this indicator and more particularly of M2 has changed a lot over our sample period (series of adjustments have been done by the Federal Reserve). As a consequence, this indicator cannot be conclusive in the short run, but its implications in the long run turn out to be fairly stable. We thus expect clearer impacts when using lags.

14.4.4 Inner Dynamics of the Default Cycle

A striking feature of the default cycle might not be captured by the above variables. After the last two recessions, strong persistence in default rates has been observed. The number of defaults remained high even during economic recoveries. The default cycle seems to exhibit its own dynamics. We believe that the set of predictive variables should be expanded with default-endogenous variables. Kavvathas (2000) used the weighted log upgrade–downgrade ratio and the weighted average rating of new issuers as explanatory variables. He actually only took into account the first PCA (principal components analysis) factor of these variables. The average rating of financial institutions may be of primary interest in describing the short-term trend of the global economy (in terms of credit crunch). This trend can also be captured by the ratio of downgrades over all nonstayer

transitions. For instance, Jónsson and Fridson (1996) show that the credit quality of speculative issuers explains a large proportion of annual aggregate default rates. As representative of the default cycle trend, we choose to include the following rating-based variables:

- IG and NIG upgrade rates: Both variables should include information on economic health.*
- IG and NIG downgrade rates: Downgrades should be higher in bad conditions. Differences between upgrade and downgrade impacts should capture a potential asymmetry in the default cycle.

14.4.5 Statistics

Table 14.1 presents basic statistics on the set of retained factors. Obviously, some of the above variables such as real GDP growth and industrial production growth are highly correlated, which would deteriorate statistical significance on the full set of variables. Many of these factors are redundant, and will be eliminated at the estimation stage in multifactor analysis. Taking a pragmatic approach, we will first identify the most relevant factors and then construct parsimonious multifactor models.

TABLE 14.1 Statistics on Covariates

(%)	Mean	Minimum	Maximum	Volatility	Skewness	3 Year Autocorrelation
S&P500 return	9.3	−32.4	43.9	15.8	−0.57	−12.2
S&P500 volatility	15.4	6.3	62.8	7.3	3.02	8.2
Treasury 10 year yield	7.9	3.3	15.3	2.8	0.84	−23.4
Term structure slope	1.3	−2.1	3.3	1.1	−0.19	−8.1
Real GDP growth	3.0	−2.8	8.1	1.9	−0.57	17.0
DIP growth[a]	0.2	−1.8	2.0	0.6	0.02	22.8
CPI growth	3.5	1.1	11.8	1.9	1.89	60.4
Personal income growth	6.0	1.5	13.4	2.3	0.67	−32.6
BBB yield	10.1	6.2	17.2	2.7	1.06	62.7
BBB spread	2.2	1.3	3.8	0.6	0.77	9.1
BBB–AAA spread	1.1	0.6	2.7	0.4	1.33	44.0
Treasury net issues	14.4	−39.1	74.8	16.3	−0.48	−12.6
Money lending growth	6.2	−3.9	12.8	3.9	−1.00	16.2
IG upgrade rate	0.5	0	2.0	0.3	1.29	9.3
NIG upgrade rate	1.0	0	10.9	0.9	5.90	6.5
IG downgrade rate	1.1	0	3.1	0.7	0.73	4.9
NIG downgrade rate	1.7	0	13.1	1.3	3.62	−5.4

Note: Basic statistics on retained factors. Figures are given on an annual basis, and in percentages (except for the skewness). All variables but upgrade and downgrade rates are U.S. indicators.

[a] DIP growth—Deseasonalized industrial production growth.

* The investment-grade class (IG) gathers the AAA, AA, A, and BBB classes, whereas BB, B, and CCC classes are collected in the noninvestment grade (NIG) class.

Although we do not explicitly include firm-specific factors, our approach does not fully rule out nonsystematic risk as we will examine rating classes separately. Ratings should indeed constitute stable and good proxies for firm-specific components and a fair alternative to specific variables, which are not always available or are not updated frequently. Leverage targeting, jointly with the way ratings are reviewed, justifies this point. From an accounting perspective, default cannot realistically be initiated by small changes in earnings, leverage, or any balance sheet information, but rather by negative trends or by unexpected large changes in cash flows. Any negative trend should have been incorporated in issuer ratings. One may argue that ratings do not react quickly enough to new information, but there is no consensus on this issue (Altman and Rijken 2004 or Löffler 2004) and we will assume that ratings capture significant changes in firm-specific components.

14.5 PREDICTORS AND INDICATORS OF THE DEFAULT CYCLE

In this section, we investigate time-dependent covariates which embed the impact of successive shocks of the economic environment on intensities: when the factor updates, hazard rates are shifted and the conditional distribution of times-to-default is updated too. We distinguish the impacts of current and past economic conditions on default time distributions. We examine the sensitivity of hazard rates to each factor to determine which factors are the most relevant. We also explore their persistency. Surprisingly, the issue of lagged information has been ignored in most works on default probabilities, although Koopman and Lucas (2004) have reported lagged effects between the market and the bankruptcy cycle.

We analyze the explanatory power of each factor through maximum likelihood estimations. For each covariate, we run distinct lagged estimations to examine the persistency of its effects. In all cases, we look at 95% and 99% confidence tests, and likelihood ratios. The alternative model of the likelihood ratio test (LR test) corresponds to unconditional exponentiality, i.e., the case of constant intensity. We also break our dataset into several samples, namely IG, NIG, AA, A, BBB, BB, B, and CCC samples.

Tables 14.2 through 14.4 present results on the broad IG and NIG samples respectively over financial, business, and credit indicators. In addition to sensitivities, implied percentage changes in the hazard rates are provided when the factor changes by minus or plus one standard deviation. These changes are not symmetric as the factor impacts the hazard rates through the exponential function. Table 14.5 reports estimates of sensitivities with respect to upgrade and downgrade rates over rating classes. To test the stability of the results, we have estimated the models on various subsamples and found that the sample used makes little difference in most cases. For example, considering durations up to the first exit from a risk class (rather than default) keeps sensitivities almost unchanged and only lowers the significance of parameters. Focusing solely on the U.S. subsample does not modify our estimates by more than 10% on average, and does not alter signs. Such robustness could be expected as risk classes are quite stable and the whole sample is made of 66% U.S. firms. From a general perspective, all included variables are significant. We observe that most

TABLE 14.2 Sensitivities with respect to Financial Markets Information

Lag	S&P500 Return	S&P500 Volatility	Treasury Yield	Term Structure Slope
Investment Grades				
−1M	−2.25**	3.36**	*−15.98***	*40.69***
	42.69/−29.92	−21.75/27.80	56.43/−36.07	−36.08/56.45
0M	−2.72**	3.80**	−15.33**	38.44**
	53.69/−34.93	−24.22/31.97	53.61/−34.90	−34.48/52.63
1M	*−2.85***	*3.99***	−15.35**	33.11**
	56.88/−36.26	−25.27/33.81	53.70/−34.94	−30.53/43.94
3M	−2.61**	3.50**	−13.90	28.29**
	51.04/−33.79	−22.55/29.11	47.58/−32.24	−26.74/36.50
1Y	−1.96**	2.59**	−11.07**	−3.56
	36.30/−26.63	−17.23/20.81	36.34/−26.65	3.99/−3.84
3Y	1.33**	3.85**	−11.75**	−20.84**
	−18.95/23.39	−24.50/32.45	38.96/−28.04	25.76/−20.49
Noninvestment Grades				
−1M	−1.89**	1.81**	−10.91**	*22.54***
	34.80/−25.8	−12.38/14.13	35.73/−26.32	−21.96/28.14
0M	−1.96**	2.10**	−10.41**	19.66**
	36.30/−26.63	−14.21/16.57	33.84/−25.28	−19.45/24.14
1M	*−2.05***	2.39**	−10.44**	16.93**
	38.25/−27.67	−16.01/19.06	33.95/−25.35	−16.99/20.47
3M	−2.01**	2.47**	−10.03**	10.98**
	37.38/−27.21	−16.50/19.76	32.42/−24.49	−11.38/12.84
1Y	−1.02**	1.99**	−7.05**	−19.00**
	17.49/−14.88	−13.52/15.64	21.82/−17.91	23.24/−18.86
3Y	1.66**	*3.60***	*−12.21***	−21.08**
	−23.07/29.99	−23.11/30.06	40.76/−28.96	26.10/−20.70

Note: Estimations of log-linear intensities $\lambda(u, t_i)$ on investment (IG) and noninvestment grades (NIG) with time-varying covariates over the whole sample up to last days of observation. The table displays sensitivities $\widehat{\beta}$ from univariate specifications $\lambda(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}(u + t_i)]$ where the default arrival is assumed to be piecewise exponential conditional on factor realizations. We consider lags from 5 years backward to 2 months forward, only main lags are reported. Constants $\widehat{\gamma}$ are not reported. * and ** stand for significance at 95% and 99% confidence level, respectively. For each factor, the lag offering the highest likelihood ratio has been stressed in italics. Other figures display percentage changes in the hazard rate for minus/plus one standard deviation changes in the factor.

lagged factors are significant at all stages too. In addition, these findings do not change across ratings. Estimated intercepts (γ) lie around −13.1 for IG and −8.8 for NIG. Results on lags between 2 and 5 years are similar than the ones of the 3 year lagged factors. These results are not reported here.

14.5.1 Influence of Financial Markets

Table 14.2 shows that financial markets impact default probabilities as predicted by structural models (lags up to 3 months). Increases in the market index decrease the

probability of default, while increases in volatility raise default probabilities. Increases in long-term interest rates are good news because they reflect growth expectations. These findings are consistent with the results of Duffee (1998) and Collin-Dufresne et al. (2001) on spreads.

Unlike the above authors, we find that the slope of the term structure is significant and has a large effect on hazard rates. In particular, Collin-Dufresne et al. (2001) show a significant and negative relationship only with long maturity bonds with reasonable leverage, whereas it becomes positive for short maturities. Table 14.2 explains this result. Decreases in short-term yield raise default probabilities as low rates are strongly correlated with recessions. A steep contemporaneous or recent slope of the term structure of riskless rates tends to be associated with higher intensities of default, while past steep slopes (over 1 year lags) tend to decrease intensities. Therefore, effects on bond spreads are the results of these conflicting phenomena, which dampens significance. Short maturity bonds are mainly affected by recent changes in slope, so that an increase in the slope leads to an increase in hazard rates and spreads. The only exception to this short-term/long-term interest rates split is for the CCC class, for which a steep slope is always associated with lower intensities, irrespective of the lags. This can be due to two reasons. First, low short-term interest rates can indicate a slowdown of economic activity and an increased competition in the corporate bond market. Second, increases in long-term interest rates are often interpreted as expectation of higher growth. Future growth is the main determinant of survival for junk issuers, as these companies are highly levered and require strong business conditions to move up the rating ladder.

Some of the studies of corporate bond spreads mentioned above indicate a higher impact of the market index than interest rates. Looking at implied percentage changes on the hazard rate, our results are more contrasted. For investment grades, a decrease in the long-term yield and an increase in the slope lead to shocks of the same order of magnitude than a decrease in the S&P500 return. Even if all market indicators have smaller effects on NIG, contemporaneous changes in the slope have lower impacts than changes in the S&P500 return, and impacts of the market index are a bit more persistent. It may explain why interest rates have usually been found to be less economically significant. The results on the impact of volatility are interesting and challenging for the structural models a la Merton on an aggregated pool of firms. We indeed find that volatility has a lower impact on hazard rates for NIG firms, which is the opposite of what is predicted by structural models. We will come back to this point later on.

14.5.2 Business Cycle Effects

The business cycle appears to have large effects on the default cycle. Of course, business expansion tends to decrease intensities, as found by Fons (1991) and Helwege and Kleiman (1997) for bankruptcy rates. Consistently with Duffie et al. (2007), the personal income growth (PIG) has larger impacts than real GDP growth. But Table 14.3 shows that GDP effects are more persistent and that the causality between the PIG and the changes in the hazard rate is unclear. From likelihood ratios, the results suggest that decreases in hazard rates imply future increases in the PIG. The effects of the PIG and the CPI growth

TABLE 14.3 Sensitivities with respect to Business Cycle Information

Lag	Real GDP Growth	Industrial Production Growth	CPI Growth	Personal Income Growth
Investment Grades				
−1M	−17.47**	−48.37**	−15.85**	*−27.13***
	39.37/−28.25	33.67/−25.19	35.14/−26.00	86.64/−46.42
0M	−19.23**	−38.40**	−15.08**	−26.74**
	44.10/−30.61	25.91/−20.58	33.18/−24.91	84.97/−45.94
1M	−20.81**	−46.55**	−12.77*	−25.14**
	48.50/−32.66	32.22/−24.37	27.46/−21.54	78.29/−43.91
3M	*−21.91***	*−57.27***	−9.85**	−22.16**
	51.63/−34.05	41.00/−29.08	20.58/−17.07	66.48/−39.93
1Y	−12.62**	−47.77**	−2.76	−9.72**
	27.10/−21.32	33.19/−24.92	5.38/−5.11	25.05/−20.03
3Y	15.32**	5.99	*−16.07***	−.40
	−25.25/33.79	−3.53/3.66	35.71/−26.31	.92/−.92
Noninvestment Grades				
−1M	−21.44**	−39.38**	−7.87**	*−19.82***
	50.28/33.46	26.65/−21.04	16.13/−13.89	57.75/−36.61
0M	−22.66**	−36.20**	−5.87**	−18.78**
	53.81/−34.98	24.26/−19.52	11.80/−10.55	54.02/−35.08
1M	*−23.01***	−50.57**	−4.47*	−17.26**
	54.84/−35.42	35.45/−26.17	8.86/−8.14	48.73/−32.77
3M	−22.58**	*−54.06***	−2.02	−13.16**
	53.58/−34.89	38.31/−27.70	3.91/−3.77	35.35/−26.12
1Y	−6.78**	−35.55**	.53	−.45
	13.75/−12.09	23.78/−19.21	−1.00/1.01	1.04/−1.03
3Y	11.71**	13.65**	*−15.50***	1.34
	−19.95/24.92	−7.86/8.53	34.25/−25.51	−3.03/3.13

Note: Estimations of log-linear intensities $\lambda(u, t_i)$ on investment (IG) and noninvestment grades (NIG) with time-varying covariates over the whole sample up to last days of observation. The table displays sensitivities $\widehat{\beta}$ from univariate specifications $\lambda(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}\ (u + t_i)]$ where the default arrival is assumed to be piecewise exponential conditional on factor realizations. We consider lags from 5 years backward to 2 months forward, only main lags are reported. Constants $\widehat{\gamma}$ are not reported. * and ** stand for significance at 95% and 99% confidence level, respectively. For each factor, the lag offering the highest likelihood ratio has been stressed in italics. Other figures display percentage changes in the hazard rate for minus/plus one standard deviation changes in the factor.

diminish as the credit quality lowers. Conversely, the GDP becomes more important for NIG, indicating a closer link with the global economic health. The positive skewness of the PIG and the negative skewness of the GDP (see Table 14.1) reinforce the relevance of the GDP. Remark that the PIG is strongly correlated with the long-term interest rate (74%) and with the slope of the term structure (−59%), and therefore with parallel shifts in the term structure.

Allowing for business cycle effects enables us to compare the relative usefulness of financial market information and business cycle information. Janosi et al. (2002) stress the

difficulty to extract default information from stock markets. Our implied percentage changes in the hazard rates are in line with this claim and shows that business information is more reliable. The lower volatility of business indicators makes 1% changes more economically significant: the sensitivity to the GDP is −22.66 for NIG, while the sensitivity to the S&P500 return is −1.96. Taking into account the variability of these factors, lowers the differences but still shows that the GDP is more important, especially for NIG firms: an increase by one standard deviation in the GDP reduces intensities by 34.98% for NIG, whereas an increase by one standard deviation in the S&P500 return reduces intensities by 26.63%.

14.5.3 Impacts of Credit and Default Factors

So far, we have not considered information from credit markets. Table 14.4 shows that credit information brings significant explanatory power, in particular through the BBB spread. Its effects are similar across ratings. Signs of the money lending variable confirm that a credit crunch amplifies defaults. Nevertheless, recent changes in net treasury issues and money lending have minor but statistically significant impacts on intensities. We argue that this is due to their weak short-term informational content. Interestingly, the BBB yield and the IG spread seem to carry little information on default.

Finally, rating trend indicators (upgrade/downgrade rates) appear as major explanatory components from Table 14.5. They express the persistency of the default cycle both in declines and recoveries. Studies of rating migrations (Nickell et al. 2000) have shown that the NIG downgrade ratio is highly correlated with increases in the number of defaults. Including such endogenous factors in multifactor models should be highly relevant as downgrade ratios are not much correlated with other factors (less than ±25%).

14.5.4 Persistency and Importance of Past Conditions

Lagged information provides further insights on the explanatory power and the time-span of economic shocks over the default cycle. We only report sensitivities to 3 year lagged factors in Tables 14.2 through 14.4 because the highest likelihood ratios have been found for this particular lag. Similar results are obtained with 2, 4, and 5 year lagged factors. The 1 year lagged factor bring intermediate results between short and long lags, but are not statistically significant.

With 3 year lags, half of the nonfinancial factors become insignificant for IG hazard rates, whereas all variables but the PIG remain significant for NIG. The 3 year lagged changes in money lending have large effects on hazard rates: a one standard deviation increase induces an increase of 60.24% in IG hazard rates. As expected, the money lending contains long-term information. The significance of the factors evidences the high degree of persistence of economic shocks on the firms likelihood of default. It bears major implications from a modeling standpoint as Markovian processes are unlikely to provide such features.

Besides, we find that some factors impact default probabilities differently in the long run. The S&P500, the term structure slope, the real GDP growth, and net treasury issues appear to be leading indicators of future peaks of defaults. It supports the claim that the

TABLE 14.4 Sensitivities with respect to Credit Markets Information

Lag	BBB Yield	BBB Spread	IG Spread	Treasury Issues	Money Lending
Investment Grades					
−1M	*−10.23***	75.08***	*50.38***	1.55***	2.54
	31.81/−24.13	−36.27/56.91	−18.25/22.33	−22.33/28.74	−9.43/10.41
0M	−9.65***	74.35***	44.63***	.99***	3.39
	29.76/−22.94	−35.99/56.22	−16.35/19.54	−14.90/17.51	−12.38/14.13
1M	−9.39***	*76.89***	41.38***	.90*	4.45*
	28.86/−22.39	−36.96/58.62	−15.25/18.00	−13.64/15.80	−15.93/18.95
3M	−8.38*	72.70***	35.73***	1.09***	5.32***
	25.39/−20.25	−35.35/54.68	−13.32/15.36	−16.28/19.44	−18.74/23.06
1Y	−7.51*	49.11***	−7.02	−.75*	9.34***
	22.48/−18.35	−25.52/34.27	2.85/−2.77	13.00/−11.51	−30.53/43.94
3Y	−8.60	41.57***	−13.48	*−2.41***	*12.09***
	26.14/−20.72	−22.07/28.33	5.54/−5.25	48.12/−32.49	37.59/60.24
Noninvestment Grades					
−1M	−5.76***	62.07***	*51.84***	.46***	4.88***
	16.86/−14.40	−31.09/45.12	−18.73/23.04	−7.22/7.79	−17.33/20.96
0M	−5.54***	59.09***	45.95***	.14	5.41***
	16.13/−13.89	−29.85/42.55	−16.79/20.18	−2.26/2.31	−19.02/23.49
1M	−5.45***	*60.12***	39.87***	−.02	6.00***
	15.85/−13.68	−30.28/43.44	−14.74/17.29	0.33/−0.33	−20.86/26.36
3M	−5.02***	59.68***	28.85***	−.26*	6.56***
	14.52/−12.68	−30.10/43.06	−10.90/12.23	4.33/−4.15	−22.57/29.15
1Y	−4.94***	28.91***	−13.84***	−1.61***	7.55***
	14.27/−12.49	−15.92/18.94	5.69/−5.39	30.01/−23.08	−25.51/34.24
3Y	*−9.70***	30.16***	−15.08***	*−1.76***	*11.01***
	29.94/−23.04	−16.55/19.84	6.22/−5.39	33.23/−24.94	−34.91/53.61

Note: Estimations of log-linear intensities $\lambda(u, t_i)$ on investment (IG) and noninvestment-grades (NIG) with time-varying covariates over the whole sample up to last days of observation. The table displays sensitivities $\widehat{\beta}$ from univariate specifications $\lambda(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}\ (u + t_i)]$ where the default arrival is assumed to be piecewise exponential conditional on factor realizations. We consider lags from 5 years backward to 2 months forward, only main lags are reported. Constants $\widehat{\gamma}$ are not reported. * and ** stand for significance at 95% and 99% confidence level, respectively. For each factor, the lag offering the highest likelihood ratio has been stressed in italics. Other figures display percentage changes in the hazard rate for minus/plus one standard deviation changes in the factor.

default cycle lags the economy (Koopman and Lucas 2004). The interesting point lies in the signs of these factors which come as warnings: expansion peaks of the financial market or of the business cycle seem to announce increases in the number of defaults 3 years later. It has to be taken with care as it could only represent the singularities of the global economy over the past 25 years and will not necessarily apply to the future. For example, early repayments and small levels of issues by U.S. Treasury signaled the peak of the U.S. cycle, which was later followed by a major default crisis. However, we stress that these lagged effects could also be significant because migration from distress to default takes time as reported by Altman (1989). From that perspective, the default cycle has to remain high

TABLE 14.5 Sensitivities to Aggregate Default Indicators

Default Factors/Class	IG	AA	A	BBB	NIG	BB	B	CCC
IG upgrade rate	−55.70**	−6.34	−16.69	−38.85	−68.35**	−42.94**	−68.57**	−70.05**
	18.19/−15.39	1.92/−1.88	5.13/−4.88	12.36/−11.00	22.76/−18.54	13.75/−12.09	22.84/−18.59	23.39/−18.95
NIG upgrade rate	−18.87**	−5.05	−7.65	−22.78	−30.30	−28.42	−33.66**	−41.23**
	18.51/−15.62	4.65/−4.45	7.13/−6.65	22.75/−18.53	31.35/−23.87	29.14/−22.57	35.38/−26.13	44.93/−31.00
IG downgrade rate	72.91**	38.52	53.03**	64.62**	50.22**	59.40**	50.42**	34.75**
	−39.97/66.59	−23.63/30.95	−31.01/44.95	−36.39/57.20	−29.64/42.13	−34.02/51.56	−29.74/42.32	−21.59/27.54
NIG downgrade rate	23.80**	22.43	24.38**	23.82**	24.91**	25.41**	26.21**	23.99**
	−26.61/36.26	−25.29/33.85	−27.16/37.29	−26.63/36.30	−36.48/57.43	−28.13/39.14	−28.87/40.59	−26.79/36.60

Note: Estimations of log-linear intensities $\lambda(u, t_i)$ with time-varying covariates over the whole sample up to last days of observation. The table displays sensitivities $\widehat{\beta}$ from univariate specifications $\lambda(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}(u + t_i)]$ where the default arrival is assumed to be piecewise exponential conditional on factor realizations. Constants $\widehat{\gamma}$ are not reported. * and ** stand for significance at 95% and 99% confidence level, respectively. Other figures display percentage changes in the hazard rate for minus/plus one standard deviation changes in the factor.

after economic recoveries, generating explanatory power for past conditions and persistency for economic shocks. We argue that lagged factors when used as supplementary information could at least help in capturing business and market trends, which constitute the essential information on future default probability.* Section 14.6.4 examines this issue.

14.6 EFFICIENT HAZARD MODELS ACROSS RATING CLASSES

In the previous section, we identified some major default determinants using univariate hazard models. We now turn to multifactor models. By doing so, we can first compare the pertinence and the complementarity of the various economic components. We show that explained variations in intensities could be severely underestimated because of inappropriate choices in the information set. For instance, leaving aside information provided by the business cycle can be damaging for the performance of the model. Second, we can identify relevant parametric conditional distribution and propose a specification test in finite samples. We finally show the critical importance of economic trends and past information.

14.6.1 Failure of Contemporaneous Financial Market Factors

Table 14.6 presents estimates of sensitivities $\widehat{\widehat{\beta}}$ for different specifications on the basis of contemporaneous financial predictors over rating classes. It enables us to test whether models based solely on stock prices or interest rates or on both (joint model) are sufficient to explain historical default intensities.

The table presents three multifactor models. For robustness checks, we included a dummy indicating non-U.S. firms. Sensitivities to this non-U.S. indicator were not significant. LR tests identify the joint model as the best one, whereas interest rates alone provide the poorest fits. This last result confirms findings of Driessen (2005) or Janosi et al. (2002) on credit spreads. Interestingly, stock market volatility is not always significant and its relative impact on default is minor with respect to other factors. Such a finding is highly challenging for structural models as the volatility determines the dynamics of equities and as a consequence default probabilities. However, BBB and BB classes are significantly affected by market volatility. This may be explained by the economic impact (in terms of increased funding cost) of a migration from IG to NIG. Some fund managers systematically rule out NIG corporate bonds from their portfolios: at the time of a downgrade from BBB, numerous funds closed their positions, resulting in a jump in credit spread and the cost of debt. The market volatility could be a good proxy for that kind of market segmentation behavior and consequently has to be a key indicator for these classes.

Figure 14.1 focuses on IG and NIG classes displaying baseline hazard rates for nonparametric (solid line), semi-parametric (solid bold line), and parametric counterparts (dashed lines). Unreported baseline hazards for the Interest Rates model show that our interest rate factors, when considered as a group, are unhelpful determinants of default across rating classes. Conversely, stock market information brings significant explanatory

* Indeed, from time-series cycle analysis between the GDP and the bankruptcy rates, Koopman and Lucas (2004) observe differences in magnitude and lengths. The default cycle being much smoother and persistent than the financial market or the business cycles, transitory shocks should not represent the most relevant information.

TABLE 14.6 Contemporaneous Financial Multifactor Models

Model (Factors/Class)	Stock Market	Interest Rates	Financial Markets	Stock Market	Interest Rates	Financial Markets
		IG			NIG	
S&P500 return	−2.26**		−1.40**	−1.87**		−1.62**
S&P500 volatility	2.35**		2.53**	.50*		.33
Treasury yield		−10.67**	−4.37**		−7.97**	−4.21**
Term structure slope		32.22**	24.87**		15.52**	4.09**
		AA			BB	
S&P500 return	−3.17**		−3.03*	−1.92**		−1.44**
S&P500 volatility	.03		.10	.76**		1.15**
Treasury yield		−10.55	−5.70		−3.05*	−1.14
Term structure slope		19.72	12.01		25.80**	17.11**
		A			B	
S&P500 return	−2.03**		−1.27*	−1.96**		−1.69**
S&P500 volatility	.35		.35	.33		.17
Treasury yield		−9.95**	−6.67**		−8.36**	−4.53**
Term structure slope		27.33**	19.68**		16.93**	4.91**
		BBB			CCC	
S&P500 return	−2.11**		−1.44**	−1.27**		−1.27**
S&P500 volatility	2.13**		2.55**	.97*		.35
Treasury yield		−4.47	−1.90		−14.61**	−10.25**
Term structure slope		33.25**	25.66**		−12.85*	−10.98**

Note: Estimations of semi-parametric models of default hazard rates with time-varying factors over rating classes for durations up to the first exits and all countries. The table displays sensitivities $\widehat{\beta}$ from multivariate specifications $\lambda(u, t_i) = \lambda^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]$. We focus on financial market information. * and ** stand for significance at 95% and 99% confidence level, respectively.

power on the IG class. In a Merton-like intensity model with additional stochastic liabilities, it could be interpreted as evidence of the level and higher variability of assets being the main determinants of the default probability changes. The baseline hazard is leveled down by 29% on Figure 14.1a, thanks to stock indicators. But for each bump in Figure 14.1a and c, deviations from the constant (bold lines) remain significant, implying that the S&P500 return and volatility do not succeed in capturing all shocks of the economy which affect the default riskiness. In particular, for the NIG category, the small difference between $\widehat{\alpha}^{\circ}_{\mathrm{NP}}(u)$ and $\widehat{\alpha}^{\circ}(u)$ shows that these joint financial components perform poorly on NIG issuers as univariate results suggested.

The analysis of the Financial Markets model evidences how correlated factors can be damaging. We concentrate on IG and NIG classes on Table 14.6 and on Figure 14.1b and d. Sensitivities to the long-term yield are insignificant. From a statistical standpoint, it can be explained by high correlation with other factors. From an economic standpoint, the

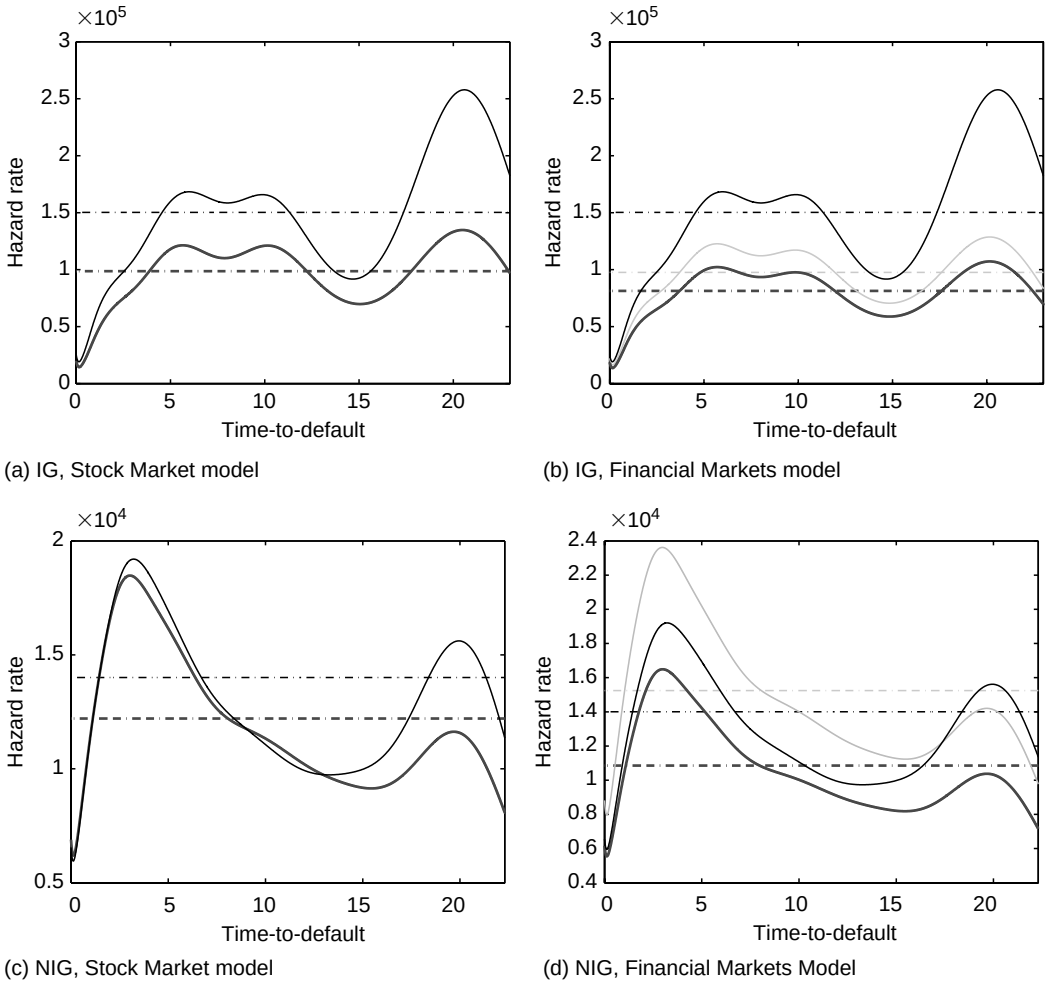


FIGURE 14.1 Baseline hazards of multifactor models relying on financial information. Estimated nonparametric baseline hazard rates $[\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}(u), \widehat{\widehat{\alpha}}^{\circ}(u)]$ and corresponding means over investment (IG) and noninvestment grades (NIG). Thin lines denote the full nonparametric model $[\alpha(u, t_i) = \alpha^{\circ}(u)]$ and bold lines show semi-parametric specifications $\{\alpha(u, t_i) = \alpha^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]\}$ including only significant factors, whereas light lines keep all factors. Dashed lines represent averages of baselines—they are not statistically different from the estimated constants e^{γ} of parametric model counterparts $\{\alpha(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}(u + t_i)]\}$. The Stock Market model uses the contemporaneous return and volatility on the S&P500. The Financial Markets model includes in addition the U.S. long-term yield and term structure slope.

long-term yield is less informative on current economic conditions than the market index, and on future growth opportunities than the term structure slope. Figure 14.1d proves that mis-specified and correlated factors dramatically reduce the explanatory power. The light curve (all financial factors) lies above the nonparametric model, meaning that the model introduces additional noise in hazard rates instead of improving the fit. The bold line, on

the other hand, shows the baseline hazard when the sensitivity to the treasury yield is constrained to be null. For the IG category, the other coefficients become −1.53**, 2.64*, and 27.12**. And for the NIG category, they become 1.67**, 0.62*, and 6.29**. Both Figure 14.1b and d prove in this constrained case the benefit to extract additional information from the term structure slope. The variations in hazard rates explained by this specification are about 43% for IG and 12% for NIG.

Finally, we observe that a constant hazard either unconditional or conditional on financial information does not represent the data correctly. Short-term default probabilities are completely overstated by standard hazard models (the dashed line is higher than $\widehat{\widehat{\alpha}}^{\circ}(u)$ up to 2–4 years) as well as long-term default probabilities for NIG. Introducing financial factors leads to overestimating long-term hazard rates on a larger part of the debt life. In an attempt to capture effects of the 2001 default peak, the model grants too much weight to covariations between the financial markets and the defaults, yet still undershoot this peak. Notice that given our sample window, the 2001 recession is responsible to a large extent for the first hump of $\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}(u)$ (among NIG observed times-to-default which range between 1 and 5 years approximately, one fourth faced the 1991 recession at these horizons while one half faced the 2001 recession).*

14.6.2 Models Based on Nonfinancial Information

Table 14.7 reports multifactor specifications using business and credit information. All factors enter the various models with the same signs. Unlike Duffie et al. (2007) who worked on bankruptcy rates, we observe that the real GDP and the PIG are complementary explanatory variables for our dataset. The impact of the GDP increases as the credit quality decreases, whereas that of the PIG decreases. This is in accordance with univariate results. All factors from the credit cycle are significant. In particular, the impact of the BBB spread is homogenous across rating classes, and so are the effects of the NIG downgrade rate. This homogeneity can indicate the influence of changes in the default risk premium. Higher premia increase future coupon rates as most of the companies issue floating rate bonds, pushing some firms up to their limit. As one could expect, the IG downgrade rate brings default information for IG hazard rates. But looking at rating classes, we observe that the effects are starker around BBB and BB firms. This factor captures local demand and supply shocks, whereas the NIG downgrade rate captures global health of issuers.

For each rating class, we report as the Nonfinancial model the best model using these five factors. The model is selected from LR tests. Figure 14.2a and c plots corresponding baseline hazards for IG and NIG categories. It is striking to observe that the GDP does not add information on the IG likelihood of default, whereas the situation reverses for NIG (actually, substituting the PIG by the GDP do not modify significantly the likelihood for the BBB class).

* We check this overfitting problem because of the 2001 recession by estimating the Stock Market model on the subsample constituted by firms entered in the process after 03/01/1991. This basically cuts durations higher than 10 years, and thus the second hump on intensity graphs. On this reduced sample, the model delivers higher sensitivities of −2.82 on the S&P500 return and of 3.14 on its volatility for IG, −1.99** and 0.88** respectively for NIG. At the same time, it also exhibits a higher overestimation of instantaneous probabilities from 6 to 10 remaining years after the peak.

TABLE 14.7 Contemporaneous Nonfinancial Multifactor Models

Model (Factors/Class)	Business Cycle	Credit Cycle	Nonfinancial	Business Cycle	Credit Cycle	Nonfinancial
		IG			NIG	
Real GDP growth	−8.80**			−16.83**		−13.90**
Personal income growth	−22.91**		−15.53**	−11.63**		
BBB spread		39.04**	14.54**		30.24**	18.42**
IG down grade rate		39.34**	30.77**		17.21**	15.13**
NIG down grade rate		15.53**	14.23**		18.76**	15.04**
		AA			BB	
Real GDP growth	−7.87			−16.13**		−12.45**
Personal income growth	−10.85*		−6.96	−13.28**		
BBB spread		25.31*	14.11		33.47**	23.17**
IG down grade rate		9.93	6.26		24.93**	23.59**
NIG down grade rate		17.67*	17.73*		18.03**	14.6**
		A			B	
Real GDP growth	−13.42**			−16.89**		−13.98**
Personal income growth	−21.30**		−19.83**	−12.20**		
BBB spread		35.23**	2.58		33.35**	22.01**
IG down grade rate		18.01*	7.02		15.29**	13.02**
NIG down grade rate		18.80**	19.11**		19.33**	15.34**
		BBB			CCC	
Real GDP growth	−10.67**			−12.39**		−8.81**
Personal income growth	−17.47**		−11.35**	−8.36**		
BBB spread		34.51**	15.96**		26.18**	22.01**
IG down grade rate		33.39**	28.54**		3.31	1.50
NIG down grade rate		13.77**	11.81**		18.56**	15.45**

Note: Estimations of semi-parametric models of default hazard rates with time-varying factors over rating classes for durations up to the first exits and all countries. The table displays sensitivities $\widehat{\beta}$ from multivariate specifications $\lambda(u, t_i) = \lambda^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]$. We focus on business and credit information. The Nonfinancial model displays the best model from this set of factors, on the basis of likelihood ratio tests. * and ** stand for significance at 95% and 99% confidence level, respectively.

As we previously mentioned, this split between the GDP and the PIG over rating classes can be explained by the effect of interest rates and their correlation with the PIG. Adding the term structure slope makes the PIG insignificant on IG firms, but does not change impacts of the GDP. Our results explain the findings of Duffie et al. (2007) since they do not include interest rates information. It could also reflect that their sample consists mainly of IG firms.

Figures 14.1b and 14.2a show that IG firms are more closely linked to financial markets than to the other components of the economy. The semi-parametric baseline hazard is indeed higher for the Nonfinancial model than for the Financial Markets model.

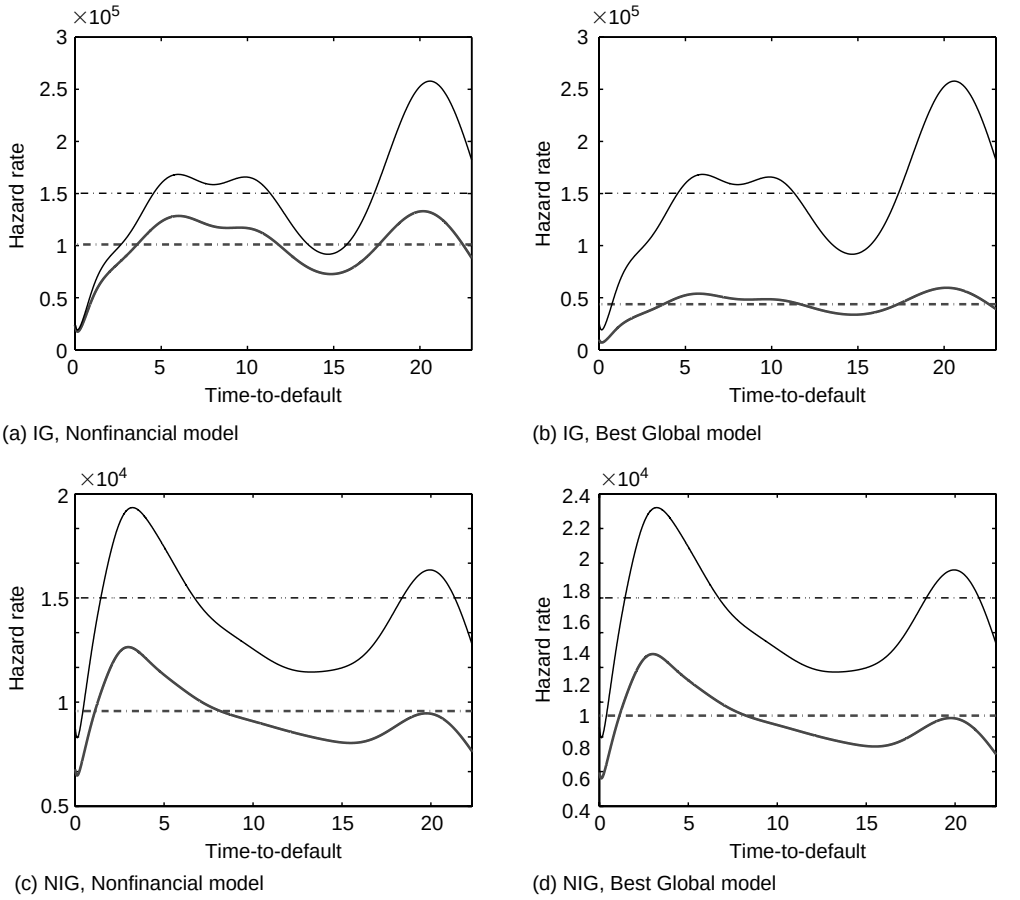


FIGURE 14.2 Baseline hazards of Nonfinancial and Global Multifactor models.
Estimated nonparametric baseline hazard rates $[\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}(u), \widehat{\widehat{\alpha}}^{\circ}(u)]$ and corresponding means over investment (IG) and noninvestment grades (NIG). Thin lines denote the full nonparametric model $[\alpha(u, t_i) = \alpha^{\circ}(u)]$ and bold lines show semi-parametric specifications $\{\alpha(u, t_i) = \alpha^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]\}$. Dashed lines represent averages of baselines—they are not statistically different from the estimated constants e^{γ} of parametric model counterparts $\{\alpha(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}(u + t_i)]\}$. The Nonfinancial model uses the BBB spread, the IG, and the NIG downgrade rates. In addition, the IG model relies on the personal income growth and the NIG model on the real gross domestic product growth. The Best Global model relies on the S&P500 volatility, the term structure slope, the GDP, the BBB spread, and both IG and NIG downgrade rates.

Conversely, considering NIG firms, the business cycle and credit-specific information have much larger impacts on default hazard rates than stock and interest rates information. The Nonfinancial model explains 48% of the variation in NIG hazard rates, and 34% of the changes in IG hazard rates.

14.6.3 Benefits of Considering All Current Economic Indicators

In traditional structural or intensity-based models, market factors are expected to integrate information on the business cycle or on credit-specific phenomena. However, these factors may merely supply a noisy signal of the business cycle and may not be the most efficient vehicle of financial information. We have shown above that financial information is of prime importance for explaining default probability changes of IG firms. Nonfinancial information is more relevant for NIG firms. We will now test whether financial and nonfinancial variables are complementary or substitute predictors of the default cycle. The top part of Table 14.8 presents our Best Global models, over IG, NIG, and BBB to CCC classes.

Both broad categories of factors (financial and nonfinancial) form part of the most relevant information set. However, some individual factors are no longer statistically significant. The return of the S&P500 is not relevant for all classes except the AA (unreported). Both its correlations with the business cycle (36% with the GDP) and with the BBB spread (−46%) explain this result. It confirms that the market index is too noisy to provide fully reliable information on changes in overall default risk. Because of the significant impact of the term structure slope, the PIG is also insignificant and does not contribute to the explanatory power of the model, according to LR test. The GDP, on the other hand, is a key instrument. As before, the IG downgrade ratio has significant impact on the BBB and BB classes. No variable should be added from LR test outputs.

Figure 14.2b and d shows the baseline hazards for the Global models. The IG baseline is strongly shifted downwards by 62% compared to the previous specification. The NIG baseline is very similar to the baseline of the Nonfinancial model, explaining 49% of the variations in hazard rates. These results prove that financial and nonfinancial factors are complementary and should be included in any model of default probabilities under the

TABLE 14.8 Global Multivariate Proportional Hazard Models and Specification Tests

Model (Factors/Class)	**IG**	**NIG**	**BBB**	**BB**	**B**	**CCC**
Best Global Model						
S&P500 volatility	3.17**	.61**	3.34**	.67*		
Term structure slope	26.24**	3.22*	25.99**	10.91**	4.18*	−10.67**
GDP	11.16**	−14.48**	−12.89**	−13.05**	−14.02**	−9.13**
BBB spread	8.82*	12.21**	11.01**	12.44**	23.74**	29.89**
IG down grade rate	23.95**	12.94**	19.42**	18.01**		
NIG down grade rate	13.18**	14.92**	8.74**	14.36**	16.42**	15.99**
Specification Tests						
Financial Markets	.119	.000	.150	.077	.000	.000
Nonfinancial	.003	.121	.122	.068	.097	.064
Best Global	.038	.075	.124	.065	.059	.042

Note: Estimations of semi-parametric models of default intensities with time-varying covariates over investment grade (IG) and noninvestment grade (NIG) classes for durations up to the first exits and all countries. The first part of the table displays sensitivities $\widehat{\beta}$ from multivariate specifications $\lambda(u, t_i) = \lambda^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]$. * and ** stand for significance at 95% and 99% confidence level. The second part presents p-values of specification tests on conditionally log-logistic counterparts.

historical measure. Nevertheless, we remark that the informational content of stock and interest rates markets on the default likelihood of NIG issuers is highly limited. For that kind of issuers, it is crucial to rely on business factors. The proportions of explained hazard variations of our models are much higher than the reported 25%–30% by Collin-Dufresne et al. (2001) on credit spreads. This evidence that the new factors are key determinants of the default regime changes even if the NIG model is not fully satisfactory. In particular, the level of the first hump (Figure 14.2d) suggests that the default cycle is longer than the business cycle and that some persistency has still to be incorporated.

To verify our observations, we test whether the different models are mis-specified or not. Well-specified models could indeed be used in risk management and pricing applications. There exists no full specification test of semi-parametric hazard models a la Cox (test of the parametric part could be done), but Fernandes and Grammig (2005) have proposed an asymptotic specification test of parametric hazard models. We can therefore assume a functional form for the baseline hazard rate (i.e., specify a conditional distribution of times-to-default) and test the model using their approach. Figures 14.1 and 14.2 show that the conditional exponentiality is strongly mis-specified. Other standard distribution of durations are the Weibull, the log-normal, and the log-logistic distributions. Full maximum likelihood estimations show that for all baseline hazards across rating classes, similar fits are obtained with the log-logistic and the log-normal distribution. Hereafter, we check mis-specification of conditionally log-logistic factor models. Keeping previous notations, hazard models with log-logistic baseline are defined by

$$\lambda(u, t_i) = \frac{\delta p(\delta u)^{p-1}}{1 + (\delta u)^p} \exp\left[\beta' \mathbf{Z}(t_i, u + t_i)\right] \quad \forall i, \tag{14.8}$$

δ and p being the parameters of the baseline hazard. The asymptotic theory doubtfully applies to our limited samples. We rather propose a specification test in finite samples, starting from the Fernandes and Grammig statistic and using bootstrap:

1. Estimate $\widehat{\theta}[\theta = (\delta, p, \beta)]$ and $\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}(u)$ on the observed durations $\mathbf{U} = (U_1, \ldots, U_N)$.
2. Compute the statistic $\Lambda(\widehat{\theta}) = \int\left\{\int_0^u [\alpha(u;\, t.,\, \widehat{\theta}) - \widehat{\widehat{\alpha}}_{\mathrm{NP}}(u;\, t.)]\mathrm{d}u\right\}^2 \mathrm{d}F(u)$, where F is the true probability function of U_is.
3. Draw a new uncensored duration sample $\mathbf{D}^{(i)} = \left[D_1^{(i)}, \ldots, D_N^{(i)}\right]$ from the estimated distribution $f(u; t., \widehat{\theta})$, called a bootstrap sample.
4. Apply a uniform right censoring scheme matching the censoring percentage of the observed sample. It creates simulated durations $\mathbf{U}^{(i)} = \left[U_1^{(i)}, \ldots, U_N^{(i)}\right]$.
5. Estimate $\widehat{\theta}^{(i)}$ and $\widehat{\widehat{\alpha}}^{\circ(i)}_{\mathrm{NP}}(u)$ on the simulated sample $\mathbf{U}^{(i)}$, and compute the corresponding statistic $\Lambda\left[\widehat{\theta}^{(i)}\right]$.
6. Repeat steps 3–5 S times, and obtain the empirical distribution of the statistic $\Lambda\left[\widehat{\theta}^{(\cdot)}\right]$, called the bootstrap distribution.
7. Reject the null hypothesis of correct specification at significance level 5% if $\Lambda\left(\widehat{\theta}\right)$ is larger than the 95% percentile of the bootstrap distribution.

This type of bootstrap procedures is known to work extremely well in finite samples. Notice that dates of entry into the risk classes are kept fixed (parametric bootstrap). The bottom part of Table 14.8 displays *p*-values based on 1000 bootstrap samples for the Financial Markets, the Nonfinancial, and the Global models, where nonsignificant sensitivities have been constrained to zero. Results confirm previous graphical outputs and LR tests. The Nonfinancial model performs quite well over classes from BBB to CCC and for the broader NIG category. As expected, in the case of NIG firms and junk issuers, modeling hazard rates by means of financial factors is strongly rejected. Nevertheless, stock and interest rates information are sufficient to capture changes in hazard rates of IG firms. Unreported results corroborate the inability of conditionally exponential models to represent the data at a 99.9% confidence level. We also checked that simple unconditional log-logistic models are rejected. As a consequence, it is necessary both to rely on a complete set of economic factors and to take ageing effects (from the most recent rating review) into account so as to correctly capture and predict hazard rates. When using all sources of information, results are less conclusive. The Global model is indeed rejected at 95% over CCC and IG classes. Correlation between the different factors may be at the origin of these mis-specifications.

14.6.4 Trends and Persistency of Shocks

So far, all selected variables have been contemporaneous, but potential lead–lag effects between the economy and the default cycle should also be considered. Looking back at Table 14.2, we can argue that lagged variables bring information upon current economic conditions, and some variables lead the default cycle by an average of 3 years. To address the benefits of past information, we concentrate on the Financial Markets model which is the most akin to structural Merton-type models. We add 3 year lagged volatility and stock market return, 10 year treasury yield and term structure slope. Estimated parameters $\widehat{\widehat{\beta}}$ are provided in Table 14.9 using past information only, and using both contemporaneous and past information.

In the complete case, we also report for each factor two additional sensitivities which deliver another decomposition of the usefulness of information. The trend is the sensitivity which can be attributed to the differential between the contemporaneous and the 3 year lagged factor, whereas the persistency is the total sensitivity to the lagged factor. Using past information only, results are in line with univariate analyses. Past increases in volatility and 10 year treasury yield respectively increase and decrease the default likelihood, but the effects are stronger than that of contemporaneous factors. It supports the presence of a significant lag between the financial cycle and the default cycle. The sensitivity to past changes in the term structure slope becomes negative. We discussed that point for the particular CCC class: 3 years later, the remaining informational content of past slopes lies in growth anticipation. The positive sign of the sensitivity to the S&P500 return is less obvious. We still advocate for anticipation of changes in the business cycle which is confirmed by the Full model. Let us consider a concrete situation. Assume that we were at the top of an economic expansion 3 years earlier, i.e., the persistency factor is positive. The more the trend factor is negative (the smaller today returns are with respect to past

TABLE 14.9 Multivariate Proportional Hazard Models with Past Information

Financial Markets Models	Contemporaneous	Past	Full	Contemporaneous	Past	Full
Factors/Class		IG			NIG	
S&P500 Return						
Contemporaneous	−1.40**		−.41	−1.62**		−.40**
3 year lagged		1.02**	1.22**		1.51**	1.77**
Trend			−.41			−.40**
Persistency			.81*			1.37**
S&P500 Volatility						
Contemporaneous	2.53**		3.14**	.33		.77**
3 year lagged		4.09**	2.93**		3.89**	3.25**
Trend			3.14**			.77**
Persistency			6.07**			4.01**
Treasury 10 year yield						
Contemporaneous	−4.37**		2.53*	−4.21**		5.63**
3 year lagged		−8.79**	−6.42**		−6.91**	−9.30**
Trend			2.53*			5.62**
Persistency			−3.89**			−3.68**
Term Structure Slope						
Contemporaneous	24.87**		30.31**	4.09**		14.403**
3 year lagged		−20.11**	−4.24		−14.59**	−6.0858*
Trend			30.31**			14.40**
Persistency			26.07**			8.32*
Specification Test	.119	.000	.071	.000	.029	.053

Note: Estimations of semi-parametric models of default intensities with time-varying covariates over rating classes for durations up to the first exits and all countries. The table displays sensitivities $\widehat{\widehat{\beta}}$ from multivariate specifications $\lambda(u, t_i) = \lambda^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]$. * and ** stand for significance at 95% and 99% confidence level, respectively. If β_c^F is the sensitivity to the contemporaneous factor F and β_p^F to its 3 year lag, the sensitivity to the trend component is given by β_c^F and the one to the persistency component by $\beta_c^F + \beta_p^F$.

ones), the higher the default likelihood is. Further assume that 3 years earlier we were at the very beginning of the expansion period. The more the trend factor is positive, the lower the default likelihood is, provided that the growth or the differential is higher than a given point depending on past conditions (e.g., from Table 14.9 $\frac{-[1.37 \times \mathbf{Z}(t-3\ yr)]}{(-0.40)}$ for the NIG case).

From a global perspective, Table 14.9 shows that both trend and past information are determinants of default probabilities. The Full models are not rejected while the Past models are. This shows that it is not sufficient to consider the lag between the economy and the default cycle. Figure 14.3a and b confirms the poor explanatory power of past information alone, but Figure 14.3b and d diagnoses high explanatory power when it is used with contemporaneous factors to capture economic trends. Explained variations in hazard rates reach 78% for IG and 59% for NIG categories. The improvement is even more substantial than that achieved by additional business and credit indicators for the NIG class.

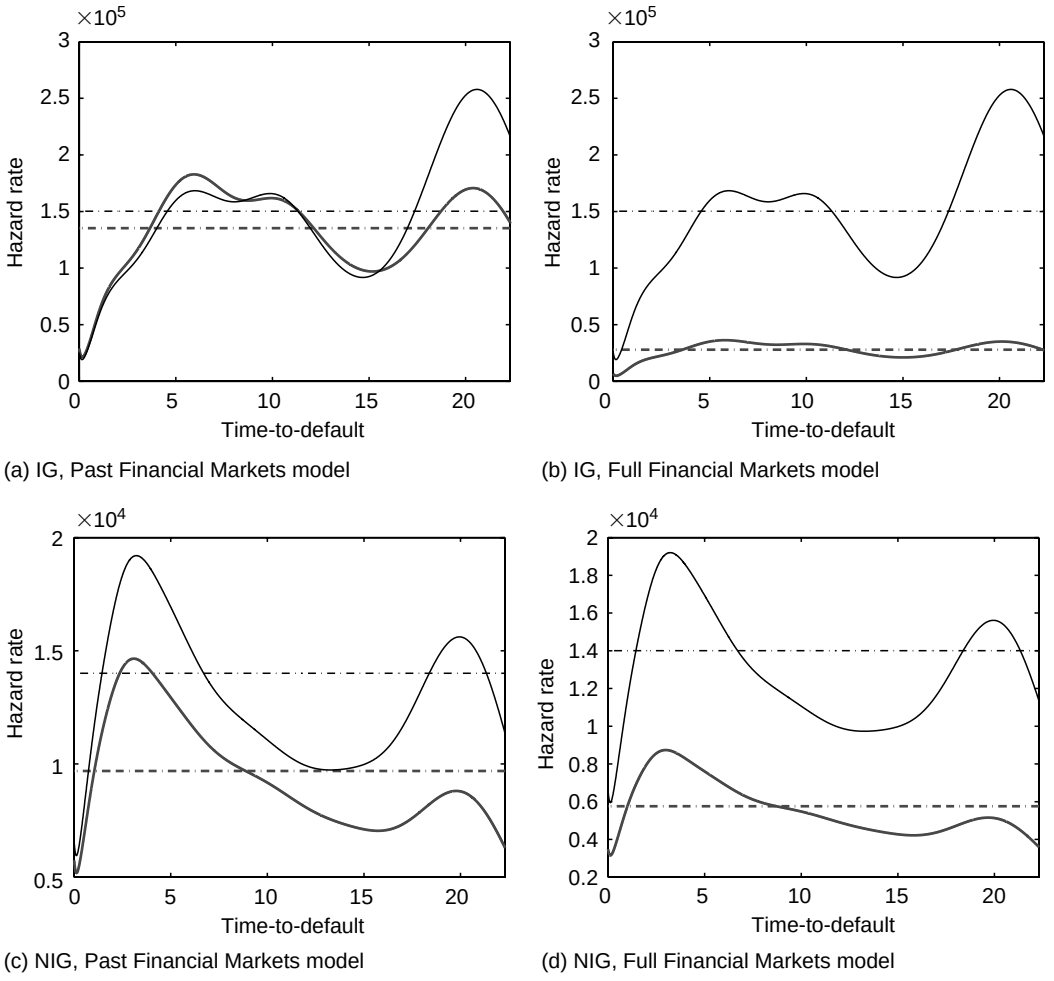


FIGURE 14.3 Baseline hazards of multiFactor models with past information.
Estimated nonparametric baseline hazard rates $\left[\widehat{\widehat{\alpha}}^{\circ}_{\mathrm{NP}}(u), \widehat{\widehat{\alpha}}^{\circ}(u)\right]$ and corresponding means over investment (IG) and noninvestment grades (NIG). Thin lines denote the full nonparametric model $[\alpha(u, t_i) = \alpha^{\circ}(u)]$ and bold lines show semi-parametric specifications $\{\alpha(u, t_i) = \alpha^{\circ}(u) \exp[\beta' \mathbf{Z}(u + t_i)]\}$. Dashed lines represent averages of baselines—they are not statistically different from the estimated constants e^{γ} of parametric model counterparts $\{\alpha(u, t_i) = \exp[\gamma + \beta' \mathbf{Z}(u + t_i)]\}$. The Past model uses 3 year lagged factors (S&P500 return and volatility, 10 year treasury yield, and term structure slope). The Full models add the same comporaneous factors.

We can propose two explanations for such findings. First, it may reflect some cyclicality in equity returns, and an idiosyncrasy of the period we are studying. We have found that high current equity returns tend to be associated with low current default rates. If there is cyclicality in equity returns, with a peak-to-trough time of approximately 3 years, it is plausible that high returns will be associated with high default rates 3 years on. Nonetheless,

we have found no evidence of such a cyclicality since Past models do not perform in explaining hazard rate variations. An alternative, more likely, explanation would be that in good times (when the equity market is performing well), companies can afford to raise large amounts of debt while preserving acceptable levels of leverage. Several years later, this level may become unsustainable for some firms, thereby raising the default rate. Table 14.9 and Figure 14.3b and d do not contradict such a hypothesis. If the stock market has an upward trend, the market appreciation induces a decrease in hazard rates. If the market depreciates or stagnates, hazard rates increase as some firms start experiencing difficulties.

14.7 SUMMARY

In this chapter, we study times-to-default in the S&P's rated universe using hazard models. We rely on a tractable framework that enables us to analyze the behavior of default probabilities with respect to changes in various economic indicators. This is done under the historical measure to focus precisely on probabilities without any perturbations because of changes in the default risk premium. The chapter concentrates on common default determinants and investigates default probabilities, rather than bankruptcy rates as considered in most previous research. Our setting also highlights the distribution of explanatory errors through time. The models we propose can be operated to forecast default probabilities at any horizon, and as a consequence, as inputs to standard credit risk applications.

We examine different economic components which should alter the default cycle and quantify their explanatory power: the financial market information, the business cycle and endogenous proxies from credit markets, and the default cycle. We explore further the sensitivity of default probabilities to past economic conditions and show their persistency as well as impacts of subsequent trends. Specification tests for parametric hazard rate models are also proposed and applied to time-to-default modeling.

Our initial empirical results show that the business cycle and financial market factors offer comparable explanatory powers. Business cycle changes have larger influences on NIG issuers, whereas stock and bond market indicators are key determinants of future default probabilities of IG companies. Leverage ratio-targeting and business diversification provide some intuition for this phenomenon. We show that selecting a set of factors from each economic components (financial, business, and credit) outperforms other specifications in capturing movements in default probabilities.

Results on impacts of the market volatility are challenging for structural models as stock price volatility is only really significant for the BBB and BB classes which are more strongly impacted by supply and demand phenomena. However, a lag between the market and the default rates can explain these results as lagged volatility is a significant determinant of default probabilities for all issuers.

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APPENDIX

For parametric hazard rate models, the standard estimation procedure relies on the maximum likelihood technique and works in the following way. Assuming that structural dynamics variables are independent, the likelihood is separable into two terms, one related to the dynamics of factors and the other dealing with conditional durations. Therefore, if we are not interested in factors dynamics, we can ignore this part and focus purely on time-to-default. For a given firm i, the likelihood l of observed duration u_i can be written conditionally on factor realizations at firm's death or exit, but the whole construction of the risk classes information set has to be known up to that calendar time:

$$l(u_i) = l_1\left(u_i|\mathcal{F}^Z_{t_i+u_i}\right) \times l_2\left(\mathcal{F}^Z_{t_i+u_i}\right)$$

where l_1 is the univariate likelihood of the conditional duration and l_2 the likelihood associated with the factor dynamics. From that point, letting L_1 and L_2 denote the multivariate counterparts of l_1 and l_2, the multivariate likelihood function for a sample of n firms observed up to time $t = \max_i \{t_i + u_i\}$ is defined by

$$L(u_1, \ldots, u_n) = L_1\left(u_1, \ldots, u_n|\mathcal{F}^Z_t\right) \times L_2\left(\mathcal{F}^Z_t\right)$$

with

$$L_1\left(u_1, \ldots, u_n|\mathcal{F}^Z_t\right) = \prod_{i=1}^{n} \exp\left[-\int_0^{u_i} \lambda(s,t_i)\mathrm{d}s\right] \left[\mathbb{I}_{(d_i>c_i)} + \lambda(u_i,t_i)\mathbb{I}_{(d_i<c_i)}\right] \tag{14.A1}$$

where

$c_1, \ldots, c_n$ are realizations of censoring variables $C_1, \ldots, C_n$
$d_1, \ldots, d_n$ are default durations $D_1, \ldots, D_n$

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CHAPTER 15

Managing Credit Risk for Retail Low-Default Portfolios*

Gabriele Sabato

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* The material and the opinions presented and expressed in this chapter are those of the author and do not necessarily reflect views of ABN AMRO Bank.

15.1 INTRODUCTION

Low-default portfolios (LDPs) can be defined as portfolios where the bank has no or a very low level of defaults and is therefore unable to estimate and validate probability of default (PD), loss given default (LGD), and exposure at default (EAD) on a basis of a proven statistical significance. Several types of portfolios may have low numbers of defaults due to different reasons: for example, portfolios generally considered to be of low risk (such as exposures to sovereigns, banks, insurance companies, or highly rated corporates) or commonly small in size, either globally or at an individual bank level (such as project finance or shipping).* However, most of the literature does not consider retail portfolios in this list or consider only retail mortgages.

Unfortunately, in the real world, banks often have to face the problem of relatively sparse default data for many retail portfolios. Contingency issues (e.g., banks are recent market entrant for a given retail portfolio or have started to collect data only from a small period) or the nature of the product itself (e.g., mortgages) are the most common reasons. Considering that retail banking is generally based on transactional data (hard information, as Berger [2006] observes) and needs advanced credit risk management tools to be managed in an efficient way, the lack of internal data to develop or validate meaningful credit risk models must be regarded as more dangerous for retail portfolios than for non-retail ones.

Although retail mortgages, one of the best examples of LDP, are commonly considered a low-risk product, they usually represent a very high percentage (often more than 50%) of bank retail assets. Indeed, banking organizations should pay a special attention to the way they manage this business, either in the application or in the behavioral process, since any kind of inefficiency (such as the lack of a scoring system or the use of a low-quality one) can have significant effects on the overall bank profitability.

Financial institutions should be aware that credit risk management for retail LDPs is becoming a strategic issue in general and for Basel II purposes in particular. The new Basel Capital Accord and concerns raised by the industry that the lack of sufficient statistical data and the resulting difficulty in backtesting risk parameters will result in LDPs being excluded from the IRB treatment have caused a special attention to this topic. Many recent studies,† mainly focusing on the bank's wholesale portfolio, conclude that it seems inconsistent with the spirit of the new Basel Accord, to exclude LDPs from the IRB treatment only because they have suffered so few defaults. However, I believe that managing LDPs in a more efficient way, using scoring systems based as much as possible on internal data, will likely be the next challenge for banks to improve their internal efficiency and profitability more than only to be allowed to apply the Basel II IRB approach.

The prime objective of this chapter is to present a new technique that allows banks to use internal data to develop or validate default prediction models even if the number of defaults in the selected retail portfolio is low or equal to zero. In LDPs, the dependent variable (if a regression analysis is used) or the variable to define the different groups (if the

* See Basel Committee on Banking Supervision (2005).

† See BBA, LIBA, and ISDA (August 2004, January 2005) discussion paper about low-default portfolios.

multivariate discriminant analysis is used) cannot be defined due to the lack of information about one group, defaulted clients. Usually, in the past, banking organizations have solved the problem by using generic scoring models (expert scorecards based on subjective weights or developed on pooled data) or having their processes based on simple policy rules, all manually checked by the employees. Both these solutions are extremely inefficient in a retail context. The performance of generic scorecards is frequently very low and often the selected variables are not able to satisfactorily explain the credit risk of a specific retail product. The use of policy rules to manage retail credit risk is highly cost inefficient considering the large volume and the low profit margin of retail products.

Avery et al. (2004) examine the potential value associated with incorporating situational data, such as local economic circumstances (e.g., unemployment rate) or personal situations (e.g., divorce) into credit risk evaluation. Their empirical models yield strong inferences that situational circumstances significantly influence an individual's propensity to default on a new loan. Their findings demonstrate that adverse, temporary economic, or personal shocks, such as income disruptions, are important factors influencing payment performance even after accounting for an individual's ex ante credit quality.

Considering the results of the study of Avery et al. (2004), I argue that when the credit quality of the individuals in the selected bank portfolio is unknown (because, for example, there was not enough time to observe their behavior), temporary economic or personal shocks can be used as factors to infer customers' future credit quality. I propose to utilize shocks that typically cause the default (trigger events, as Avery et al. stipulate, such as the loss of the job, a divorce, or the death) to model the default event inside the portfolio. Statistics about these shocks are regularly published from the national institutions of all the countries.

I test the proposed methodology on two different retail portfolios: a mortgage portfolio of a big Polish bank and a consumer finance portfolio of a small Czech bank.* My findings show that the performance of the models developed using the development sample with inferred defaults is, in both cases, significantly higher than the performance of the generic scorecards used by the analyzed banks. Moreover, I show that the performance of the models developed using the inferred defaults is considerably close to the performance of the models developed using the actual defaults.

In this study, I focus on retail portfolios containing products for private individual clients, but I believe that a similar methodology can be applied, with some realistic assumption, also to small and medium enterprise (SME) clients. Actually, SMEs have been included by banks in the retail segment only recently, often forced by national regulators and to follow the new Basel Capital Accord rules.† This means that many SME portfolios have a small history within the bank and contain few or zero defaults,

* The choice of using data from two Eastern European countries is well considered. Actually, even if the issue of LDPs is addressed by banks all over the world, however, it is extremely relevant and important in economic regimes where modern banking is a new feature in the nation's economic development (such as Eastern Europe).

† Most large banks have only recently been motivated to develop models specifically for SMEs since the new Basel Accord explicitly differentiates capital requirements between large corporates and SMEs. See Altman and Sabato (2007) for further discussions.

hence we can reasonably consider most of them as LDPs. Moreover, many recent studies (Schwaiger 2002, Saurina and Trucharte 2004, Udell 2004, Jacobson et al. 2005, Kolari and Shin 2004, Altman and Sabato 2005, Berger 2006, and Altman and Sabato 2007) demonstrate that the use of scoring systems for SMEs clients is a significant strategic and competitive issue for banking organizations to achieve internal efficiency and maximize profits linked to the SME business. For this reason, the credit risk management for SME LDPs can be reasonably considered another important challenge for banks.

In Section 15.2, a survey of the most relevant literature about failure prediction methodologies is provided. First, the choice of using a logistic regression to develop credit risk models is addressed and justified. Then, I give an overview of the most recent studies about LDPs and I investigate their findings. In Section 15.3, I extensively analyze the proposed methodology from both a theoretical and an empirical point of view. Data from a mortgage portfolio of a big Polish bank and from a consumer finance portfolio of small Czech bank will be utilized to test the prediction power of the proposed technique. In Section 15.4, I submit my conclusions.

15.2 REVIEW OF THE RELEVANT RESEARCH LITERATURE

In this section, I review some of the most important works about default prediction methodologies. First, I analyze the different alternative statistical techniques that can be used to develop credit risk models and then I focus on the authors who have investigated the problem of modeling credit risk for LDPS.

15.2.1 Default Prediction Studies

The literature about default prediction methodologies is extensive since many authors during the last 40 years have examined several possible realistic alternatives to predict customers' default or business failure. The pioneers in this field were Altman (1968) and Beaver (1967), who developed models to predict business failures using a set of financial and economic ratios. Beaver (1967) used a dichotomous classification test to determine the error rates a potential creditor would experience if he classified firms on the basis of their financial ratios as failed or nonfailed. He used a matched sample consisted of 158 firms (79 failed and 79 nonfailed) and he analyzed 14 financial ratios. Altman (1968) used the multiple discriminant analysis (MDA) technique to solve the inconsistency problem linked to the Beaver's univariate analysis. Since the Altman study, the MDA was the prevalent statistical technique applied to the default prediction models and it was used by many authors (Deakin 1972, Edmister 1972, Blum 1974, Altman et al. 1977, Eisenbeis 1977, Taffler and Tisshaw 1977, Bilderbeek 1979, Micha 1984, Gombola et al. 1987, Altman et al. 1995, Lussier 1995). However, in most of these studies, authors pointed out that the two basic assumptions* of the MDA are often violated when it is applied to the default

* The MDA is based on two restrictive assumptions: (1) the independent variables included in the model are multivariate normally distributed; (2) the group dispersion matrices (or variance-covariance matrices) are equal across the failing and the nonfailing groups. See Barnes (1982), Karels and Prakash (1987), and McLeay and Omar (2000) for further discussions about this topic.

prediction problems. Moreover, in MDA models, the standardized coefficients cannot be interpreted as the slopes of a regression and hence do not indicate the relative importance of the different variables. Considering these significant MDA's problems, Ohlson (1980) for the first time applied the conditional logit model to the default prediction's study.* The practical benefits of the logit methodology are that it does not require the restrictive assumptions of MDA and allows working with disproportional samples. From the statistical point of view, the logit analysis seems to fit perfectly the characteristics of the default prediction problem, where the dependant variable is binary (default/nondefault) and with the groups being discrete, nonoverlapping, and identifiable. Then, the logit score is a score between 0 and 1, which easily gives the probability of default of the client. Lastly, the estimated coefficients can be interpreted separately as the importance or significance of each of the independent variables in the explanation of the estimated PD.

After the work of Ohlson (1980), most of the academic literature (Zavgren 1983, Gentry et al. 1985, Keasy and Watson 1987, Aziz et al. 1988, Platt and Platt 1990, Ooghe et al. 1995, Mossman et al. 1998, Becchetti and Sierra 2002, Charitou and Trigeorgis 2002, Lizal 2002) started to use logit models to predict default. After careful consideration of the nature of the problems and of the purpose of this study, I have decided to choose the logistic regression as the appropriate statistical technique.

15.2.2 Low-Default Portfolio Studies

Only recently, concerns raised by the industry that the lack of sufficient statistical data and the resulting difficulty in backtesting risk parameters will result in LDPs being excluded from the Basel II IRB treatment have caused a special attention to the topic of LDP credit risk management. However, most of the recent literature focuses on the nonretail LDPs of the banks (such as exposures to sovereigns, banks, insurance companies, highly rated corporates, project finance, or shipping) that are considered to be the portfolios most exposed to the problem of low number of defaults.

During the almost 40 years of existence of the scoring techniques, many alternative scoring methodologies have been tested to improve the performance of the models. Balcaen and Ooghe (2004) show more than 10 different techniques† that have been used to develop scoring models, but none of these has been specifically addressed to solve the problem of LDPs. Between the described techniques, the multilogit model and the dynamic event history analysis seem to be of some interest for our purposes. The multilogit model (introduced by Peel and Peel 1988) allows to simultaneously use data from several years before failure and to simultaneously discriminate between failing and non-failing clients for several reporting periods before failure. This statistical process can be useful when the number of defaults in 1 year for one product is too low to develop a robust 1 year default

* Zmijewski (1984) was the pioneer in applying probit analysis to predict default, but, until now, logit analysis has given better results in this field.

† Between the all listed methodologies, the following are the most interesting: survival analysis, machine learning decision trees, artificial neural networks, fuzzy rules-based classification model, multilogit model, dynamic event history analysis, and linear goal programming.

prediction model, but it requires assumptions that are very likely to be violated in practice.* The dynamic event history model (DEHA), applied by Hill et al. (1996), sees failure as a process and looks at the transitions to and from the stable and financially distressed states and from the latter to the bankrupt state, using multiperiod data. A transition or a change in a firm's financial status (e.g., stable, financially distressed, or bankrupt) is measured by means of a "transition rate" or a "conditional probability." In this way, instead of considering clients who went bankrupt as defaulted (that can be few in our portfolio), we can classify clients who had a major shift in their financial ratios as defaulted. Although it is a very interesting approach, it is difficult to apply this technique to private individuals.

Allen et al. (2003) provide an analysis of the possible alternative scoring techniques that can be used to model credit risk focusing on the retail business. They examine the traditional approaches to credit risk measurement (expert systems, rating systems, and credit scoring models), but the issue of LDPs is only mentioned. In their study, Loffler et al. (2004) propose a Bayesian methodology that enables banks with small datasets to improve their default probability estimates by imposing prior information on the estimates. As prior information, they use coefficients from credit scoring models estimated on other datasets. They demonstrate that the performance of the model developed using the Bayesian technique is slightly higher than the performance of the model developed with the logistic analysis.† The analysis is done on a sample of nonfinancial firms contained in the S&P500 index. Their findings show how banking organizations can increase the accuracy of their logistic scoring models by using prior information. However, the issue of using the Bayesian methodology when the number of defaults in the portfolio is low or equal to zero is not addressed. Schuermann and Hanson (2004) examine the possibility of applying the duration method to estimate the PD by means of migration matrices. Their proposal is a good solution when a certain number of defaults, in at least some (usually the low-quality) rating grades, is available. However, their approach cannot be used for portfolios without any defaults or with a very low number of defaults scattered in the overall portfolio.

The only study that I am aware of that focused on estimating probabilities of default specifically for LDPs is a recent chapter by Pluto and Tasche (2005). They observe the issue of LDPs in terms of lack of a sufficient degree of conservatism to reflect the prudential risk management style that banks should apply when they develop internal models to estimate PD. As an attempt to overcome this issue, the authors suggest to use the most prudent estimation principle. This methodology provides confidence intervals for the PD of each rating grade. Then, the PD range can be adjusted by the choice of an appropriate confidence level. Their results show monotone PD estimate, but, again, no examples of how to apply this methodology to retail portfolios are shown.

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* The multilogit method depends entirely on the assumption of signal consistency. This assumption requires that for a firm the data for the consecutive years before the failure give consistent signals about the status of the firm.

† The Bayesian estimators have been introduced by Adkins and Hill (1996) applied to a probit analysis.

Most of the recent literature about LDPs explored different alternative statistical techniques that increase the value of the information given by a small number of defaults in the sample or by external sources. However, the possibility of inferring, in a nonarbitrary way, a sufficient number of defaults so as to develop a default prediction model based on internal data, has never been investigated. This paper focuses on a possible realistic technique that banks can use to increase the number of defaults in their LDPs. In this way, banking organizations will be able to utilize a classical statistical methodology (e.g., MDA or logistic analysis) to develop default prediction models using their own internal data.

15.3 METHODOLOGY

Low-default portfolios can present a significant obstacle in developing credit risk models. Especially for the better rating grades, but even entire portfolios with low or no defaults are not uncommon. The issue of retail LDPs has been addressed by banks using generic scoring models (expert scorecards based on subjective weights or developed on pooled data) or having their processes based on simple policy rules. These practices, while widespread in the industry, do not both entirely satisfy the national supervisors' desire for a statistical foundation of the scoring models and provide banks with efficient credit risk management tools. In LDPs, a sufficient number of internal data for the selected portfolio are available, but few or no defaults prevent the definition of the binomial dependent variable (default/nondefault) needed to run the regression.

The value of the information contained in the available internal data is significantly high, but this is generally not used to develop scoring models due to the sparse number of defaults. In this section, I introduce a new methodology that can be used to model credit risk for retail LDPs.

15.3.1 Shock Methodology

Following the literature discussed in Section 15.2, I use a conditional probability model, logit model, to estimate the 1 year PD through a range of variables by maximizing the log-likelihood function. This procedure is used to obtain the estimates of the parameters of the following logit model (Hosmer and Lemeshow 1989, Gujarati 2003):

$$P_1(X_i) = \frac{1}{\left[1 + \lambda^{-(B_0 + B_1 X_{i1} + B_2 X_{i2} + \ldots + B_n X_{in})}\right]} = \frac{1}{\left[1 + \lambda^{-(D_i)}\right]}$$

where

$P_1(X_i)$ = PD given the vector of attributes X_i
B_j = coefficient of attribute j (with $j = 1, \ldots, n$) and B_0 = intercept
X_{ij} = value of the attribute j (with $j = 1, \ldots, n$) for customer i
D_i = the logit for customer i

The logistic function implies that the logit score P_1 has a value in the [0,1] interval and is increasing in D_i. If D_i approaches $-\infty$, P_1 will be 0 and if D_i approaches $+\infty$, P_1 will be 1. This peculiarity of the logistic function has made its success between default prediction

techniques,* where the response variable is binary (default/nondefault). Moreover, the advantage of this method is that it does not assume multivariate normality and equal covariance matrices, as discriminant analysis does.† Before running the regression, the binary (0,1) dependent variable must be associated to the default event. To define the default event, I use the definition contained in the new Basel Capital Accord (June 2004, par. 452) considering in default an obligor that is 90 or more days past due.

There is not a minimum number or an optimal number of defaults in a sample to develop a default prediction model. In many studies (Altman 1968, Blum 1974, Bilderbeek 1979, Zavgren 1983, Platt and Platt 1990, Mossman et al. 1998, Charitou and Trigeorgis 2002), authors used matched samples of default and nondefault observations. Usually, pairing was made on criteria of size, industry, and age and, even if the number of defaults was low (i.e., Altman used 33 defaults, Deakin [1972] 32 defaults, and Ohlson [1980] 105 defaults), the models presented high prediction power and soundness. However, other studies (Piesse and Wood 1992) have pointed out that, when a default prediction model is based on nonrandom samples, the accuracy results of the model cannot be generalized. The use of matched samples that differ significantly from the original population proportions is likely to lead to biased coefficients in logit models and can cause misleading indications of the model's predictive accuracy (Zavgren 1983, Zmijewski 1984, Ooghe and Verbaere 1985, Keasey and Watson 1991). As such, I develop the 1 year PD prediction models utilizing random, nonmatched samples.

Concerns have been raised that the default prediction model methodologies applied to the retail business have been based on corporate financial model techniques. The behavior of corporate business is not similar to that of an individual. Considerable evidence suggests that individuals may be subject to shocks (change in marital status or employment) rather than continuous process of degradation. Avery et al. (2004) examine the potential value associated with incorporating situational data, such as local economic circumstances (e.g., unemployment rate) or personal situations (e.g., divorce), into credit risk evaluation. Their empirical models yield strong inferences that situational circumstances significantly influence an individual's propensity to default on a new loan. Their findings demonstrate that adverse, temporary economic, or personal shocks, such as income disruptions, are important factors influencing payment performance even after accounting for an individual's ex ante credit quality. Hence, I suggest that, if the information about default is not available, we can use the information about one of these shocks as predictor of the possible future personal financial distress.

Statistics about the shocks that can cause the default (e.g., unemployment rate, divorce rate) are regularly published from the national institutions of all the countries. We can use these statistics to model the probability of the shock for one of the variables contained in our sample (typically age, level of education, or region where the client lives). Following

* See Back et al. (1996) for a clear discussion about the topic.

† See Altman (1968), Dimitras et al. (1996), Altman and Narayanan (1997), and Altman and Sabato (2007) for further discussion about multiple discriminant analysis (MDA) technique. The work of Lo (1986) compares the multiple discriminant analysis and the logistic analysis in the credit risk modeling and concludes that the results of both are very similar.

this process, we will obviously overestimate the discriminatory power of the chosen variable inside our sample since we will construct the dependent variable (shock/non-shock) based on its relationship with the shock event. However, I believe that this is a realistic assumption considering that we are not trying to predict the default event, but the trigger event that usually causes the default. In this way, I transform a certain number of nondefaulted clients in defaults and, instead of assigning them randomly, I assign defaults following the indication given by some event that we know that is often linked with the default (such as the loss of the job or the divorce). Once the sample is ready, I will run a logistic regression and use as predictors all available variables.

15.3.2 Empirical Tests

To test the proposed methodology, I use a mortgage data sample of a Polish bank, which contains 11,646 mortgages granted during the years 2000–2002, and a consumer finance portfolio of a small Czech bank, containing 3,383 clients over the period 2000–2001. In both samples, a sufficient number of defaults are available to develop an internal default prediction model (Tables 15.1 and 15.2), but, in the first step, I do not consider the actual defaults. I keep only nondefaulted clients and I use a trigger event (the unemployment rate) to randomly transform the expected number of nondefaulted clients in defaulted ones. As input, I need to know the expected average default rate for the analyzed portfolios. Usually, this is an information that banking organizations are able to find internally (if the product is already been managed for enough time) or externally if the bank has just launched the product. In this case, the expected average default rate for the Polish mortgage portfolio was around 5%* and for the Czech consumer credit portfolio it was around 10%.† These expected average default rates can be considered as a prior probability; hence I should assign 568 defaults (11,366 × 0.05) in the Polish portfolio and 324 defaults (3,240 × 0.1) in the Czech portfolio.

TABLE 15.1 Original Polish Mortgage Dataset

Groups	Number	Percentage (%)
Nondefaulted clients	11,366	97.6
Defaulted clients (actual)	280	2.4
Total	11,646	100

Note: This table shows the original Polish mortgage sample, such as extracted from the bank database. The actual defaults will be excluded from the analysis in the first step to infer a new number of defaults more liked to bank's risk expectations associated with this portfolio. In the first and second rows, the number and the percentage of nondefaulted and defaulted clients are shown.

* The current experienced default rate (2.4%) for the mortgage portfolio was considered not realistic, since the bank applied a very strict selection process to the applicants during the first 2 years after the launch of the mortgage business.

† This expected default rate was based on the competitor's experience and the bank's perceptions about the business' riskiness.

TABLE 15.2 Original Czech Consumer Finance Dataset

Groups	Number	Percentage (%)
Nondefaulted clients	3240	95.8
Defaulted clients (actual)	143	4.2
Total	3383	100

Note: This table shows the original Czech consumer finance sample, such as extracted from the bank database. The actual defaults will be excluded from the analysis in the first step to infer a new number of defaults more liked to bank's risk expectations associated with this portfolio. In the first and second rows, the number and the percentage of nondefaulted and defaulted clients are shown.

For both samples, I observe the distribution of the unemployment rate versus two variables contained in the development samples: age and level of education (see Tables 15.3 through 15.6). For each band (i) of the chosen variables (columns 1 in Tables 15.3 through 15.6), a different value of the unemployment rate can be observed (U_i) (columns 2 in Tables 15.3 through 15.6). In the first step, I calculate for each band the distance of the unemployment rate from the average unemployment rate of the overall portfolio (AVGU). I call this value shock index (SI_i) (columns 3 in Tables 15.3 through 15.6), where:

$$SI_i = \frac{U_i}{AVGU}$$

The shock index measures the distance of each variable's class from the trigger event mean and I assume that this distance would be the same if we were using the default event instead of its trigger event. Then, I calculate the expected default rate for each variable's

TABLE 15.3 Construction of the Shock Index between the Unemployment Rate and the Age of the Client in Poland

Age	Unemployment Rate	Shock Index	Expected Default Rate (%)	% Population	# Population	Expected Number of Defaults
≤24	25.4	1.5119	5.58	4.95	563	31
25–34	27.8	1.6548	6.10	33.08	3,760	230
35–44	21.7	1.2917	4.77	28.93	3,288	157
45–54	21.1	1.2560	4.63	27.70	3,148	146
≥55	3.6	0.2143	0.79	5.34	607	5
Total average	16.8		5.00	100.00	11,366	568

Note: This table shows how the shock index and the expected number of defaults for each band are calculated. In the first and second columns, the average Polish unemployment rate per age class is shown. These figures have been found in the Polish National Statistical Institute Report for 2004 and 2005. In the third column, the shock index is calculated dividing the unemployment rate for each band by the total average unemployment rate. In the fourth column, the expected default rate for each band is shown. This has been obtained by multiplying the shock index by the expected average default rate at portfolio level, weighted by the percentage of population in each band. In the fifth and sixth columns, the percentage and the number of customers for each portfolio band are depicted. In the last column, the expected number of defaults for each band is calculated multiplying the expected default rate per band by the number of clients in each band.

TABLE 15.4 Construction of the Shock Index between the Unemployment Rate and the Level of Education of the Client in Poland

Level of Education	Unemployment Rate	Shock Index	Expected Default Rate (%)	% Population	# Population	Expected Number of Defaults
Basic, preprimary, and no education	32.6	1.9405	17.90	3.05	347	62
Secondary education (vocational)	22.4	1.3333	12.30	14.17	1,611	198
Secondary education (general)	7.3	0.4345	4.01	38.54	4,380	176
Tertiary education (university)	4.8	0.2857	2.64	44.24	5,028	133
Total average	16.8		5.00	100.00	11,366	568

Note: This table shows how the shock index and the expected number of defaults for each band are calculated. In the first and second columns, the average Polish unemployment rate per level of education class is shown. These figures have been found in the Polish National Statistical Institute Report for 2004 and 2005. In the third column, the shock index is calculated by dividing the unemployment rate for each band by the total average unemployment rate. In the fourth column, the expected default rate for each band is shown. This has been obtained by multiplying the shock index by the expected average default rate at portfolio level, weighted by the percentage of population in each band. In the fifth and sixth columns, the percentage and the number of customers for each portfolio band are depicted. In the last column, the expected number of defaults for each band is calculated multiplying the expected default rate per band by the number of clients in each band.

band (DF_i) multiplying the shock index (SI_i) by a default rate (X) that allows the expected average default rate for the entire portfolio (AVGDF), weighted by the percentage of population in each band (a_i), to be equal to the expected value (columns 4 in Tables 15.3 through 15.6):

$$DF_i = SI_i \cdot X$$

where

$$X = \frac{AVGDF}{SI_1 \cdot a_1 + SI_2 \cdot a_2 + SI_3 \cdot a_3 + \cdots + SI_n \cdot a_n}$$

Once this expected default rate for each band has been calculated, I multiply it by the number of the population contained in the band (columns 6 in Tables 15.3 through 15.6) and I get the expected number of defaults that I should infer in that band (columns 7 in Tables 15.3 through 15.6). Then, I randomly transform inside each band, the found number of nondefaulted clients in defaulted ones and I obtain new development sample with a number of defaults as sufficient as to develop a meaningful credit risk model using own internal data.

For the Polish sample, I assign the default using the level of education, while, for the Czech sample, I use the age.* In Tables 15.7 and 15.8, the new structure of the two

* The choice is not based on any specific reason. My intention was only to choose a different variable for each sample.

TABLE 15.5 Construction of the Shock Index between the Unemployment Rate and the Age of the Client in Czech Republic

Age	Unemployment Rate	Shock Index	Expected Default Rate (%)	% Population	# Population	Expected Number of Defaults
15–19	38.4	4.9272	49.42	0.40	13	6
20–24	14.8	1.9024	19.08	9.75	316	60
25–29	8.1	1.0384	10.41	17.50	567	59
30–34	7.5	0.9668	9.70	17.10	554	54
35–39	7.1	0.9080	9.11	15.56	504	46
40–44	6.2	0.7976	8.00	14.69	476	38
45–49	5.8	0.7456	7.48	10.77	349	26
50–54	6.9	0.8908	8.93	7.99	259	23
55–59	4.9	0.6246	6.27	4.26	138	9
60–64	2.8	0.3590	3.60	1.45	47	2
$\geq$65	3.7	0.4795	4.81	0.52	17	1
Total average	7.8		10.00	100.00	3240	324

Note: This table shows how the shock index and the expected number of defaults for each band are calculated. In the first and second columns, the average Czech unemployment rate per age class is shown. These figures have been found in the Czech National Statistical Institute Report for 2004 and 2005. In the third column, the shock index is calculated dividing the unemployment rate for each band by the total average unemployment rate. In the fourth column, the expected default rate for each band is shown. This has been obtained by multiplying the shock index by the expected average default rate at portfolio level, weighted by the percentage of population in each band. In the fifth and sixth columns, the percentage and the number of customers for each portfolio band are depicted. In the last column, the expected number of defaults for each band is calculated multiplying the expected default rate per band by the number of clients in each band.

TABLE 15.6 Construction of the Shock Index between the Unemployment Rate and the Level of Education of the Client in Czech Republic

Level of Education	Unemployment Rate	Shock Index	Expected Default Rate (%)	% Population	# Population	Expected Number of Defaults
Basic, preprimary, and no education	20.5	2.6312	25.96	19.14	620	161
Secondary education	6.2	0.7949	7.84	54.97	1781	140
Tertiary education (University)	2.2	0.2828	2.79	25.90	839	23
Total average	7.8		10.00	100.00	3240	324

Note: This table shows how the shock index and the expected number of defaults for each band are calculated. In the first and second columns, the average Czech unemployment rate per level of education class is shown. These figures have been found in the Czech National Statistical Institute Report for 2004 and 2005. In the third column, the shock index is calculated dividing the unemployment rate for each band by the total average unemployment rate. In the fourth column, the expected default rate for each band is shown. This has been obtained by multiplying the shock index by the expected average default rate at portfolio level, weighted by the percentage of population in each band. In the fifth and sixth columns, the percentage and the number of customers for each portfolio band are depicted. In the last column, the expected number of defaults for each band is calculated multiplying the expected default rate per band by the number of clients in each band.

TABLE 15.7 New Polish Mortgage Dataset

Groups	Number	Percentage (%)
Nondefaulted clients	10,798	95.0
Defaulted clients	568	5.0
Total	11,366	100

Note: This table shows the new Polish mortgage sample, containing the new defaults inferred with the proposed technique. This sample will be used to develop a default prediction model to be compared with a model built on the original sample. In the first and second row, the number and the percentage of nondefaulted and defaulted clients are shown.

development samples is depicted. I use these two new samples to run the logistic regression entering all the other available variables and I develop 1 year PD models.

15.3.3 Validation Results

To test the performance of my models, I apply them on the original samples, the ones containing the actual defaults. Using this samples, I am able to prove my initial assumption (i.e., default event and its trigger event are highly correlated). Moreover, I believe that results cannot be satisfactory evaluated only on an absolute basis and, for this reason, I show for both portfolios the performance of other two models applied on the same dataset. One is a model developed on the original sample using the actual defaults. For the Czech and the Polish portfolios, I run a logistic regression utilizing the same variables entered in the models developed with the inferred defaults. Then, I also apply the two generic scorecards used by the analyzed banks at the moment of the data extraction.*

In Tables 15.9 and 15.10, I summarize the results in terms of prediction accuracy of the six different models (the two logistic models developed with actual and inferred defaults

TABLE 15.8 New Czech Consumer Finance Dataset

Groups	Number	Percentage (%)
Nondefaulted clients	2916	90.0
Defaulted clients	324	10.0
Total	3240	100

Note: This table shows the new Czech consumer finance sample, containing the new defaults inferred with the proposed technique. This sample will be used to develop a default prediction model to be compared with a model built on the original sample. In the first and second rows, the number and the percentage of nondefaulted and defaulted clients are shown.

* The Polish scorecard was subjectively designed by the bank internal analysts about 3 months after the launch of the mortgage business, when already about 2000 data was available, but no defaults had occurred at that time. The Czech generic scorecard was provided to the bank from a consultant company. The model was developed pooling together data from consumer finance portfolios of different Western European countries.

TABLE 15.9 Comparison of the Classification Accuracy of the Different Models Applied on the Polish Mortgage Portfolio

	Type I Error Rate (%)	Type II Error Rate (%)	Accuracy Ratio (%)
Logistic model developed with actual defaults	18.60	27.46	56
Logistic model developed with inferred defaults	22.34	34.12	52
Expert model	25.82	43.30	37

Note: This table shows the misclassification rates and the accuracy ratios of the three different models applied to the Polish original sample fixing an arbitrary cutoff rate of 40%. The first column shows the type I error rate, i.e., the percentage of defaulted clients classified as nondefaulted. In the second column, the type II error rate is illustrated. This rate represents the percentage of nondefaulted clients classified as defaulted. In the last column, the accuracy ratio, defined as the ratio of the area between the cumulative accuracy profile (CAP) of the rating model being validated and the CAP of the random model, and the area between the CAP of the perfect rating model and the CAP of the random model, is shown.

and a generic model for both countries) tested on the original sample. Following the large literature about model classification accuracy (Beaver 1967, Altman 1968, Deakin 1972, Edmister 1972, Altman et al. 1977, Ohlson 1980, Taffler 1982, Verbaere 1985, Koh 1992, Ooghe et al. 1995, Steele 1995, Kraft et al. 2004), I present the type I and type II error rates and the accuracy ratio of each model.

The error rates shown in Tables 15.9 and 15.10 are calculated fixing an arbitrary cutoff rate of 40% of the population.* I acknowledge that the chosen cutoff rate is possibly not the optimal rate since the different misclassification costs for the type I and type II error rates are not taken into account (Altman et al. 1977, Taffler 1982). However, I point out that the

TABLE 15.10 Comparison of the Classification Accuracy of the Different Models Applied on the Czech Consumer Credit Portfolio

	Type I Error Rate (%)	Type II Error Rate (%)	Accuracy Ratio (%)
Logistic model developed with actual defaults	15.30	24.32	62
Logistic model developed with inferred defaults	21.74	29.56	57
Generic scorecard	27.12	31.60	43

Note: This table shows the misclassification rates and the accuracy ratios of the three different models applied to the Czech original sample fixing an arbitrary cutoff rate of 40%. The first column shows the type I error rate, i.e., the percentage of defaulted clients classified as nondefaulted. In the second column, the type II error rate is illustrated. This rate represents the percentage of nondefaulted clients classified as defaulted. In the last column, the accuracy ratio, defined as the ratio of the area between the cumulative accuracy profile (CAP) of the rating model being validated and the CAP of the random model, and the area between the CAP of the perfect rating model and the CAP of the random model, is shown.

* Applying the developed model to all the individuals contained in the sample, a score is calculated for each one of them. Then, the 40% of the sample with the lowest scores is considered rejected to check the accuracy of the model to correctly and incorrectly classify the clients (as defaulted and nondefaulted) between accepted and rejected applicants. The choice to fix a cutoff at 40% is not based on any specific reason; any other percentage could have been chosen for the analysis.

purpose of this work is to use a common, arbitrary, fixed cutoff rate only to compare the prediction accuracy of the different models and not to find the best cutoff strategy.

15.3.4 Comparison of Results

To compare models results, I provide the type I and type II error rates and an index (columns 4 in Tables 15.9 and 15.10). The index is called the accuracy ratio (AR) and is defined as the ratio of the area between the cumulative accuracy profile (CAP) of the rating model being validated and the CAP of the random model, and the area between the CAP of the perfect rating model and the CAP of the random model (see Engelman et al. [2003] for further details). Indeed, it measures the ability of the model to maximize the distance between the defaulted and nondefaulted clients.

For both portfolios, the performance of the model developed using the original actual defaults is the highest in terms of prediction accuracy (56% and 62% accuracy ratio for the Polish and Czech sample, respectively). This result was expected considering that the performances of the models have been measured on the original samples, the samples containing the actual defaults that, for these models, are also the development samples. Moreover, the value of the internal data, when is available, is extremely high and it is unquestionable that leads to the most powerful default prediction models.

I find that the prediction accuracy of the models developed on the samples manipulated with the proposed technique is significantly higher than the accuracy of the generic scorecards used by the banks and considerably close to the performance of the models developed using the actual defaults (52% accuracy ratio versus 37% for the Polish portfolio and 57% versus 43% accuracy ratio for the Czech portfolio).

These results demonstrate the added value associated with the use of internal data to develop meaningful default prediction models, even when defaults are low or not available. Utilizing the drivers of the default event to model the dependent variable can be reasonably considered a very useful data-enhancing methodology for retail LDPs. In any case, the proposed technique has led to solutions notably more efficient than generic scoring systems.

15.4 CONCLUSIONS

In this chapter, I introduce a methodology that can be used to manage credit risk for retail LDPs. In the literature, I have found several studies addressing the topic of LDPs, but few focus on the retail side. When retail portfolios experience a low-default situation, banks generally solve this problem using generic scorecards or applying only policy rules. The proposed methodology would allow banking organizations to develop powerful credit risk models utilizing their own internal data even if default data is missing.

I update the existing literature with several findings. First, I analyze the problem of LDPs focusing on retail assets.

Second, I prove my initial assumption about the inefficiency of generic scorecards that present a prediction accuracy, always significantly lower than the accuracy of the scorecards developed on internal data. I demonstrate the added value associated with internal data even if defaults are missing. I also find, as expected, that scorecards developed on

samples with enough actual defaults have the highest prediction accuracy. Indeed, any solution adopted to solve the issue of LDPs should be temporary and must be substituted as soon as enough defaults in the portfolio are available.

Last, even if LDPs are not mentioned as such in the Basel II framework, they have been the reason of many industry questions and concerns and have led the Basel Committee Accord Implementation Group's Validation (AIGV) subgroup to publish a newsletter (number 6 of September 2005) specifically addressed to the LDP matter. In this document, it is extensively recognized that relatively sparse default data might require increased reliance on alternative data sources and data-enhancing tools and techniques. Moreover, the AIGV makes clear that the PD estimates should be forward looking for all portfolios and that it should be possible to base them not only on recent loss data (that can be scarce), but also on additional information about the drivers of losses.

I believe that the methodology proposed in this chapter can be classified as data-enhancing tool with the purpose of modeling credit risk for retail LDPs. As suggested in the AIGV paper, the lack of information about defaulted clients is addressed taking into account the drivers of the default event. These trigger events are used to infer defaults inside the selected retail portfolio. Using this technique, banking organization will be able to develop powerful models to estimate PD utilizing their own internal data even if a sufficient number of defaults are still not available.

I think that the suggested methodology can be considered as a Basel II compliant realistic alternative to develop meaningful default prediction models for LDPs. However, I want to emphasize that improving the credit risk management techniques used for the LDPs banks will reap benefits especially in terms of internal efficiency of their retail portfolios.

Developing advanced credit risk management tools is likely to become a significant competitive advantage for banks that will want to compete globally. Improve the quality of the decision in the application process and rely on automated decision systems are essential elements of the retail business. Banks that will be able to manage their LDPs using efficient scoring systems will likely experience higher internal efficiency and profitability.

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CHAPTER 16

Tests on the Accuracy of Basel II

Simone Varotto

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16.1 INTRODUCTION

From the beginning of 2007 European banks are required to assess the credit risk in their portfolios with a set of new rules collectively known as the New Accord or Basel II. The new regulatory framework has introduced radical changes to the existing regulation and is quickly becoming a benchmark worldwide with all the major non-EU economies and most emerging markets planning to introduce it within the next few years.

Basel II entails a three-pronged approach to bank capital regulation: (1) a comprehensive set of rules designed to measure the risks in banks' portfolios and to produce minimum capital requirements, (2) a supervisory review process setting out the role of

bank supervisors in ensuring that the new framework is correctly implemented, and (3) disclosure requirements to induce banks to make available more information about the key risks in their books with the objective of improving bank accountability and market monitoring. Under the first "pillar," banks can calculate credit-risk capital requirements by choosing one of the three approaches. This flexibility was introduced in answer to the criticism leveled at the "one-size-fits-all" approach of Basel I, the previous regulation. All three options fundamentally depart from Basel I in that they employ credit ratings as a way to assess individual exposures' credit risk. The three options are called the standardized approach (SA), the foundation internal rating-based approach, and the advanced internal rating-based approach (collectively called IRBA hereafter). The SA is based on external ratings as assigned by recognized rating institutions whereas the other two approaches, as their names suggest, rely on ratings internally derived by the banks.* National supervisors allow banks to adopt the latter approaches only if they are satisfied that the internal rating assignment process is sufficiently accurate.†

The Basel Committee has conducted several "quantitative impact studies" to test the effect of the new rules on banks' regulatory capital. However, the main focus of such studies was to compare the new and the old regulation and to determine whether the new regulation would yield substantially lower capital requirements. On the other hand, most of the available studies on the actual accuracy of Basel II refer to earlier drafts of the new framework which has been substantially modified since the release of the first consultative papers in 1999 and 2001, also to incorporate some of the points made in these studies. Examples are, Altman and Saunders (2001) who show that the risk weights associated with the risk categories in the SA do not accurately represent the credit risk of assets that fall in such categories and Sironi and Zazzara (2003) who measure the inconsistencies between the asset correlation assumptions in the IRBA and asset correlations implied from empirically observed default correlations. The final version of the New Accord (Basel Committee on Banking Supervision 2006a) indeed presents new risk weights for the SA and a revised treatment of asset correlation in the IRBA. However, following the latest changes, Resti and Sironi (2007a) take another look at the SA and conclude, in agreement with Linnell (2001), that the different treatment of banks and nonbank corporations in the New Accord is not justifiable. Their empirical evidence suggests that the credit risk of banks and nonbanks with the same rating is statistically indistinguishable. They also conclude that the current risk categories are too coarse and should be more granular to increase accuracy, and that the risk-weight curve across risk categories should be steeper.

Empirical tests on the IRBA focus mainly on the accuracy of the regulatory approach when measuring the risk of loans to small and medium enterprises and retail exposures.

* Krahnen and Weber (2001) and Crouhy et al. (2001) describe the principles and technical aspects one should consider when devising an internal rating system. Carey (2001) and Jacobson et al. (2006) present some evidence on the consistency of internal rating systems across banks.

† In this respect, the Basel Committee has produced a comprehensive survey of techniques for the validation of internal rating systems (Basel Committee on Banking Supervision 2005). Carey and Hrycay (2001) on the other hand investigate the potential pitfalls of internal rating systems and suggest that a combination of several rating methods would help reduce their most common weaknesses.

Perli and Nayda (2004) develop a credit-risk model for retail exposures that accounts for future margin income and find that the negative relationship between probability of default and asset correlation established in the IRBA does not necessarily hold empirically. Calem and LaCour-Little (2004) show that geographical diversification, which is not directly addressed in pillar 1 of Basel II, has an important role in determining the economic capital of mortgage loan portfolios. Jacobson et al. (2005), through a re-sampling method applied to a large database of bank loans from two major Swedish banks, find no evidence in support of the IRBA assumption that SME loans and retail credits are systematically less risky than wholesale corporate loans. With the same data and analytical approach Jacobson et al. (2006) show that IRBA capital can be 6–9 times higher than economic capital.* The re-sampling technique they adopt has the advantage that it allows them to derive portfolio loss distributions without making any of the assumptions (e.g., on correlations) underlying a credit-risk model. However, the usefulness of their approach is reduced in portfolios of wholesale corporate loans. First, wholesale credits are larger in size and smaller in number than retail and SME credits in a typical bank portfolio, which will lessen the accuracy and significance of re-sampling. Second, conducting re-sampling on a database where only default losses are recorded (but not losses in loan value arising from rating downgrades) will not produce meaningful risk measures when there are no or few historical defaults in the sample. Furthermore, the practice of marking-to-market portfolio exposures and hence directly account for downgrade losses is becoming increasingly relevant today with the widespread use of securitizations, also in the retail lending sector.

The contribution of this chapter is to provide new ways in which to test the accuracy of SA and IRBA on a portfolio of wholesale corporate exposures. We do so by comparing the regulatory capital obtained from the Basel II approaches with the economic capital resulting from CreditMetrics, a popular credit-risk model which provides a natural way to relax the assumptions underlying the IRBA. Furthermore, we single out the effect of correlation and downgrade risk, as well as, loan maturity, probability of default, and credit rating on the estimation bias produced by the regulatory models. We do so dynamically over a 10 year period on portfolios of Eurobonds with different granularity and risk characteristics. We find that the IRBA (as well as the SA) can produce regulatory capital that is as much as three times larger than the economic capital generated with Credit-Metrics. This bias has the same order of magnitude as in Jacobson et al. (2006) but is smaller, probably because of the explicit inclusion in our analysis of downgrade losses which cause economic capital to rise. The fact that previous findings about the conservativeness of the new capital regulation are broadly confirmed reinforces concerns about the economic impact of the new rules especially in the light of the ongoing debate about the propensity of the New Accord to aggravate credit rationing in recessions (Gordy and Howells 2006).

This chapter is organized as follows. Section 16.2 summarizes the data used for our analysis. In Section 16.3 we describe the Basel II models for credit risk. Section 16.4

* This result is derived from Table 12 in their paper. A typo in the text of the paper refers to regulatory capital being 6% to 9% higher than economic capital rather than 6–9 times higher.

introduces the benchmark model that we employ to test the accuracy of the regulatory models. In Section 16.5 we present our results and Section 16.6 concludes the chapter.

16.2 DATA

The data we use for this study were obtained through Reuters and include U.S. dollar-denominated bonds issued by 502 firms.* Our criteria in selecting the bonds are (1) that they were neither callable nor convertible, (2) that a rating history was available, (3) that the coupons were constant with a fixed frequency, (4) that repayment was at par, and (5) that the bond did not possess a sinking fund.

The composition of the total portfolio is shown in Table 16.1. The obligors domiciled in the United States constitute 46.4%. A further 27.5% of the companies are headquartered in Japan, the Netherlands, Germany, France, or the United Kingdom; 54% of the companies in our sample are in the financial services or banking sectors.

To implement the benchmark model, we also needed: (1) transition matrices, (2) default spreads and default-free yield curves, (3) equity index data, and (4) a set of weights linking individual obligors to the equity indices. Transition matrices were sourced from Standard and Poor's (Vazza et al. 2005). Default-free interest rates and spreads for different rating categories were taken from Bloomberg.† We also created an equity index dataset going back to 1983 and comprising 243 country- and industry-specific MSCI indices. For each

TABLE 16.1 Portfolio Characteristics

Country	**No. of Firms**	**Percentage**
United States	233	46.4
Japan	42	8.4
Netherlands	36	7.2
United Kingdom	27	5.4
Germany	17	3.4
France	16	3.2
Other	131	26.1
Industry	**No. of Firms**	**Percentage**
Financial services	160	31.9
Banking	111	22.1
Utilities	27	5.4
Energy	20	4.0
Merchandising	15	3.0
Telecoms	15	3.0
Other	154	30.7
Total	502	

* Of these, 90% were Eurobonds, the remainder are national bonds from several countries.

† We used spreads for U.S. industrials since these had the longest series and the fewest missing observations.

obligor, based on the domicile and industry classification provided by Reuters, we then chose one of these indices as the source of systematic risk.

16.3 BASEL II

In this section we briefly summarize the main features of the alternative approaches to computing credit-risk capital requirements in Basel II, the SA, and the IRBA. The SA and IRBA are implemented with variations that depend on the type of borrower namely, large corporations, sovereigns, banks, and retail borrowers. We shall focus on large corporations and banks only, as the data we employ for the empirical analysis are bonds issued by these types of obligors.

The SA produces a capital requirement with a method similar to that in use in the first Basel Accord (BCBS 1988), also called Basel I. Each claim is assigned a risk weight and regulatory capital is given by the sum of the values of all claims, multiplied by their respective risk weights, and a constant factor of 8%. In Basel I weights vary in relation to the type of borrower. In the SA of Basel II, risk weights vary according to both borrower type and the borrower's external credit rating. Different risk charges are applied to unrated companies.* For claims on banks, the SA offers two options and all banks in a country will be subject to one of the two at the discretion of the national supervisor. Under option one, the risk weight depends on the risk category assigned to the country of incorporation of the borrowing bank. Risk categories are identified by the range of sovereign ratings that attract the same risk weight. So, for example, sovereign ratings from AAA to AA– represent one category as all have a 0% risk weight. Option two is particularly suitable for countries whose banks are mostly unrated. Under option two, the risk weight depends on the borrowing bank's own credit rating.† A summary of SA risk weights for large corporations, sovereigns, and banks is reported in Table 16.2.

The credit-risk capital charge under the two internal rating-based approaches is based on the same analytical framework. However, it is only in the advanced IRBA that banks are responsible for estimating all the four parameters of the model, probability of default (PD), loss given default (LGD), exposure at default (EAD), and effective maturity (M). Banks under the foundation IRBA will have to provide own estimates of the PD, but they are required to follow specific computation rules for the other three parameters. For both approaches, the minimum capital requirement is given by,

$$\text{IRBA Capital} = \sum_{s}^{N} \text{RWA}_s \tag{16.1}$$

where

RWA_s denotes the risk-weighted value of asset s

N is the total number of assets in the bank's credit-risk portfolio

* All firms in our sample are rated; so the unrated weight has not been used.

† Also paragraph 62 in Basel II states "…a preferential risk weight that is one category more favorable may be applied to claims with an original maturity of three months or less, subject to a floor of 20%. This treatment will be available to both rated and unrated banks, but not to banks risk weighted at 150%."

TABLE 16.2 Risk Weights under the Standardized Approach in Basel II

Rating	Sovereign	Banks (Option 1)	Banks (Option 2)	Corporates
AAA	0	0.2	0.2	0.2
AA+	0	0.2	0.2	0.2
AA	0	0.2	0.2	0.2
AA−	0	0.2	0.2	0.2
A+	0.2	0.5	0.5	0.5
A	0.2	0.5	0.5	0.5
A−	0.2	0.5	0.5	0.5
BBB+	0.5	1	0.5	1
BBB	0.5	1	0.5	1
BBB−	0.5	1	0.5	1
BB+	1	1	1	1
BB	1	1	1	1
BB−	1	1	1	1
B+	1	1	1	1.5
B	1	1	1	1.5
B−	1	1	1	1.5
<B−	1.5	1.5	1.5	1.5
Unrated	1	1	0.5	1

The RWA_s is defined as follows,

$$RWA_s = UL_s \cdot EAD_s \tag{16.2}$$

where

UL_s is the percent unexpected loss in assets

EAD_s is the asset's exposure at default

From this formula we can infer that regulatory capital under the IRBA is designed to cover only for unexpected losses. This is because banks systematically set aside provisions for expected losses. UL_s is defined as the difference between the expected loss from borrowers in a downturn scenario and the expected loss under a normal scenario. It is formally defined as*

$$UL_s = CF \cdot MA_s \cdot LGD_s \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD_s) + \Phi^{-1}(0.999)\sqrt{R_s}}{\sqrt{1-R_s}}\right) - PD_s\right] \tag{16.3}$$

where CF is a calibration factor that was introduced to broadly maintain the aggregate level of regulatory capital that was in place across the banking industry before the

* A comprehensive description of the rationale behind the UL formula can be found in Resti and Sironi (2007b), Chapter 20, and Basel Committee on Banking Supervision (2005).

introduction of Basel II. The factor is currently equal to 1.06. Φ denotes a cumulative standard normal. $\Phi(\cdot)$ is an expression for the PD in a downturn scenario and is fully derived in the Appendix. MA_s is a maturity adjustment which grows with effective maturity, M, and falls as PD increases. The idea behind it is that longer maturity bonds, which are riskier, should attract a higher capital charge. However, if PD goes up, the MA_s will fall because lower quality assets are exposed to downgrade risk to a lower extent than higher quality assets. In other words, the scope for loss in value due to a downgrade is larger for an AAA asset than for an asset with lower credit rating.* The maturity adjustment is given by

$$MA_s = \frac{1 + (M_s - 2.5)b(PD_s)}{1 - 1.5b(PD_s)} \tag{16.4}$$

where

$$b(PD_s) = (0.11852 - 0.05478 \ln (PD_s))^2 \tag{16.5}$$

$b(\mathrm{PD}_s)$ has been calibrated on market data to produce the downgrade effect discussed above. M is obtained with a simplified duration formula and is defined as

$$M_{\mathrm{s}} = \left(\sum_t t \cdot c_t \right) \Big/ \sum_t c_t \tag{16.6}$$

where c_t denotes the cash flow of asset s at time t. R_s in Equation 16.3 is the squared correlation between the return of a borrower's assets and the return on a systematic risk factor.† By analyzing a large database of United States, Japanese, and European firms, Lopez (2004) found R_s to be a decreasing function of the PD and an increasing function of the size V of borrower s. These findings have been implemented in the IRBA with the following formula:

$$\begin{aligned} R_s = {} & 0.12 \cdot (1 - \exp(-50 \cdot \mathrm{PD}_s))/(1 - \exp(-50)) \\ & + 0.24 \cdot [1 - (1 - \exp(-50 \cdot \mathrm{PD}_s))/(1 - \exp(-50))] \\ & - 0.04 \cdot (1 - (V - 5)/45) \end{aligned} \tag{16.7}$$

where V is measured in terms of the firm's annual sales in million Euros. The size adjustment does not apply to companies that have a turnover of more than 50 million Euros, and is set at −0.04 for those with annual sales lower than 5 millions. Although we do not have turnover data for the companies in our sample, it is plausible to assume that their size is considerable, since most of them are Eurobond issuers. Therefore, we do not

* This does not mean however that higher quality exposures will attract higher capital charges. Although, all else equal, they will have a higher MA, their PD, which has a dominant effect on the unexpected loss in Equation 16.3, will drive down their risk weight. Therefore, the impact of MA as credit quality improves is to make the fall in capital requirement less sharp.

† Provided that the systematic risk factor is the same across borrowers, the factor cancels out in the derivation of the UL_s formula and so it need not be explicitly identified (see the Appendix for details).

apply the size adjustment. If we ignore the size adjustment, Equation 16.7 indicates that R_s is a weighted average of, i.e., it varies between, 0.12 and 0.24, with the parenthetic values being normalized weights that sum 1. The weights are a function of PD and cause R_s to decline as PD increases.

Interestingly, regulatory capital under the IRBA is additive—as is in the SA and in Basel I—in the sense that to arrive at the total capital requirement one needs to sum the individual capital charge of each asset in the portfolio. However, while additivity in Basel I and the SA in Basel II follow from the implicit assumption that assets in the portfolio are perfectly correlated, the IRBA additivity is the result of the assumptions that the portfolio is perfectly diversified (or infinitely granular) and that only one systematic risk factor drives the correlation among the assets in the portfolio.* Therefore, in the IRBA, additivity obtains even though correlation is less than 1.

We implement as closely as possible the advanced IRBA by using the information in our data sample. M is estimated as in Equation 16.6 and subject to a lower and upper boundary of 1 and 5 years, respectively, as indicated in Basel II.† For the probability of default, as we lack internal rating data, we assume that the bank's internal rating system perfectly replicates that of a recognized rating agency, Standard and Poor's. This is not implausible as Basel II allows banks to map their internal ratings to agency ratings and employ the default probabilities of the latter.‡ The PDs fed into the IRBA model are, at each point in time and for each credit rating, the averages of point-in-time default rates for that rating over the previous 5 years.§ As prescribed in Basel II, we constrain default probabilities to be greater than or equal to .03%.** As we do not have enough information on LGD for different seniorities over the sample period, we use a flat LGD of 50% across the whole sample.†† The exposure at default is equal to 1 for each firm as we aim to form portfolios in which assets are equally weighted.

16.4 BENCHMARK

The IRBA in Basel II is derived under some restrictive assumptions, namely that (1) the credit risk of individual exposures is driven by one systematic risk factor only and that (2) the portfolio of exposures is fully diversified. To remove such assumptions we use CreditMetrics, a popular credit-risk model proposed by Bhatia et al. (1997). The model allows for multiple systematic factors and portfolio concentration. Below we describe its main features.

The main contribution of CreditMetrics is a simple way to model the correlation of illiquid credit exposures. Most bank loans are illiquid in the sense that they are not traded, in an exchange or over the counter, and hence their prices are not available. CreditMetrics models the price of a credit exposure as a discrete process. The exposure's future value will

* See Gordy (2003) for a detailed derivation of this result and an analysis of the properties of the IRBA model.

† See BCBS (2006a), para. 320.

‡ See BCSB (2006a), para. 462.

§ According to Basel II default probabilities should be estimated by taking average default rates over a minimum period of 5 years (see BCBS 2006a, para. 447, 463).

** see BCSB (2006a), para. 285.

†† More than 70% of our bonds are unsecured (57.34% of unsecured proper and 14.14% of senior unsecured), which according to Carty and Lieberman (1996) have an average recovery rate of 51.13%. Bonds with lower seniority (that is, subordinated), and hence with a lower recovery rate, account for only 1.55% of the total sample.

depend on the credit rating of the borrower at the chosen horizon (typically 1 year). The correlation between any two exposures will then depend on how their future rating scenarios are correlated. The idea in CreditMetrics is to model changes in rating scenarios of borrowers with a standard normal latent variable Q_s that represents the return of the assets of that borrower. Positive and large asset returns will push the borrower towards a higher rating, while large negative returns will cause a downgrade and, in severe cases, default. The thresholds $Z_{i,j}$ that determine transitions from an initial rating i to a rating j are computed by employing the transition probabilities $\pi_{i,j}$ found in rating transition matrices. These are regularly published by all the major rating agencies. The thresholds can be estimated directly using the recursive equations below:

$$\begin{cases} \pi_{i,D} = \Phi(Z_{i,D}) \\ \pi_{i,CCC} = \Phi(Z_{i,CCC}) - \Phi(Z_{i,D}) \\ \dots \\ \pi_{i,AAA} = 1 - \Phi(Z_{i,AA}) \end{cases} \tag{16.8}$$

Transition probabilities in a given year are estimated by taking the average of the previous 5 years' point-in-time transition matrices. To take into account indirect migration and generate nonzero default probabilities for ratings at the top end of the rating scale we use the "generator" approach introduced by Jarrow et al. (1997) and refined by Israel et al. (2001).

The value of the asset return Q_s is assumed to depend on one or more systematic factors. Under the simplifying assumption that each borrower's asset return depends on one factor only, the asset return will be given by

$$Q_s = \theta_s X_s + \varepsilon_s \tag{16.9}$$

where X_s and ε_s denote the standardized return on the factor and the idiosyncratic component of the firm's asset return, respectively. Then, the correlation between the asset return of borrower s and v will be $\theta_s \theta_v \rho_{s,v}$ where $\rho_{s,v}$ is the correlation between factors X_s and X_v. In CreditMetrics $\rho_{s,v}$ is proxied with the correlation of the equity indices that most closely represent borrower s and borrower v. Typically, the chosen index is that of the country and industry sector in which a borrower operates. However, it is unclear how the systematic factor loading θ_r should be estimated. For simplicity, we use the approach in the IRBA which expresses θ_r as $\sqrt{R_r}$. This has the additional advantage of simplifying the comparison between the regulatory model and the benchmark, as it eliminates one of the potential sources of difference.*

Since the latent variable Q_s is standard normal, one can easily apply Monte Carlo methods to generate correlated rating scenarios for all the loans in the portfolio and derive the distribution of portfolio values.† Then, the portfolio loss distribution is obtained by taking

* This rather ad hoc assumption implies that asset correlation under the benchmark will always be lower than that under the IRBA. Indeed, recent findings indicate that asset correlation as computed in the IRBA may be too conservative (see Rösch and Scheule, 2008).

† See Bhatia et al. (1997), Chapters 10 and 11 for details.

the difference between the 1 year forward value of the portfolio under the assumption that all exposures maintain their current rating and the generated portfolio values.* By computing the Value-at-Risk at the 99.9% confidence level on the loss distribution, as prescribed by Basel II, the unexpected loss can then be estimated as the difference between the VaR and the loss mean value. The comparison of the unexpected loss so derived with the regulatory models' capital charge will be the focus of our discussion in the reminder of the chapter.

16.5 RESULTS

In this section we compare the capital charge produced by the benchmark with the capital charges of Basel II's SA and IRBA. The starting point of our analysis is the asymptotic single risk factor model (ASRF) of Gordy (2003), a special case of both, the IRBA and the benchmark. By adding the assumptions that distinguish the two models from the ASRF, one at a time, we will be in a position to study how they diverge and why.

Table 16.3 summarizes our findings. The three features that cause the IRBA to depart from the ASRF are the maturity adjustment, the .03% default probability floor, and the calibration factor. All of them increase the capital requirement relative to the requirement one obtains with a straightforward application of the ASRF. The maturity adjustment has the largest impact with an average increase in capital of 82.4%. The default probability floor follows with a 50.4% increase, while the calibration factor adds a further 14.0%. The benchmark, on the other hand, introduces granularity effects, (explicit) downgrade risk,

TABLE 16.3 Impact of IRBA and the Benchmark Models' Assumptions on Capital Charge

Assumptions	**Marginal Capital Charges, % of ASRF**
	IRBA
Maturity adjustment	82.4
PD floor	50.4
Calibration factor	14.0
IRBA total	146.8
	Benchmark
Granularity	36.9
Multiple factors	−41.5
Downgrade risk	108.6
Benchmark total	104.0

Note: This table reports the marginal capital charges produced by the various assumptions underlying the IRBA model in Basel II and the benchmark model. Estimates are averages over the sample period based on the implementation of the models on our bond data. Marginal effects are percentages expressed in terms of the capital charge of the ASRF model of Gordy (2003).

* See, for example, Crouhy et al. (2001), Section 2.4.

and multiple systematic risk factors to the basic ASRF. The largest rise in capital is caused by downgrade risk, +108.6%, followed by granularity effects with +36.9%. The introduction of multiple factors on the other hand reduces capital by a substantial 41.5%.* The total of the above modeling assumptions for the IRBA is +146.8% which is higher than the total for the benchmark +104.0%. Therefore, from these initial observations it appears that the regulatory model produces more conservative capital requirements than the benchmark. Below, we shall investigate the causes behind this result in more detail.

Figure 16.1 shows that CreditMetrics with no downgrade risk and a single systematic factor actually converges to the ASRF as portfolio diversification increases. The figure also reveals two important points. First, credit risk may be hard to diversify. In portfolios with 100 exposures, the UL of CreditMetrics may still exceed the UL based on full diversification (ASRF) by a nonnegligible amount, due to concentration effects. Second, in high-quality portfolios, convergence is slower. For example, with an average portfolio PD of .1%, the discrepancy between CreditMetrics and ASRF is still 15.9% when the number of exposures is as high as 200.

Figure 16.2 shows the granularity effect when the benchmark is applied to our bond data. The benchmark is implemented in the simplest way, that is, without downgrade risk and with a single systematic risk factor. The difference in unexpected loss is larger at the

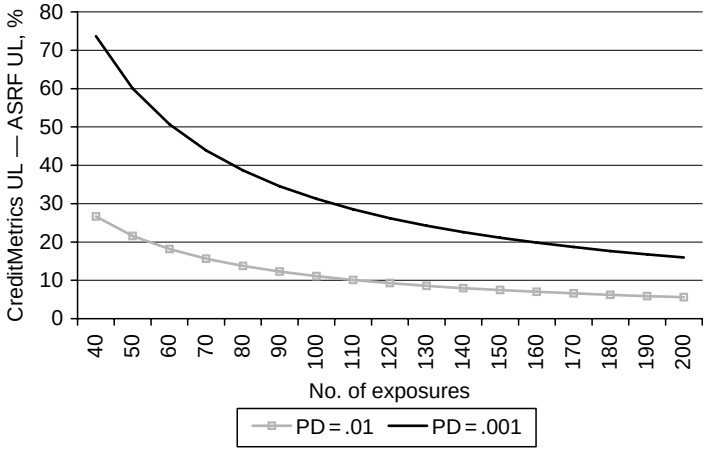


FIGURE 16.1 Convergence between CreditMetrics and asymptotic single risk factor (ASRF). In this graph we report the percentage difference in unexpected loss (UL) between CreditMetrics and the ASRF model as the number of exposures in the portfolio increases. Results are based on a sample of fictitious exposures with identical default probability and exposure at default, and a loss given default of 50%. The confidence level used to derive UL with both models is 99.9%. CreditMetrics has been implemented without downgrade risk and with a single risk factor.

* Within the framework of the benchmark model, using a single systematic factor is equivalent to using multiple perfectly correlated factors. Hence, as risk factors are always less than perfectly correlated, it follows that portfolio correlation, and as a result, unexpected losses, go down when multiple factors are introduced in the model.

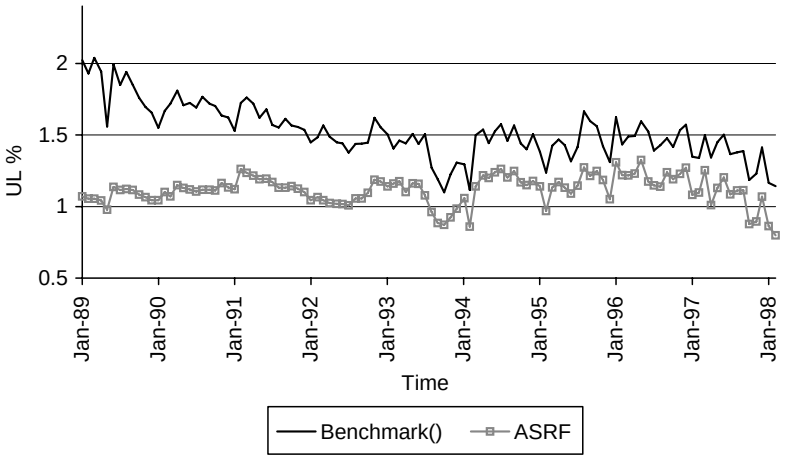


FIGURE 16.2 Unexpected loss comparison with no downgrade risk and a single systematic factor. The benchmark model is implemented with no downgrade risk and a single factor. ULs are expressed as percentages of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

beginning of the sample period because the sample size increases with time. The negative correlation between the number of exposures and the unexpected loss difference between the two models is easily inferred from Figure 16.3.

We shall now present a temporal breakdown of the impact on the unexpected loss of those assumptions of the IRBA and the benchmark that differentiate these models from the

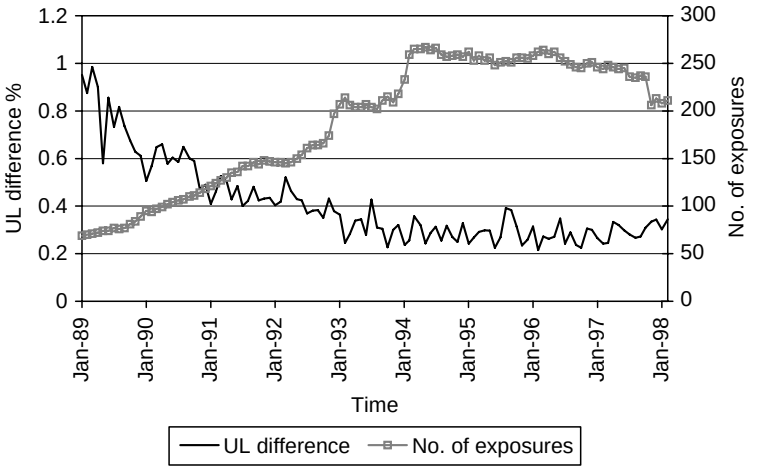


FIGURE 16.3 Unexpected loss difference and granularity. The UL difference is the difference between the benchmark and the ASRF ULs. ULs are expressed as percentages of expected portfolio value at time $t + 1$ year. The benchmark model is implemented with no downgrade risk and a single factor. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

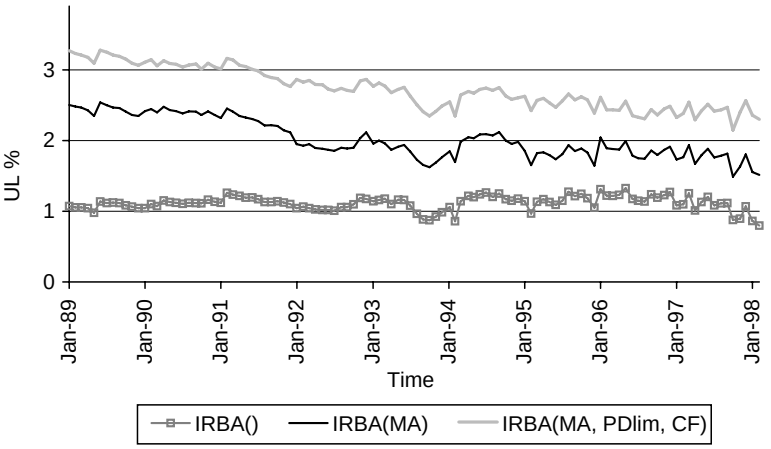


FIGURE 16.4 Internal rating-based approach (IRBA) unexpected loss under different assumptions. We report the unexpected loss of the IRBA under three scenarios: IRBA (MA, PDlim, CF) which includes maturity adjustment, default probability floor, and calibration factor; IRBA(MA) with only the maturity adjustments; and IRBA() with none of the above which is the same as the ASRF model. ULs are expressed as percentages of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

ASRF and whose average effect was reported in Table 16.3. Figure 16.4 shows how the IRBA unexpected loss increases when the maturity adjustment, the PD lower bound, and the calibration factor are introduced. The noticeable fall in the capital requirement of the full IRBA is mainly due to a declining average effective maturity of the portfolio which goes from 4.6 years at the beginning of the sample period to 3.2 years at the end of the period. Lower effective maturity directly reduces the maturity adjustment in Equations 16.3 and 16.4 thus generating the pattern we observe. Figure 16.5 illustrates how the benchmark reacts to the introduction of downgrade risk and multiple systematic risk factors. The figure reveals that downgrade risk falls markedly between January 1991 and April 1993 causing the UL to drop by about 50%.

Finally, we investigate the core question of this study which pertains to the difference in capital charges produced by the full implementation of both models as well as the SA. Figure 16.6 shows that both Basel II approaches yield higher capital requirements than the benchmark and increasingly so as time goes by. The divergence is particularly striking from 1993 onward and peaks in April 1997 when IRBA and SA are about 196% and 322% higher than the benchmark, respectively.

The figure also indicates that SA and IRBA start to diverge from the beginning of 1993. The SA may only produce higher capital charges when the average rating of the assets in the portfolio declines. From the above one may infer that the SA may become substantially more conservative than the IRBA as credit quality deteriorates. This point is summarized in Figure 16.7, which shows how the difference between IRBA and SA appears to be tightly related to the average rating in the portfolio. It should be stressed however, that our sample

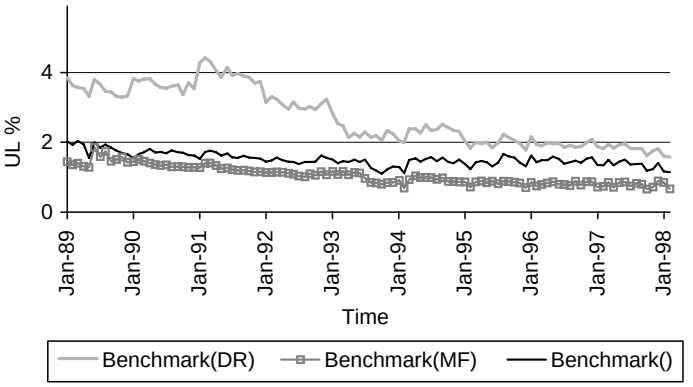


FIGURE 16.5 Benchmark unexpected loss under different assumptions. We report the unexpected loss of the benchmark model under three scenarios: benchmark(DR) with downgrade risk and one systematic factor, benchmark(MF) with multiple factors and no downgrade risk and benchmark() with no downgrade risk and one systematic factor. ULs are expressed as percentages of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

does not cover the whole rating scale and hence our conclusions may not apply to portfolios that have a very low credit quality. Indeed, Resti and Sironi (2007b)* show that for a given maturity (2.5 years) and recovery rate (45%) and given default rates associated with each rating category, the difference between the SA and the (foundation)

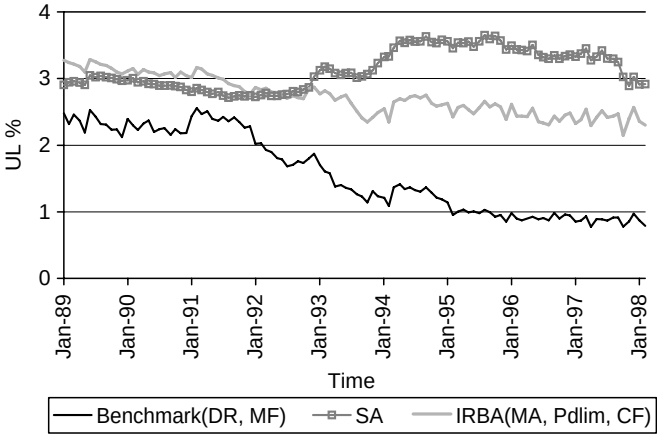


FIGURE 16.6 Unexpected loss (UL) comparison: Benchmark, SA, and IRBA. The benchmark is implemented with downgrade risk and multiple factors while the IRBA includes the maturity adjustment, the default probability floor, and the calibration factor. The UL is expressed as a percentage of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

* See Figure 20.4, page 617.

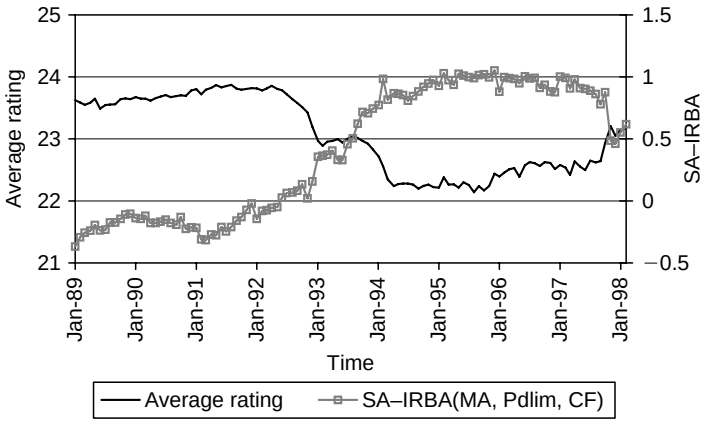


FIGURE 16.7 Average rating and discrepancy between IRBA and Standardized approach (SA). The line with squares is the difference in capital charge between SA and the IRBA implemented with all adjustments. The thin line is the average rating in the portfolio. To compute the average rating, numerical values have been assigned to individual rating categories as follows: AAA = 27, AA+ = 25, AA = 24, AA− = 23, A+ = 22, A = 21 . . . where each further notch down causes the numerical value to decrease by 1.

IRBA may go up or down. In their example, the SA produces increasingly higher capital requirements than the IRBA as the average quality of the portfolio declines from AA to A and from A to BBB, while the distance between SA and IRBA falls as one moves further down the rating scale. Eventually, the IRBA overtakes the SA in portfolios with B and CCC average rating.

Next, we test the performance of the three models in portfolios with different risk profile. We construct a high-risk and a low-risk portfolio with 40 exposures each, the ones with the lowest and highest ratings, respectively, among those available in our sample at each point in time. The results are reported in Figures 16.8 and 16.9. Although, judging from Figure 16.1, the effect of portfolio concentration on these small size portfolios should produce a noticeable increase of the benchmark UL relative to the IRBA UL, it turns out that, as before, the IRBA yields, in most cases, capital charges that are higher (and often much higher) than the benchmark. So, the maturity adjustment, PD floor, and calibration factor in the IRBA more than compensate for the granularity effect in the benchmark. Also, since the number of assets in the portfolio remains constant over time, it follows that the increasing discrepancy between the regulatory models and the benchmark cannot be the result of changes in granularity. We will explore the causes of such differences with regression analysis in the next section.

In the low-risk portfolio the IRBA systematically overshoots the benchmark. The SA also produces higher capital charges except for the first few months in 1989 when it falls below the benchmark. Also, as the credit quality of the portfolio does not fluctuate much and it always lies in the first rating bucket, the SA capital charge is flat at 1.6%, given by the product of the top rating bucket risk weight 20% and the 8% Cooke ratio.

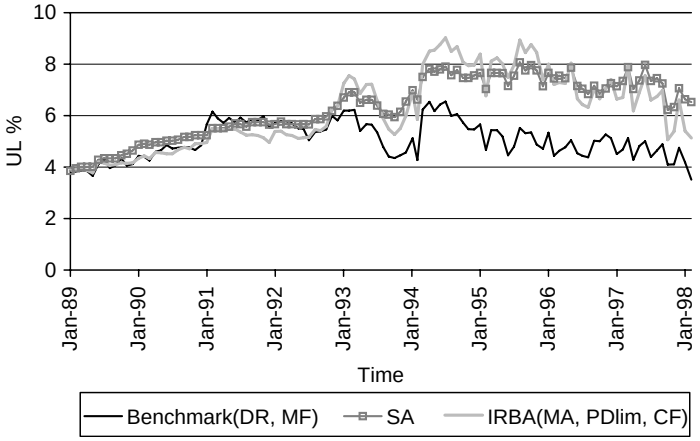


FIGURE 16.8 UL comparison in high-risk portfolio. The benchmark is implemented with downgrade risk and multiple factors while the IRBA includes the maturity adjustment, the default probability floor, and the calibration factor. The high-risk portfolio includes, at each point in time, the 40 bonds with the lowest rating in the sample. The UL is expressed as a percentage of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

Figures 16.6, 16.8, and 16.9 show that none of the Basel II models consistently yields a higher capital charge. However, according to the last quantitative impact study run by the Basel Committee before the final approval of the Basel II framework in 2006 (BCBS 2006b)

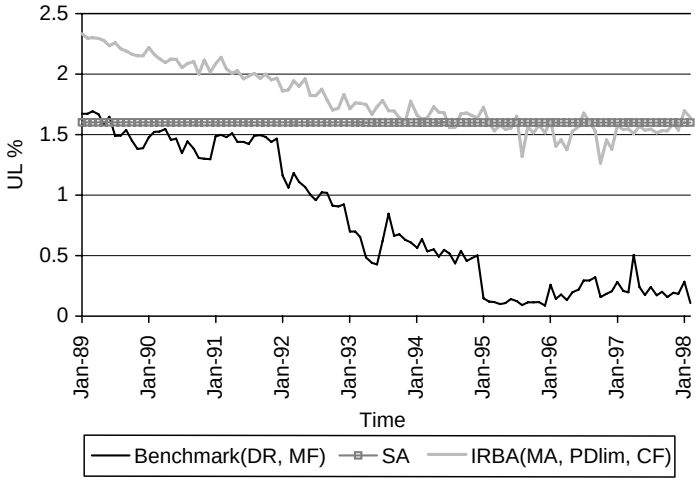


FIGURE 16.9 UL comparison in low-risk portfolio. The benchmark is implemented with downgrade risk and multiple factors while the IRBA includes the maturity adjustment, the default probability floor, and the calibration factor. The low-risk portfolio includes, at each point in time, the 40 bonds with the highest rating in the sample. The UL is expressed as a percentage of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

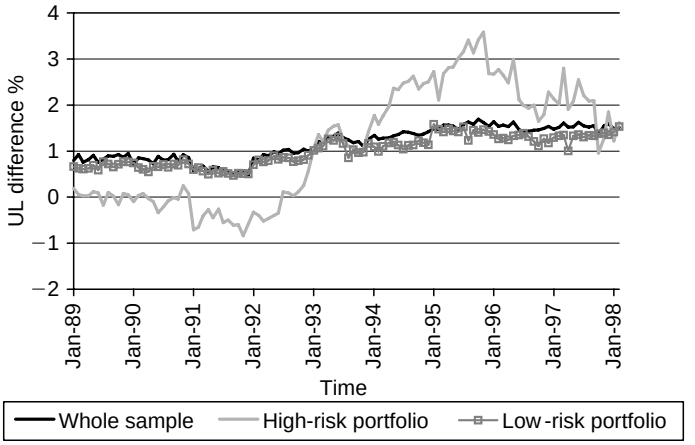


FIGURE 16.10 Internal rating-based approach (IRBA) overestimation bias. We report the difference between the unexpected loss derived with the IRBA with all the adjustments and the benchmark implemented with downgrade risk and multiple factors. The UL difference is expressed as a percentage of expected portfolio value at time $t + 1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

the SA yielded on average, across all the 382 banks in the 32 countries that participated in the exercise, a higher capital requirement than either implementation of the IRBA (foundation or advanced).* But, it should be noted that the impact study was done at a particular point in time and hence its result need not apply in general. The SA will produce higher regulatory capital than the IRBA when economic conditions are particularly benign, as was the case during the last impact study. This is because internally derived default probabilities within each rating category will fall and thus will compress the unexpected loss of the IRBA. On the contrary, the SA capital charges cannot reflect such change in PD as they are fixed for each rating category.

Figures 16.10 and 16.11 report the difference between the two Basel models and the benchmark and are derived directly from Figures 16.6, 16.8, and 16.9. Under the IRBA, the overestimation bias is always positive for the whole portfolio and the low-risk sub-portfolio and its average over the sample period is 95% and 82% above the benchmark, respectively. In the high-risk sub-portfolio the bias is much more volatile and, surprisingly, becomes considerably negative in 1991. The lowest IRBA bias is −0.84% of (expected) portfolio value and occurs in November 1991. This is 37% lower than the benchmark UL. On the other hand, the maximum overestimation occurs in November 1995 with a whopping 375% increase on the benchmark level.

In the SA the bias exhibits a wider variation than the IRBA bias when we look at the whole portfolio and the low-risk sub-portfolio. This is no doubt the result of the lower risk sensitivity of the SA. Indeed the fixed risk weights of the SA do not change when the default probability, asset correlation, or downgrade risk vary. So, the SA bias fully reflects the

* See Table 1 page 2 of Basel Committee on Banking Supervision (2006b).

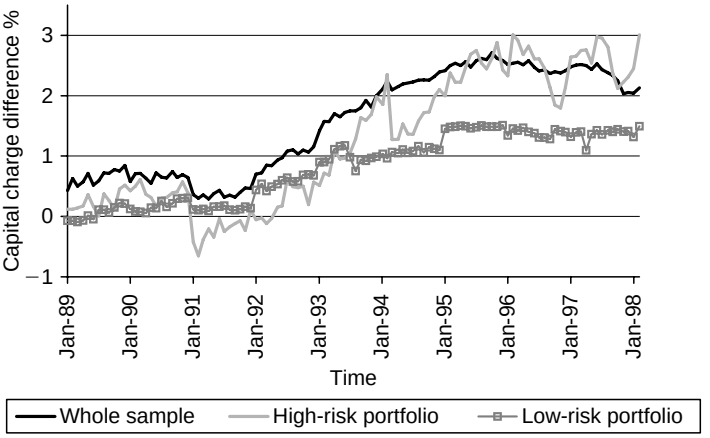


FIGURE 16.11 Standardized approach overestimation bias. We report the difference between the unexpected loss derived with the SA and the benchmark implemented with downgrade risk and multiple factors. The capital charge difference is expressed as a percentage of expected portfolio value at time $t+1$ year. Time refers to the date at the VaR horizon, i.e., current time plus 1 year.

changes in the benchmark due to fluctuations in these factors. On the other hand, the range of variation of the SA bias for the high-risk portfolio is comparable to that of the IRBA and is in fact, slightly lower. This indicates that both regulatory models share comparable (and substantial) inaccuracy when measuring the risk in low-quality portfolios. Similarly to the IRBA, the lowest and highest SA biases occur in the high-risk portfolio. The lowest bias comes about in February 1991 and equals −0.65% of the expected portfolio value, 25% below the benchmark. The highest bias happens in December 1995 and is equal to 3.01%, which is 335% above the benchmark.

In conclusion, Figures 16.6, 16.8, and 16.9 reveal that the regulatory models are closer to each other than to the benchmark. Then, it appears that although the IRBA is a step in the right direction from a fixed risk-weight approach such as Basel I and the SA in Basel II, still it has not yet gone half the way towards a full portfolio model. It is also clear that the difference between the two Basel models and the benchmark (Figures 16.10 and 16.11) may vary substantially over time, an indication that they exhibit difference sensitivity to mutating economic conditions. This suggests that the extent of the overestimation (and occasional underestimation) cannot be simply fixed by a scaling factor. Moreover, the alternating sign of the bias shows that the regulatory models may lead to capital requirements that are not necessarily conservative, as often believed, but could in fact be too low at times.

16.5.1 Regression Analysis

In this section we explore the causes behind the time variation in the overestimation bias of the Basel II models discussed in the previous section. This will help us to identify the factors that may cause the regulatory models to misrepresent portfolio credit risk. We carry out this analysis by regressing changes in the IRBA and SA biases on changes in the

characteristics of the portfolios. These include (1) average effective maturity, (2) average rating, which denotes both the internal rating for the IRBA and the external rating for the SA,* (3) average default probability, (4) portfolio concentration as measured by the Herfindahl index, (5) systematic factor correlation, and (6) the level of downgrade risk. The rating variable has been constructed by assigning a numerical value to each rating as follows, AAA = 27, AA+ = 25, AA = 24, AA− = 23, and so on. The wider gap between AAA and AA+ was present in conversion tables supplied by Reuters and denotes the absence of the AAA− rating and probably the greater difference in terms of financial strength between AAA and AA+ than between any other pair of adjacent ratings. Variable (5) is measured indirectly as the impact of changes in factor correlation on the benchmark UL. Downgrade risk is estimated as the difference in the unexpected loss of the benchmark when diagonal rating transition matrices (i.e., without downgrade risk) are replaced with full transition matrices. Regression results are reported in Table 16.4.

TABLE 16.4 Regression of Changes in Basel II Overestimation Biases on Changes of Portfolio Characteristics

	IRBA Overestimation Bias			SA Overestimation Bias		
	Whole Portfolio	**High-Risk Portfolio**	**Low-Risk Portfolio**	**Whole Portfolio**	**High-Risk Portfolio**	**Low-Risk Portfolio**
Constant	0.006	−0.007	0.002	0.001	0.017	0.002
Concentration	−10.32	—	—	−80.77	—	—
Rating	−0.162	−0.541***	−1.111***	−0.357***	−0.154	−1.111***
Default probability	21.74***	0.147	−23680.2***	−52.14***	−40.69***	−23685.6***
Effective maturity	0.482***	0.411**	0.319***	0.010	−0.093	−0.112***
Downgrade risk	−0.565***	−0.233*	−0.331***	−0.649***	−0.332***	−0.331***
Factor correlation	0.016	−0.287***	−0.226	0.055	−0.195***	−0.226
Adjusted R-squared	0.452	0.609	0.644	0.481	0.387	0.531

Note: The table reports the estimation results of regressions that aim to explain changes in the overestimation bias in Basel II's internal rating-based approach (IRBA) and standardized approach (SA) relative to the benchmark model. The estimation bias is measured as the difference between the unexpected loss obtained from one of the Basel II approaches and that of the benchmark model. The benchmark model is CreditMetrics implemented with the inclusion of downgrade risk and multiple systematic factors. The sample includes monthly observations over the period from January 1989 to February 1998. Explanatory variables are in first differences and include: portfolio "concentration" as measured by the Herfindahl index; average rating; average default probability; average effective maturity; downgrade risk which is estimated as the difference in unexpected loss of the benchmark model when diagonal rating transition matrices (i.e., without downgrade risk) are replaced with full transition matrices; and the variable that measures systematic "factor correlation." The "high-risk portfolio" (low-risk portfolio) columns denote a portion of the sample including, at each point in time, the 40 firms with the lowest (highest) rating. Parameters are estimated with ordinary least squares.

***, **, * indicate significance at the 1%, 5%, and 10% confidence level, respectively. Confidence intervals are estimated with standard errors adjusted for autocorrelation and heteroscedasticity.

* An interesting area of investigation, which is beyond the purpose of this paper, would be an assessment of systematic differences between agency ratings and banks' own ratings and how they impact on SA and IRBA regulatory capital. Differences may occur, for example, because of different emphasis on the rating time horizon (point-in-time or through-the-cycle) and differences in the definition of default.

The results indicate that the regulatory models do not share the same sensitivity to all explanatory variables. If the bias of different regulatory models is pushed in different directions by the same variable then banks using different models may be induced, as a result of regulation, to adopt different portfolio allocation and lending policies. For instance, the coefficient of effective maturity has a positive sign and is highly significant for the IRBA in the whole portfolio as well as the high and low-risk ones. So, banks can decrease the IRBA overestimation bias and hence their capital requirement by decreasing the duration of the assets in their portfolios. This may be achieved for example with policies that favor short-term lending or quick amortization of long-term loans. On the other hand, the coefficient of effective maturity is either not significant (whole portfolio and high-risk sub-portfolio) or negative and highly significant (low-risk sub-portfolio) in the SA. This implies that a bank that adopts the SA and has a high-quality portfolio can reduce overestimation bias and its capital charge through long-term lending, which is exactly the opposite outcome to that obtained with the IRBA.

Higher effective maturity makes the maturity adjustment defined in Equation 16.4 to go up which ultimately produces higher capital requirements. Given the positive sign of effective maturity in the IRBA regression, higher effective maturity also causes the IRBA estimation bias to increase, an indication that the influence of maturity on portfolio credit risk appears to be overestimated within the IRBA. This observation is consistent with Kalkbrener and Overbeck (2003) who reached a similar conclusion when investigating the maturity effect in the 2001 version of the New Accord. On the other hand, the negative and highly significant coefficient of effective maturity under the SA in the low-risk portfolio results from the influence of maturity on the benchmark model. Longer maturity produces higher volatility of forward bond values and hence higher unexpected loss from the benchmark. Since the SA is not affected by maturity, then, when maturity goes up, the gap between the SA and the benchmark will shrink which is the effect captured by the regression coefficient. However, such effect does not seem to be so prominent in the whole sample and in the high-risk portfolio.

Downgrade risk has a negative and always significant sign in both the IRBA and SA regression. In other words, when downgrade risk goes up, the benchmark model will reflect the change with higher unexpected losses and the IRBA and SA overestimation bias will fall. It is interesting to note that the maturity adjustment was introduced in the IRBA to account for downgrade risk.* But, the fact that the two variables are lowly correlated (0.28 correlation in first differences in the whole sample) and that each can only explain a fraction of the volatility of the other† suggest that the maturity adjustment does not accurately capture downgrade risk.

Although the *level* of granularity is important in explaining the difference between IRBA and benchmark (Figure 16.2), regression results show that *changes* in granularity are not significant in explaining changes in such difference, probably because they have been

* See, for example, Resti and Sironi (2007b), p. 611.

† When we regress one variable on the other, plus a constant, for the whole sample we find the adjusted R-squared to be 7.2%.

gradual over time and hence overshadowed by the effect of other variables. Even though the sign of the concentration variable, which is a measure of granularity, is not significant, still, as one might expect, it is negative. This means that higher concentration pushes the UL of the benchmark higher and closer to the regulatory models' ULs (as the latter remain unaffected by concentration).

The correlation between the external rating and default probability variables is negative, as expected.* However, the level of correlation is not as high as to cause concern for multicollinearity (−41% in the whole portfolio, −79% in the high-risk portfolio, and −61% in the low-risk portfolio). The reason for including both variables in the regression is that default probability may change within the same rating category over time. So default probability and rating need not move in unison. Also, changes in the external rating produce an impact on SA while changes in default probability do not.

The rating variable influences both the IRBA and the benchmark in a similar way since a change in rating brings about a change in the rating's associated default probability. However, the negative sign of the variable's coefficient suggests that the distance between the IRBA and the benchmark increases as the rating deteriorates. This may follow because of the parameterization of the IRBA which appears to become more and more conservative as credit quality goes down.

Similarly, higher values of the default probability variable bring about higher IRBA overestimation bias in the whole (average quality) portfolio. The direction of the relationship is the same as in the high-risk portfolio but the default probability coefficient in that instance is not significant. The result in the low-risk portfolio is more curious as the default probability coefficient is negative and highly significant. This implies that higher credit quality (i.e., lower PD) causes the gap between IRBA and benchmark to grow, which appears to contradict our previous conclusions. In fact, the finding seems to be the result of the PD floor of .03% in the IRBA. As the credit quality improves and the PD keeps falling below the .03% limit, the benchmark UL will fall while the IRBA UL will not be affected. Hence, the gap between the two opens up. According to this line of reasoning, however, the rating's coefficient in the low-risk portfolio should be positive, while it is negative and highly significant. The reason for it is that the rating variable for the low-risk portfolio only changes in the first 2 years of the sample period, when the usual negative relationship between credit quality and overestimation bias still applies. After that, the average rating of the portfolio is flat at the AAA level (as all assets in the high-risk portfolio are AAA rated from January 1991). Therefore, the negative relationship between credit quality and overestimation bias does not carry into the remaining years (i.e., most of the sample period) when the PD limit effect kicks in.

Unlike in the IRBA, in the SA both the rating and default probability coefficients are always negative. The two variables have the same sign because the SA is only affected by ratings and not by default probability. The negative sign of the rating variable means that, similarly to the IRBA, the SA appears to become more conservative than the benchmark as the credit rating deteriorates. On the other hand, a higher value of the default probability

* The correlation was estimated on the first difference of the variables.

variable causes the benchmark UL to increase while leaving the SA UL unaffected, which results in a lower SA overestimation bias.

The factor correlation variable is significant only in the high-risk portfolio under both regulatory models. Its sign is negative in the high-risk portfolio because if factor correlation goes up (down) the benchmark UL will go up and the overestimation bias will have to fall (increase) as the variable does not have any bearing on the regulatory models.

It should be noted that the low-risk portfolio under both IRBA and SA produces coefficients for all the explanatory variables, with the exception of effective maturity, that have very similar values (often identical up to three digits). This is because the IRBA and SA overestimation biases in the low-risk portfolio are highly correlated (65% on first differences) as they are overwhelmingly influenced by the behavior of the benchmark model (Figure 16.9).*

16.6 CONCLUSION

In this chapter we test the accuracy of the new credit-risk measurement techniques introduced with Basel II. We do so by comparing the credit-risk measures produced by the new regulatory framework with those obtained with a benchmark model that removes several of the restrictive assumptions of the former. We find that the discrepancies between regulatory models and the benchmark may be large. Our results may be summarized as follows: (1) Regulatory models are typically more conservative than the benchmark and may occasionally produce risk estimates that are more than three times higher than the benchmark level. This should raise some concern as the finding combined with the higher risk sensitivity of the new regulation may well exacerbate credit rationing in periods of economic recession. (2) Credit-risk underestimation is also possible although its magnitude during our observation period is small compared to the overestimation bias (37% and 25% below the benchmark under the IRBA and the SA, respectively). The implication is that banks may at times be overexposed to credit risk, despite the conservative approach taken in devising the new rules. (3) Contrary to the evidence presented in the last quantitative impact study undertaken by the Basel Committee (BCBS 2006b) we find that the SA may yield lower capital requirements than the IRBA, when default rates are high and the portfolio has long duration. (4) The difference between the regulatory models and the benchmark depends on the combined effect of several variables. As a result, the discrepancy could not be corrected easily, for example with the use of a constant scaling factor. (5) We find the more advanced internal rating-based model IRBA to yield risk measures that are on average closer to the SA than to the benchmark model. This suggests that although the IRBA and the benchmark are both portfolio models, the simplifying assumptions in the IRBA offset a large proportion of the benefits of a portfolio approach to credit-risk modeling. (6) Different regulatory models can produce different incentives and,

* The SA unexpected loss is constant over the sample period so the SA bias is entirely driven by the benchmark loss. In the case of the IRBA, although its unexpected loss changes, it does so with little volatility, so again the IRBA bias is dominated by the benchmark loss.

as a consequence, different portfolio allocation distortions in banks. For instance, the adoption of the IRBA may induce banks to shorten the duration of their assets to attract lower capital requirements and align regulatory capital with economic capital. On the other hand, the SA has little sensitivity to asset duration. So, banks could successfully engage in regulatory capital arbitrage by investing in assets with longer duration. (7) Finally, we find that the maturity adjustment appears to be a poor proxy for downgrade risk.

We should emphasize that our conclusions are based on the benchmark we employ in this study. Although the chosen benchmark provides a natural way to relax the assumptions in Basel II and is a widely popular model, its predictive ability has not been thoroughly investigated yet, partly due to data availability issues which make backtesting problematic.* Furthermore, in this work we do not consider possible generalizations of the benchmark in which recovery risk and credit spread risk are accounted for as suggested by Kiesel et al. (2001). These generalizations are likely to increase the capital requirements implied by the benchmark. However, this will not necessarily result in a greater consistency between the regulatory models and the benchmark as the additional risks, which are not captured by the regulatory models, will create further points of departure between the two. Moreover, the introduction of these risks leaves open the question of how to model the dependence between credit spreads, recovery rates, and transition rates. We leave the investigation of these issues to future research.

APPENDIX: DERIVATION OF DOWNTURN PROBABILITY OF DEFAULT IN THE IRBA

Following Vasicek (2002), let us assume that the asset return Q_s of any borrower s in a loan portfolio depends on one systematic risk factor only, X, and that the asset correlation between any two borrowers is constant and equal to R. Also, assume that the variables Q_s for $s = 1, 2, \ldots$ are jointly standard normal. Then, Q_s will be

$$Q_s = X\sqrt{R} + Z_s\sqrt{1-R}$$

where X and $Z_1, Z_2, \ldots$ are mutually independent standard normals. The unconditional probability of default for borrower s is simply

$$\mathrm{PD}_s = P(Q_s < Q_{s,d}) = \Phi(Q_{s,d})$$

where $Q_{s,d}$ is the default threshold for borrower s. On the other hand, the conditional probability of default depends on the value taken by the systematic factor X. Let us denote such value as x. Then, the conditional probability can be written as

$$PD(X = x)_s = P(Q_s < Q_{s,d}|X = x) = P\left(x\sqrt{R} + Z_s\sqrt{1-R} < Q_{s,d}\right)$$

* For a first analysis of the out-of-sample performance of such benchmark see Nickell et al. (2007).

which yields

$$PD(X = x)_s = P\left(Z_s < \frac{Q_{s,d} - x\sqrt{R}}{\sqrt{1-R}}\right) = \Phi\left(\frac{Q_{s,d} - x\sqrt{R}}{\sqrt{1-R}}\right)$$

In a downturn X will be very low (that is very negative). Let us call the value of X in a downturn, x_{down}. Basel II assumes a downturn scenario that occurs once every 1000 years. Then, x_{down} in Basel II will be

$$x_{\text{down}} = \Phi^{-1}(0.001)$$

So, the probability of default of borrower s conditional on a downturn that occurs with the above frequency will be

$$PD(X < x_{\text{down}})_s = \Phi\left(\frac{Q_{s,d} - \Phi^{-1}(0.001)\sqrt{R}}{\sqrt{1-R}}\right)$$

or equivalently

$$PD(X < x_{\text{down}})_s = \Phi\left(\frac{\Phi^{-1}(PD_s) + \Phi^{-1}(0.999)\sqrt{R}}{\sqrt{1-R}}\right)$$

which is the expression we find in Basel II documents.

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Part V

Credit Risk Dependence and Dependent Defaults

CHAPTER 17

Correlation Risk: What the Market Is Telling Us and Does It Make Sense?*

Vineer Bhansali

CONTENTS

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17.1 CORRELATION IN CREDIT MARKETS

Correlation drives the risk of credit portfolios just as volatility is the primary driver of options portfolios. This chapter discusses correlation in the context of credit derivatives markets, and focuses on interpretation relevant to investments. The basic issue is stated easily—if we could access the true joint distribution of all the assets in a credit portfolio, and simultaneously forecast their evolution, recovery levels, and timing of default, we would theoretically have no problem in the pricing and the risk management of any credit derivative product; we would know the full portfolio loss distribution at all times. The complications in practice arise due to the technical complexity of obtaining these all important loss distributions, and market practitioners have had to resort to some powerful yet limiting approximations. So, the use of the term correlation in the context of credit derivatives requires an understanding of the limitations of statistics and models that have been used to understand the joint behavior of corporate spreads.

Even without a model, we have a sense of correlation using statistical measures, i.e., by looking at historical data, and for many investment applications, this intuitive understanding is sufficient. However, recent product development has enabled the extraction of implied correlation using correlation-sensitive traded securities. The implied correlation approach adopts the risk-neutral pricing framework, and makes strong assumptions about the liquidity and risk-premium content, as well as the distributional characteristics of assets. We should note that the expected loss on straight portfolio products or baskets is not sensitive to changes in correlations, so tranches or other correlation-sensitive securities are required to extract any implied correlation content.

Statistical measures of correlation can be further divided. We can measure the correlation between asset values of the various entities, between the equity prices of the entities, or actual defaults of the entities. Default statistics by nature consist of rare event counting, and, thus, are not easily quantified. Underlying asset dynamics are not always easy to extract from equity prices, so modelers frequently assume that observed equity correlations are sufficient, and there is reasonable evidence that this is not a particularly onerous assumption (Mashal et al. 2003).

The standard model for default correlations assumes some form of structural model for the underlying entities whose net assets evolve under a stochastic process (typically taken to be jointly lognormal), and a default occurs when the net asset value falls below a default threshold. Calibration of models poses another set of problems, and as of this writing, most practitioners use a mix of structural and reduced form (hazard intensity based) approaches.

We can extract implied correlation from correlation-sensitive securities, such as tranches and default baskets, but there is no guarantee that the correlation so extracted has anything to do with the real world. The problem with naive implied correlation is actually even more severe because the results depend on the assumption that the default swap market is complete. This is a strong and incorrect assumption in reality, since there is no way to arbitrage mis-pricings in these instruments. May 2005 exposed the weakness of so-called arbitrage trades between tranches whose hedge ratios were determined by blind

reliance on model sensitivities. Figure 17.1 shows the structural break in the relationship between the equity tranche and the senior mezzanine tranche for the investment grade index. As discussed below, these two tranches carry a significant amount of spread, default, and correlation risk.

Correlation is to tranched portfolio credit losses what volatility is to plain vanilla options, so we should explore the analogy with options models a bit further to build intuition. When we price options, we can choose to use historical or implied volatilities to plug into an option pricing formula. A number of assumptions are made to obtain the price of options. First, we assume that we know the option payoff. Second, we assume that

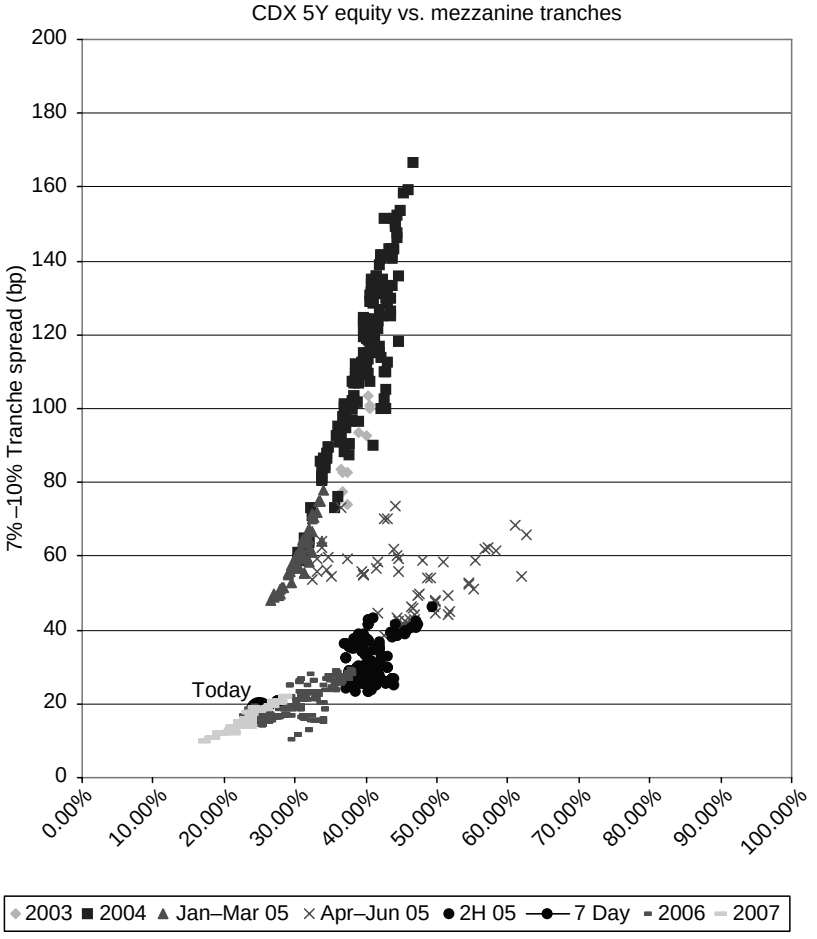


FIGURE 17.1 0%–3% Tranche vs. 7%–10% tranche for the CDX IG index. The *x*-axis shows the price of the 0%–3% equity tranche (in percent up-front points premium assuming 500 bp running spread), and the *y*-axis shows the spread of the 7%–10% mezzanine tranche. Note that the relationship changes substantially after the spring 2005 autocrisis. (From PIMCO, Newport Beach, California. With permission.)

we know the probability distribution of the underlying variables over which we can average the payoffs, and third, we assume that we know how to discount the probability-weighted payoffs. If we think of the underlying asset as being lognormal, then volatility (the second moment) is sufficient for capturing the uncertainty in the possible outcomes, i.e., the shape of the distribution. However, if we estimate these volatilities using historical data, or some other ad hoc method, and plug it into an option pricing formula, a trader who has access to the options markets can take advantage of us if our estimate is wrong.* So over time, the value of realized volatility and implied volatility should track each other closely due to arbitrage.†

Let us see what, if any, is the analog in the credit derivatives markets. For simplicity, we will use tranches (CDOs) on traded indices such as the CDX and the iTraxx, since they are the most liquid correlation products (we could also use *n*th-to-default baskets, but their liquidity and uniformity is far smaller).

Tranches on indices can be thought of as put spreads on index losses. There are 125 names in the widely traded CDX index. There are tranches that cover 0%–3% of losses (equity), 3%–7% of losses (junior mezzanine), 7%–10% of losses (senior mezzanine), 10%–15% of losses (senior), 15%–30% of losses (super senior), and, by construction, the 30%–100% tranche, which is the remainder when all the losses from the index are allocated. As correlation between the names increases, we expect that there is a higher probability that either more names default together or they do not. So the junior tranches (e.g., the equity tranche) will tighten in spread as correlations increase because there is a relatively higher probability of no defaults (note that we are not making a model dependent statement yet). On the other hand, as correlation increases, the higher rated, senior tranches will fall in value, since they are now exposed to higher tail risk. So, we should expect that as systemic risk increases, the impact on the senior tranches would be relatively

* Think of the skew in S&P500 option prices. The implied volatility of deeply out-of-the-money puts is higher than that of deeply out-of-the-money calls. The common explanations for this phenomenon are crashophobia, or Black's leverage phenomenon, but these explanations cannot uniquely determine the magnitude of the effects. What we can say with some degree of confidence is that the lognormal distribution of returns assumed for the underlying is inadequate to simultaneously explain the realized volatility of the underlying for all horizons and for all levels. We can also look at calendar spreads of options, i.e., options with same strikes but different maturities. Again, depending on the shape of the term structure of the volatility, the relative value of the two options can be quite variable. In periods of market crises, short-term volatility rises, and the term structure of volatility usually flattens. The underlying reasons for this can be traced to rise in risk-aversion, higher transactions costs for hedging, and demand for short-term crisis protection. Despite the serious shortcomings of the Black–Scholes model, these features of the implied volatility are consistent with our understanding of what risk means, and hence they are sensible.

† The arbitrageur does not necessarily have to even trade using options. Assuming that he can trade frequently, he can dynamically create a locally replicating portfolio and trade it against the option we have priced using the incorrect volatility value. Over time, he will be able to harvest the relative mis-pricing if our forecast of realized volatility is wrong. In this sense, the implied volatility indicates a good forecast of future realized volatility. We can also think about what happens when we change the strikes of the options. If we price a put spread where the strikes of the two put options are far from the forward, we can observe the prices of each of the two options and back out the one volatility number that would price the put spread to equal the traded price. In the limit that the put spread becomes a digital (i.e., the spread narrows to zero), the implied volatility is a point estimate of some realized volatility for that level of the underlying. It is still hard to develop intuition for the statistical analog of this tail volatility.

higher. The dependence of the mezzanine tranches on correlation is not that easy to explain, and, among other things, depends on their attachment and detachment levels (lower and upper strikes). There are also tranches on indices with different final maturities. We would intuitively expect that as correlation rises, the shorter maturity tranches have different sensitivity to correlation than the longer maturity tranches. To what degree this holds is discussed in a later section.

17.2 MARKET FOR IMPLIED CORRELATION

In this section, we discuss implied correlation from the perspective of the Black–Scholes analog for correlation products—the Gaussian copula model. The Gaussian copula model has more than a passing similarity with the simple Black–Scholes model for options, and has suffered its shares of attacks. However, just as its older cousin, the Gaussian copula has not only survived, but thrived. The reasons are multifold: (1) No one who trades correlation products really believes that the Gaussian copula is the last word, but by their usage they all agree that it is an excellent starting point and relatively easy to conceptualize (no one believes that Black–Scholes for options is the last word either, but the formula by virtue of its clarity and simplicity has survived 30 years in great style); (2) exact solutions are easy to obtain for products such as tranches under reasonable assumptions, and when exact solutions are not available, the framework allows for straightforward numerical simulations; (3) it is easy to communicate the value of the variables and parameters, though the meaning of these parameters is not always transparent. However, we should hasten to add that much research has continued to expand the simple copula model to make more realistic assumptions. A thorough review of this exciting field is beyond the scope of this chapter.

Tranche markets are not limited to investment grade indices. There are also tranches on the high yield indices that are reasonably liquid as of this writing, as well as tranches on indices of subprime mortgage issuers (ABX and TABX). Discussion of the subprime tranche market would require us to delve into mortgage credit and prepayment models, so in this discussion we focus mainly on pure credit tranches.

17.2.1 Tranche Pricing

Let us review how correlation enters the pricing of tranches. As discussed above, a tranche is a spread on the losses of the underlying index. A tranche is defined by an attachment point (lower strike K_d) and a detachment point (upper strike K_u). If the loss on the underlying index is $L_{index}(t)$, then the tranche loss is simply

$$L_{tranche}(t) = \max\left[L_{index}(t) - K_d, 0\right] - \max\left[L_{index}(t) - K_u, 0\right]$$

If we divide the expectation of this quantity by the tranche notional $K_u - K_d$, we obtain the tranche default probability. Finally, subtracting the resultant quantity from unity gives the tranche survival probability.

To price a tranche, we impose the condition that the present value of the expected premium payments (premium payments get reduced proportionately to the reduction in

notional upon default) should equal the present value of the expected protection received (i.e., when a name defaults we obtain 1 minus the recovery rate). The tranche survival probability, equal to the expected percentage notional of the tranche surviving at some time t, is the key quantity that is impacted by correlations. To obtain the expectations for each maturity requires knowledge of the loss distributions for each maturity, and that is not always a computationally tractable task in real time.

In a brute force model, we can take all the names in the index and generate Monte Carlo paths for defaults. In this approach, we generate default times based on default intensities extracted from a reduced form model for each name, and apply the pricing algorithm given above. When there are a lot of assets, specifying each element in the correlation matrix of default times is not only tedious but also becomes computationally intractable. In addition, if we are interested in backing out the correlations from traded products by standing up the pricing equation on its head, we are not guaranteed that the results would be unique or sensible. As an alternative, we can assume a simple factor model. Each asset return depends on the market factor, and an idiosyncratic factor. A common correlation determines how much the correlation drives asset returns. If we assume that the standardized asset returns are jointly normal with the market, then the decomposition is given by the standard Cholesky decomposition formula for generating correlated random variables from independent ones. We can, thus, generate random numbers for the realization of the market factors and the individual factors and combine them using the Cholesky formula to obtain the asset return realization. If we compute the cumulative distribution function of these asset returns, we obtain a uniformly distributed set of numbers. Since the probability of default lies between 0 and 1, we can use this uniformly distributed set of numbers as default time probabilities at each firm level after computing the probability of survival embedded in single name CDS spreads. If the asset return distributions are related in a Gaussian copula, then the default time probabilities are also related with a Gaussian copula. Note that at each stage, we are working in the risk-neutral framework, so we are throwing out any risk-premium information. So it becomes critical that the parameters are retrospectively evaluated through the lens of common-sense real-world experience.

17.2.2 Compound and Base Correlations

Regardless of whether we start with the brute force simulation of default times or a structural model, the assumption that the underlying components are related via a common market factor makes correlation a critical variable in the pricing of tranches. The market has adopted the Gaussian copula model due to its speed and clarity, and there are two different ways in which correlation based on the Gaussian copula is quoted (O'Kane and Livesey 2004):

- In the compound correlation convention, each tranche is quoted using a single tranche specific correlation number, so different tranches appear with different correlations, and we observe a compound correlation smile.
- Using the base correlation convention, we think of each of the mezzanine and senior tranches as the difference between two equity tranches. We start with the equity

tranche specified by its own unique correlation (for equity tranches the base and compound correlations are the same). For the next higher tranche, we use the fact that it is the difference between a wider equity tranche and the first equity tranche. We can solve for the correlation (holding the correlation for the first equity tranche fixed), such that the price of the difference of the two equity tranches equals the market price of the next higher tranche. We can keep repeating this process until all tranches are associated with a unique correlation specific to its detachment point. The correlations so obtained are the base correlations.

There is no extra information in the base correlation approach as compared to the compound correlation approach. The base correlation approach is preferred for some applications because it guarantees that for any tranche prices there are unique solutions for the implied base correlation. Despite the problem of multiple solutions for mezzanine tranches, compound correlation is still useful in connecting theory to observation.

17.3 DO THE MARKET'S ESTIMATES OF CORRELATIONS MAKE SENSE?

Compound correlations implied by tranche prices demonstrate a skew. As of April 16, 2007, the 5y IG8 index was trading at a spread of 37. Against this level of spreads, the tranches were at prices as displayed in the second and third columns of Table 17.1.

We can see that the junior mezzanine tranches trade at the lowest implied compound correlation. We also see that the equity tranche has positive correlation sensitivity, i.e., as correlation rises the equity tranche PV rises in value, whereas all the other tranches lose value. The fact that compound correlation exhibits a smile is not surprising—it is akin to option volatilities exhibiting a smile with strikes as one moves away from the at-the-money level. Admittedly, the simple Gaussian copula assumption is too restrictive to explain the full correlation structure of 125 names. But just as we ask the question for options, we can ask: Is there a logical explanation for this? There are a few possibilities: one possibility is simply a demand and supply mismatch for particular tranches; another explanation is the possibility of modeling error even after accounting for the limitations of the distribution. Similar to the Black–Scholes framework, the Gaussian copula has no way to account for

TABLE 17.1 IG8 5y Index Tranches and Compound Correlations Based on Gaussian Copula with Recovery at 40% for $10MM Notional of Each Tranche

	Upfront	Spread (in bp)	Compound Correlation	PV01	Spread Delta	Corr Sensitivity per 1% in $1000
0%–3%	26.10	500	13.40	3.69	21.74	73.04
3%–7%	0	101	7.34	4.58	7.77	−54.82
7%–10%	0	21	12.51	4.6	1.74	−17.13
10%–15%	0	10.30	17.79	4.60	0.78	−7.47
15%–30%	0	4.30	27.63	4.60	0.28	−2.50

Source: From PIMCO, Newport Beach, California. With permission.

jump risk. If there is a risk of jump-to-default (as in the May 2005 correlation crisis), the equity tranches would be most directly impacted and hence would command a risk premium. How might demand and supply impact the pricing of tranches and hence the correlation structure? Long-term equity correlations of investment grade names in the equity market indices are about 20%–25%. Similarly, the average equity return correlation for the underlying components of the IG8 CDX index is 22%. Then an implied correlation of 13.45% for the equity tranche implies that the equity tranche is cheap (because as correlation rises the equity tranche gains in value and its spread tightens). But as discussed above, this could be due to the inability of the model in pricing jump risk. For the mezzanine tranches, as correlation rises its value falls, so a reversion to historical correlation levels would make the tranche widen in spread. However, what is interesting is that the junior mezzanine tranche is so much tighter than historical correlations would imply. We can get a hint of why this might be so by again resorting to our options analogy. As of April 16, 2007, the index spread is 37 bp, and with spread duration of ~4.5 years, the PV of the index expected loss is approximately 1.67%, so the equity tranche with 0%–3% losses is "in-the-money" whereas the junior mezzanine tranche is just out of the money. In an environment of credit spread tightening, low volatility, and low systemic risk perceptions, selling the junior mezzanine tranche is akin to selling out-of-the-money options to collect insurance premium. Before the May 2005 correlation debacle, hedging equity tranches with mezzanine tranches was a popular trade implemented by hedge funds. The unwind of these positions, and the consequent break down of the relationships as illustrated in Figure 17.1, at steep losses for some investors who have a preference for such relative value trades could have further been responsible for the correlation mismatch we observe in the equity and junior mezzanine tranches. As we move to the senior tranches, we observe that their compound correlation begins to converge to long-term equity correlations. We also know anecdotally that in periods of market stress, equity correlations rise, as there is a general flight to quality. So in periods of crisis, we would expect equity correlations to rise, and assuming that senior tranches track this, their value to fall with correlation rising. In other words, purchasing protection on senior tranches would benefit from systemic shocks in a more dependable way. Note that insurance companies are typically sellers of synthetic insurance via super-senior tranches, and this is not very different than selling reinsurance on hurricanes or catastrophes.

Even with its severe limitations, we can use the compound correlation as a diagnostic tool. Suppose equity and mezzanine spreads were to widen drastically—widening equity spreads mean tightening compound correlation, and widening mezzanine tranche spreads mean rising compound correlation. Then the compound correlation curve would flip or invert from the shape as in Table 17.1. This could be the situation observed in reality if levels of risk aversion rise in the credit derivatives markets.

We can also discuss the sensibility of base correlations. First, as we move up the subordination structure, we find that base correlations rise (the correlation skew). This makes sense in the context of the pricing of mezzanine tranches. Note that the junior mezz can be thought of as the difference between the 0%–7% equity and the 0%–3% equity. Once the correlation for the 0%–3% is determined by the price of the equity, the richness of the

3%–7% tranche can only be possible if the 0%–7% is richer. But this is an equity tranche, which is long correlation risk, so its implied base correlation has to be higher. Continuing this for the higher tranches, it makes sense that the base correlations reflect the demand and supply mismatch as illustrated by the demand for the mezz tranches. Second, let us assume that we move in the index maturity dimension. For the IG8 series, we have the 5 year, the 7 year, and the 10 year indices, and on each one of these we have all the standard tranches discussed above. As we move to longer maturities, the implied base correlations fall for all tranches. This also makes sense, because as we go out in the maturity, the same equity tranches are more in the money (expected index losses are higher), and hence these are expected to trade cheaper. As the base correlation for the 0%–3% equity tranche falls, so does the base correlation for all the higher tranches. We can also move in the vintage dimension (assuming that the components are essentially the same). As we move to older indices, we would see that the base correlations rise for the same tranches. So, for instance, if the base correlation for the ITRAXX7 5y (ref. index 24 bp) 0%–3% is 16.4%, for ITRAXX6 5y (ref. 21) 0%–3% it is 17.5%.

Invoking demand and supply considerations to explain correlation skews and smiles might be less than convincing to some. An astute arbitrageur would ask why investors do not create combinations of the tranches to take advantage of the correlation skews/smiles. First, there is clear market segmentation—hedge funds and banks are typically equity tranche investors, whereas insurance companies are investors in (sell protection) the senior structures. Crossover buyers, pensions, retail and single tranche collateralized debt obligations (CDO) hedgers dominate the mezzanine structures. So, there is not much mobility of investors between the tranches, and different investors show preference for different parts of the capital structure (rating agency ratings also assist in habitat formation). Another explanation relies on our often-used analogy with options. We know that the implied volatility of deeply out-of-the-money puts in the equity market is much higher than the implied volatility of options close to at the money. But this does not automatically suggest that arbitrageurs should sell a lot of out-of-the-money options and hedge them with at-the-money options. The failure of such a strategy would almost be guaranteed if realized outcomes on the underlying deviated from a continuous lognormal behavior, or if there was less than perfect liquidity in the hedge instruments. However, there is considerable evidence that tranches have become a lot more efficient as carriers of correlated credit risk.

There are a number of other ways by which we can try to ascertain whether the correlation implied by the market makes sense. First, we can take a purely statistical approach. Following the work of Voronov et al. (2005), we can think of each underlying name in the index as being driven by a time series process. Different time series processes can have very complicated tail correlations. The logic here is that if we can fit the time series of index spreads to realistic models, we can generate the tail correlations organically. In practice, however, it seems unlikely why credit markets should be driven by any particular time series processes—the evidence is actually contrary. There is also the question of change of measure. Asset correlations that enter the risk-neutral valuation of tranches are under the risk-neutral or pricing measure, whereas the estimates using time

series are necessarily under the pricing measure. In so far as correlation could carry a risk premium, it is not clear how one should translate from one to the other.

Another appealing approach is the approach promoted recently by Longstaff and Rajan (2006). Their approach is a top–down approach and removes the need for specifying a copula. Three types of default events, corresponding to firm-specific, industry-, and economy-wide default events drive spreads. Attempting to fit the movements of the indices and the tranches with a single factor model fails; confirming the fact that default correlations matter. Fitting their model to traded tranche prices on the IG indices shows that the average times for firm-specific default are approximately a year, for industry-wide defaults are 40 years, and for economy-wide risk are ~750 years. For the industry-wide and economy-wide factors, they find that 10% and 35% of the index firms would default, which is in the zone of sensibility. This way of looking at correlation risk highlights the economics behind default risk embedded in credit portfolio products. Current work on extracting similar information from noninvestment grade products that we are pursuing seems to confirm and extend the intuition behind much of this top–down analysis.

17.4 CONCLUSION

In this chapter we reviewed how correlation impacts investment analysis in credit derivative products. We discussed the widely used standard model for correlation modeling, and attempted to shed some light on the sensibility of the measures obtained by fitting to the prices of traded credit derivatives. We also discussed alternative statistical and top–down or macro models to gain another perspective on default correlations. The correlation market is a natural extension of the volatility market to credit, and as any option trader knows, it is crucial to understand what drives implied volatility, and now implied correlation, in the market.

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CHAPTER 18

Copula-Based Default Dependence Modeling: Where Do We Stand?

Elisa Luciano

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The assessment of the joint default probability of groups of obligors, as well as related notions, such as the probability that the n-th one of them defaults, is a crucial problem in credit risk. To solve it, both the industry and the academia have extensively relied on copula methods. These allow to split any joint default probability into the marginal ones and a function, the copula itself, which represents only the association or dependence between defaults. Essentially, the splitting up makes both default modeling and calibration much easier, since it permits separate fitting at the univariate and joint level. At present, the use of copula techniques is a well-established fact in risk modeling and credit derivative pricing. However, the choice of the copula and its calibration is still an open issue: model risk indeed exists at both stages, and its understanding relies on a deep discussion of the current practices and model limitations.

The first part of the chapter reviews the theoretical background for different copula choices in default–no default models and discusses its static and dynamic consistency, when switching from structural to intensity-based approaches. The review proceeds as follows: we first resume some basic facts about structural and intensity-based models, and how copulas are introduced and calibrated in the two settings. We then recall how and under which conditions two multivariate models of these classes can be statically remapped into each other, keeping dependence fixed. This remapping is indeed exploited in most of the industry applications of intensity-based models, starting from the well-known Li's model (Li, 2000). However, it suffers from dynamic inconsistency, in that structural models with diffusive asset values do not have an intensity-based representation. In addition, remapping allows dependence calibration under the historical measure only, since the historical equity returns are used as proxies for the asset ones. To calibrate under a risk-neutral measure, factorization, i.e., the so-called factor copula, is introduced. We discuss factorization in joint default modeling. The survey will put into evidence the lack of dynamically consistent foundations for inferring the risk-neutral dependence of survival times for more than two obligors, without assuming equicorrelation.

The second part of the chapter suggests a possible solution to this problem: structural models, in which the asset value is a pure jump process, such as the Merton's restatement of Madan (2000), or the corresponding early default model. These models allow for an intensity interpretation, and are therefore dynamically consistent. They fit data nicely at the univariate level (Fiorani and Luciano, 2006):

- Can be extended to the multivariate case (Luciano and Schoutens, 2006; Fiorani et al., 2007).
- Can be calibrated under the risk-neutral measure and without equicorrelation.
- And have proven to fit credit data very satisfactorily (ibid.).

18.1 COPULA MODELING IN STRUCTURAL MODELS

This section makes an overview of basic copula uses in structural models. We will briefly recall how copulas are introduced, how historical calibration is usually done for a given copula choice, and finally how the best copula can (or cannot) be chosen.

We consider a portfolio of n obligors, with random default times $\tau_1, \tau_2, \ldots, \tau_n$. The indicator variable of default of obligor i, $i = 1, 2, \ldots, n$, at time t, $t \in \mathbb{R}^+$, is denoted as $Y_i(t) := 1_{\{\tau_i \leq t\}}$. Let $(\Omega, \mathcal{H}, P)$ be the probability space over which the times-to-default are defined. The space is endowed with the filtrations associated the single stochastic processes $Y_i(t)$, $\mathbb{G}_i := \{\mathcal{G}_{it}; t \geq 0\}$:

$$\mathcal{G}_{it} := \sigma[Y_i(u); 0 \leq u \leq t] \tag{18.1}$$

as well as with the joint filtration $\mathbb{G} := \vee_{i=1}^{n} \mathbb{G}_i$, with the usual properties of right continuity and completeness. Initially, all obligors are alive, so that $Y_i(0) = 0$ for all i. The default probability by time t, as determined at time 0, $P[Y_i(t) = 1 | \mathcal{G}_{i0}]$, will be denoted as $F_i(t)$, while $S_i(t)$ will denote the corresponding survival probability, $S_i(t) := 1 - F_i(t)$. Analogously, we will denote the joint default probability at time t, evaluated at time 0, as $F(t)$.

$$F(t) := P[Y_1(t) = 1, Y_2(t) = 1, \ldots, Y_n(t) = 1 | \mathcal{G}_0] \tag{18.2}$$

and the corresponding survival probability as $S(t)$

$$S(t) := P[Y_1(t) = 0, Y_2(t) = 0, \ldots, Y_n(t) = 0 | \mathcal{G}_0] \tag{18.3}$$

In describing the marginal and joint default or survival functions, we will omit, from now on, the filtration. As for the copula tool, we will denote an n-dimensional copula, defined on $I^n := [0,1] \times \cdots \times [0,1]$, as $C(u_1, \ldots, u_n)$, while its survival counterpart will be $\breve{C}(u_1, \ldots, u_n)$. We take for granted the copula definition as a joint distribution function with uniform margins, which implies that C is always between the bounds $\max(u_1 + u_2 + \cdots + u_n - 1, 0)$ and $\min(u_1, u_2, \ldots, u_n)$. We will also take for granted the fundamental Sklar's theorem, which, under continuity of the margins, allows to represent any joint distribution function of n random variates (rvs), $F(t)$, in terms of a copula C and the marginal distribution functions, $F_i(t)$, $i = 1, 2, \ldots, n$:

$$F(t) = C[F_1(t), F_2(t), \ldots, F_n(t)] \tag{18.4}$$

Analogously, any joint survival probability $S(t)$ can be represented in terms of the so-called survival copula, $\breve{C}$, and the marginal survival functions, $S_i(t) := 1 - F_i(t)$,

$$S(t) = \breve{C}[S_1(t), S_2(t), \ldots, S_n(t)] \tag{18.5}$$

This is the reason why the copula is also called dependence function, in that it isolates dependence from the margin values: independence, for instance, is represented by the so-called product or factor copula $C^{\perp}$, which is simply the product of its arguments:

$$C^{\perp}(u_1, \ldots, u_n) = u_1 \times u_2 \times \cdots \times u_n \tag{18.6}$$

Copulas can be parametrized by the linear correlation coefficient as well as by more general association measures: in the sequel, we will use for instance the so-called Gaussian and

Student's t copulas, which are characterized by linear correlation and—the second only—by the degrees of freedom parameter. They are linked by the fact that the former is the limit of the latter, when the number of degrees of freedom diverges. For further details, as well as for an overview of other common specifications, we refer the reader to either Nelsen (1999) or Cherubini et al. (2004).

In the credit risk case, since the variables τ_i are default times, the copula represents their dependence, and we could denote it as C^τ, where τ is the vector formed by the rvs τ_i:

$$F(t) = C^\tau[F_1(t), F_2(t), \ldots, F_n(t)] \tag{18.7}$$

$$S(t) = \breve{C}^\tau[S_1(t), \ldots, S_n(t)] \tag{18.8}$$

The specification of the copula depends on whether we rely on a structural approach or an intensity-based one. In the seminal structural model of Merton (1974), to start with, it is well known that default of firm i can occur only at debt maturity. Let the latter be t for all the firms in the portfolio: default is triggered by the fact that the firm's asset value $V_i(t)$ falls to the liability one, $K_i(t)$. The distribution of the time-to-default, τ_i, is therefore

$$\tau_i = \begin{cases} t & P[V_i(t) \leq K_i(t)] \\ +\infty & P[V_i(t) > K_i(t)] \end{cases} \tag{18.9}$$

while the default probability at time t is

$$F_i(t) = P[V_i(t) \leq K_i(t)] \tag{18.10}$$

If the asset value follows a geometric Brownian motion, as assumed by Merton, and the standard notation of the Black–Scholes framework is adopted, the marginal default probability can be easily computed to be

$$F_i(t) = \Phi[-d_{2i}(t)] \tag{18.11}$$

where Φ is the distribution function of a standard normal, while

$$d_{2i} := \frac{\ln[V_i(0)/K_i(t)] + \left(m - \sigma^2/2\right)t}{\sigma\sqrt{t}} \tag{18.12}$$

and m is the instantaneous return on assets, which equates the riskless rate r under the risk-neutral measure. By assuming that log assets are not only normally distributed at the individual level, but also jointly normally distributed with correlation matrix R, it follows that the joint default probability of the n assets is

$$F(t) = P[V_1(t) \leq K_1(t), \ldots, V_n(t) \leq K_n(t)] = \Phi_R[-d_{21}(t), \ldots, -d_{2n}(t)] \tag{18.13}$$

where Φ_R is the distribution function of a standard normal vector with correlation matrix R. From the fact that the arguments of the distribution Φ_R can be written as the inverses,

according to the univariate normal density, of the marginal default probabilities, i.e., $-d_{2i}(t) = \Phi^{-1}[F_i(t)]$, it follows, by simple substitution, that

$$F(t) = \Phi_R\left\{\Phi^{-1}[F_1(t)], \ldots, \Phi^{-1}[F_n(t)]\right\} \tag{18.14}$$

The latter is the copula representation of the joint default probability for the case at hand, since F is represented in terms of the marginal distributions. The copula which we have obtained is the so-called Gaussian copula:

$$C(u_1, \ldots, u_n) := \Phi_R\left[\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right] \tag{18.15}$$

Instead of joint normality, we can keep the margins fixed and assume another type of dependence, which means another copula. For instance, we can assume a Student t copula, defined as

$$C(u_1, \ldots, u_n) := t_{R,\nu}\left[t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), \ldots, t_\nu^{-1}(u_n)\right] \tag{18.16}$$

where $t_{R,\nu}$ is the standardized multivariate Student's t distribution with correlation matrix R and ν degrees of freedom, while t_ν^{-1} is the inverse of the corresponding margin. By so doing, we essentially incorporate fat tails in the asset return joint distribution. The joint default probability immediately becomes

$$F(t) = t_{R,\nu}\left\{t_\nu^{-1}[F_1(t)], t_\nu^{-1}[F_2(t)], \ldots, t_\nu^{-1}[F_n(t)]\right\} \tag{18.17}$$

In general, starting from a marginal probability of the type in Equation 18.10, and therefore from a marginal model of the Merton's type, we can write the joint default probability as

$$F(t) = C^V[F_1(t), \ldots, F_n(t)] \tag{18.18}$$

where C^V is the copula of the asset values. Let us recall however that the above expression applies only at debt maturity, and assuming that the latter is common to all firms in the basket.

Starting with Black and Cox (1976), the structural model of Merton has been extended by introducing the possibility of early distress. Default occurs as soon as the asset value $V_i(t)$ falls below a (deterministic) level $K_i(t)$, which can be, for instance, the present value of debt. As a consequence, the time-to-default is

$$\tau_i := \inf\{t : V_i(t) \leq K_i(t)\} \tag{18.19}$$

and the cumulated default probability is

$$F_i(t) = P\left[\min_{0 \leq u \leq t} V_i(u) \leq K_i(u)\right] \tag{18.20}$$

In this model, even if V_i follows a geometric Brownian motion, its minimum does not, and the departure from the plain vanilla option pricing at the individual level is immediate.

Other departures have been caused, during the development of the literature on structural models, by introducing more realistic assumptions, such as the stochastic interest rate of Longstaff and Schwartz (1995). However, the use of a Gaussian or Student version at the multivariate level has been widespread, mostly for analytical convenience.

More recently, default has been supposed to be triggered by the fall of a credit worthiness index under an empirically calibrated barrier (Hull and White, 2000). The credit index does not necessarily admit a transform which behaves as a geometric Brownian motion. Because of the lack of marginal normality, it is not natural any more to assume joint normality and to collect the univariate probabilities in a Gaussian or an elliptic copula, such as the Student: the bivariate version was outlined in Hull and White (2001), while multivariate versions are manageable through a purely numerical fit of the copula, similar to the implied volatility derivation (see the perfect copula of Hull and White, 2005).

18.1.1 Dependence Calibration (under the Historical Measure) and Best Copula Choice

The calibration of structural models is straightforward, especially if one refers to the seminal Merton one, as we will do. Under a Gaussian copula, as in the natural multivariate version of Merton's work, only the linear correlation coefficients between the asset values are needed. In case the Student copula is adopted, also the degrees of freedom parameter is required.

Since asset values are generally unobservable, the corresponding equity ones are generally used for calibration. This is the approach taken, for instance, by Mashal and Naldi (2003), together with the Student assumption: they suggest a joint maximum likelihood estimation of the degrees of freedom parameter and the correlation matrix. Obviously, the calibration is done under the historical measure: under such a measure, as Mashal et al. (2004) have shown, the dependence of assets is well proxied by the equity one. However, the coincidence between historical and risk-neutral correlation requires either no premium for default risk or particular assumptions, such as the ones listed by Rosenberg (2000). If we exit the Merton model, in which asset returns are jointly normally distributed, such a coincidence is not guaranteed.

It also follows from the meaning of the copula in the structural approach that the best copula is the one which best describes equity dependence. The current literature tends to agree on the fact that—at least for extreme co-movements—the Student dependence is more appropriate, since it encapsulates tail dependence: evidence is reported, for instance, in Mashal and Zeevi (2002) and Mashal et al. (2004).

18.2 COPULA MODELING IN INTENSITY-BASED MODELS

In the intensity-based models of Lando (1998) and Duffie and Singleton (1997, 1999), default of the obligor i occurs as soon as a counting process, which is assumed to be of the Cox or doubly stochastic type, jumps for the first time. Omitting the technical details, for which the reader is referred to the probabilistic set up of Brémaud (1981), or to the credit applications just mentioned, occurrence of default is linked to the event that the so-called compensator of the Cox process, denoted as Λ_i, be greater than or equal to a stochastic barrier. The latter is exponentially distributed, with parameter 1:

$$\tau_i := \inf\{t : \Lambda_i(t) \geq \theta_i\} \tag{18.21}$$

with $\theta_i \sim \exp(1)$. Equivalently, using the change of variable $\theta_i := -\ln U_i$:

$$\tau_i := \inf\{t : \exp[-\Lambda_i(t)] \leq U_i\} \tag{18.22}$$

where $U_i \sim U(0,1)$. The compensator is usually assumed to have an intensity, which means to admit the representation

$$\Lambda_i(t) = \int_0^t \lambda_i(s)\,\mathrm{d}s \tag{18.23}$$

where λ_i is a nonnegative process, satisfying $\int_0^t \lambda_i(s)\mathrm{d}s < \infty$ almost surely for all t. Given a filtration $\mathbb{F} := \{\mathcal{F}_t;\, t \geq 0\}$ on Ω, the intensity $\lambda_i(s)$ is assumed to be predictable, while the threshold θ_i is independent of $\mathcal{F}$. We also need to define the filtration $\mathbb{H} := \mathbb{F} \vee \mathbb{G}$, or $\mathbb{H} := \{\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t;\, t \geq 0\}$.

The default probability of the above firm at time zero turns out to be

$$F_i(t) = 1 - E\left\{\exp\left[-\int_0^t \lambda_i(s)\,\mathrm{d}s\right]\right\} \tag{18.24}$$

where $E(\cdot)$ is the usual expectation operation, conditional on $\mathcal{F}_0$.

If, in particular, the intensities are deterministic, the Cox processes describing default arrival become inhomogeneous Poisson ones and

$$F_i(t) = 1 - \exp\left[-\int_0^t \lambda_i(s)\,\mathrm{d}s\right] \tag{18.25}$$

Under the additional requirement that intensities are constant, $\lambda_i(s) = \lambda_i$ for every s, they become homogeneous Poisson processes, with $F_i(t) = 1 - \exp(-\lambda_i t)$.

When extending the Cox framework to multiple firms, i.e., taking $i = 1, \ldots, n$, the previous framework naturally leads to conditionally independent defaults, or a multivariate Cox process, as in Duffie (1998) and the literature thereafter: the intensity processes are liable to be correlated, while the thresholds θ_i, or their transform U_i, are assumed to be independent, and therefore have the product copula. It follows that the joint default probability of the n firms by time t, evaluated at time 0, is

$$S(t) = E\left(\exp\left\{-\int_0^t [\lambda_1(s) + \lambda_2(s) + \cdots + \lambda_n(s)]\,\mathrm{d}s\right\}\right) \tag{18.26}$$

This extension usually produces low levels of both (linear) correlation between the default indicators of different firms, $\rho[Y_i(t), Y_j(t)]$, for $i \neq j$ and all t, and (linear) correlation of survival times, $\rho(\tau_i, \tau_j)$, $i \neq j$. This happens, as is intuitive, as long as the intensities are diffusion-driven.

An alternative approach, introduced by Schönbucher and Schubert (2001), to whom we refer for the technical details, allows defaults not to be conditionally independent. The thresholds θ_i (or U_i) are linked by a copula which is not the product copula of Duffie (1998). With the usual notation, let C^U be the copula of the thresholds U_i, which is also the survival copula* of θ_i, $C^U = \breve{C}^\theta$. The construction of Schönbucher and Schubert (2001) gives as joint survival probability

$$S(t) = E\left(C^U\left\{\exp\left[-\int_0^t \lambda_1(s)\mathrm{d}s\right], \exp\left[-\int_0^t \lambda_2(s)\mathrm{d}s\right], \ldots, \exp\left[-\int_0^t \lambda_n(s)\mathrm{d}s\right]\right\}|\mathcal{F}_0\right) \tag{18.27}$$

which reduces to Equation 18.26, as due, in the Duffie (1998) framework.

Whenever the n firms are driven by inhomogeneous Poisson processes, since the intensities are deterministic, the survival probability in Equation 18.27 becomes

$$\begin{aligned} S(t) &= C^U\left\{\exp\left[-\int_0^t \lambda_1(s)\mathrm{d}s\right], \exp\left[-\int_0^t \lambda_2(s)\mathrm{d}s\right], \ldots, \exp\left[-\int_0^t \lambda_n(s)\mathrm{d}s\right]\right\} \\ &= C^U[S_1(t), \ldots, S_n(t)] \end{aligned} \tag{18.28}$$

which further reduces to the product

$$\begin{aligned} S(t) &= \exp\left\{-\int_0^t [\lambda_1(s) + \lambda_2(s) + \cdots + \lambda_n(s)]\mathrm{d}s\right\} \\ &= S_1(t) \times \cdots \times S_n(t) \end{aligned} \tag{18.29}$$

with independent thresholds.

18.2.1 Dependence Calibration (under the Historical Measure) and Best Copula Choice

In the general case of stochastic intensities, by choosing independent thresholds, one restricts C^U to be the product copula, as in Equation 18.26. Apart from this case, calibration is a very delicate issue, since the thresholds U_i or θ_i are not observable: the literature has concentrated on copulas which produce a shift of the remaining intensities after default of one obligor, such as the Archimedean copulas (Schönbucher and Schubert, 2001). The calibration is usually done restricting to a one-parameter copula family, such as a Gumbel one, a Clayton, or a Gaussian with equicorrelation. Once such a family has been chosen, the parameter value is perfectly fitted using, for instance, the diversity score produced by Moody's. The perfect fitting, which is illustrated for instance in Jouanin et al. (2001), does not permit to select a best copula: with respect to the structural approach

* The copula of the thresholds U_i and the one of their transforms θ_i are linked by the fact that each U_i is a decreasing function of the corresponding θ_i.

then, in which there is scope for a copula selection procedure, taking into consideration the best fit of asset or equity returns, here there is no room for a best-fit selection.

As long as intensities are deterministic, calibration at a single point in time is much easier, since, as we will now explain, the copula C^U is nothing else than the survival copula of the associated asset values. This is the approach first taken by Li (2000), made rigorous in Frey and McNeil (2003), and extremely popular in practice, together with its best copula selection. It is based on the observation that structural models with a diffusion-driven asset or credit index process on the one side and reduced form models with deterministic intensities on the other are both latent variable models: for any fixed horizon t, there exist n rvs $Z_i(t)$ and n thresholds $D_i(t) \in \mathbb{R}$ such that default occurs if and only if the random variables are smaller than or equal to the corresponding thresholds:

$$Y_i(t) = 1 \Leftrightarrow Z_i(t) \leq D_i(t) \tag{18.30}$$

Indeed, in the structural case, one has simply to choose either the asset values or their minimum and the liabilities ones as rvs and thresholds, respectively:

$$Z_i = V_i, \quad D_i = K_i \tag{18.31}$$

In the intensity-based case, one has to replace the asset/index values, V_i, and the threshold ones, K_i, with the stochastic thresholds and the compensators, respectively:

$$Z_i = \theta_i, \quad D_i = \Lambda_i \tag{18.32}$$

or with the uniform thresholds and the negative exponential of the compensators:

$$Z_i = -U_i = -\exp(-\theta_i), \quad D_i = -\exp(-\Lambda_i) \tag{18.33}$$

Having made this simple remark, we can exploit the equivalence of latent variable models to remap intensity-based models into structural ones, for a fixed horizon. Indeed, two latent variable models

$$[V_i(t), K_i(t)]_{1\leq i\leq n}, [\theta_i(t), \Lambda_i(t)]_{1\leq i\leq n} \tag{18.34}$$

are equivalent if the corresponding default indicator vectors are equal in distribution: this allows us to assess that, at any fixed time t, the copula for the thresholds θ_i, C^θ—which, since $C^U = \breve{C}^\theta$, is also the survival copula of the uniform thresholds, $\breve{C}^U$—is the same as the asset value copula, C^V. The estimate of the last one obtained from equity returns, as justified in structural models, can therefore be directly used to construct joint default probabilities, starting from marginal intensity-based ones. Formally, we can then write the joint default probability as

$$\begin{aligned} F(t) &= C^V[F_1(t), F_2(t), \ldots, F_n(t)] \\ &= C^V\left\{1 - \exp\left[-\int_0^t \lambda_1(s)\mathrm{d}s\right], 1 - \exp\left[-\int_0^t \lambda_2(s)\mathrm{d}s\right], \ldots, 1 - \exp\left[-\int_0^t \lambda_n(s)\mathrm{d}s\right]\right\} \end{aligned} \tag{18.35}$$

However, this remapping has two problems. First, it is not dynamically consistent, since in structural models with a diffusive asset value, such as the ones described above and used in common applications and commercial softwares, the times-to-default are not totally unpredictable stopping times and, as a consequence, there does not exist an intensity which can represent their arrival. Second, it leads to an historical calibration of default dependency, via equity returns, which again has no reason to coincide with the risk-neutral one.

18.3 FACTOR MODELS

While remapping, as described in the previous section, permits to reconnect the copula choice to the asset one, while paying the price of dynamic inconsistency and calibration limited to the historical measure, another popular approach to default modeling allows us to switch to the so-called product copula. The switching or reduction technique, which is widely adopted for the evaluation of losses in high-dimensional portfolios, with hundreds of obligors (Laurent and Gregory, 2003), is the standard approach of (linear) factorization, or transformation into a Bernoulli factor model.

Given a vector of random factors,

$$[X_1(t), X_2(t), \ldots, X_p(t)] \tag{18.36}$$

the first step in reducing it consists in assuming that there exist n (probability) functions $p_i^t : \mathbb{R}^p \to [0,1]$, $i = 1, \ldots, n$, such that, conditional on a specific realization of the factors $X_1 = x_1$, $X_2 = x_2, \ldots, X_p = x_p$, the default indicators $Y_i(t)$ of the n obligor are independent Bernoulli variables, with probability

$$Y_i(t) = \begin{cases} 1 & p_i^t(x_1, \ldots, x_p) \\ 0 & \text{otherwise} \end{cases} \tag{18.37}$$

In the sequel, we will work with one or two factors. In the former case, if the (unique) factor X has a density $f(x)$ on the real line $\mathbb{R}$, it follows from the definition of conditional probability that the marginal (unconditional) default probabilities can be written as

$$F_i(t) = \int_{\mathbb{R}} p_i^t(x) f(x)\, \mathrm{d}x \tag{18.38}$$

while the joint one can be represented through a (conditional) factor copula $C^{\perp}$, as desired:

$$F(t) = \int_{\mathbb{R}} \prod_{i=1}^{n} p_i^t(x) f(x)\, \mathrm{d}x = \int_{\mathbb{R}} C^{\perp}\left[p_1^t(x), \ldots, p_n^t(x)\right] f(x)\, \mathrm{d}x \tag{18.39}$$

Let us discuss the factorization separately for structural, intensity-based with nonstochastic intensity, and stochastic intensity with independent and dependent threshold ones: without much loss of generality, we will use one-factor models, in which the factor is standard Gaussian.

18.3.1 Structural Models

The key assumption is that the ith asset value or credit worthiness indicator, properly normalized, can be factorized as

$$V_i = \rho_i X + \sqrt{1-\rho_i^2}\varepsilon_i \tag{18.40}$$

where $\rho_i \in \mathbb{R}$, X and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent standard Gaussian, so that V_i is standard normal too. The common factor is evidently X, while the ε_i are to be interpreted as the idiosyncratic ones: it follows from Equation 18.40 not only that the V_is are independent, conditionally on X, but also that the unconditional linear correlation coefficient between V_i and V_j is $\rho_i\rho_j$. The conditional marginal default probabilities, $p_i^t(x)$, if K_i is the properly normalized liability or barrier level, are easily calculated:

$$p_i^t(x) = P(V_i \leq K_i|x) = \Phi\left(\frac{K_i - \rho_i x}{\sqrt{1-\rho_i^2}}\right) \tag{18.41}$$

The expressions for the unconditional ones, Equation 18.38, and the joint unconditional ones, Equation 18.39, easily follow.

18.3.2 Intensity-Based Models with Nonstochastic Intensity

To adopt a factor model, starting from a uniform threshold, i.e., from the representation Equation 18.25 of the unconditional marginal default probability, one must factorize a transform of it, which is standard Gaussian by definition: $W_i := \Phi^{-1}(1-U_i)$. By imposing

$$W_i = \rho_i X + \sqrt{1-\rho_i^2}\varepsilon_i \tag{18.42}$$

one gets as conditional marginal default probability, with a little bit of algebra

$$p_i^t(x) = P(\theta_i \leq \Lambda_i|x) = \Phi\left\{\frac{\Phi^{-1}[1-\exp(-\Lambda_i)] - \rho_i x}{\sqrt{1-\rho_i^2}}\right\} \tag{18.43}$$

Again, the unconditional marginal probabilities, Equation 18.38, and the joint unconditional ones, Equation 18.39, easily follow.*

* More interestingly, the factorized models confirm the correspondence between structural and intensity-based models discussed in Section 18.2.1, since the two models give the same conditional probabilities (and therefore unconditional ones) whenever

$$K_i = \Phi^{-1}[1-\exp(-\Lambda_i)]$$

which is the condition that characterizes the equivalence between latent variable models of the structural and intensity type.

18.3.3 Stochastic Intensity Models with Independent Thresholds

In a multivariate Cox framework, à la Duffie (1998), the variable which can be factorized is a transform of the compensator, with no factorization of the thresholds, which are already independent. Indeed, if we denote as G_i the distribution function of the compensator at time t, so that $1 - G_i(\Lambda_i)$ is uniform, we can define

$$W_i := \Phi^{-1}[1 - G_i(\Lambda_i)] \tag{18.44}$$

and the latter can easily be proven to be standard normal. Let us impose on W_i the factor structure in Equation 18.42. Noting that $\Lambda_i = G_i^{-1}[1 - \Phi(W_i)]$, the conditional marginal default probability

$$p_i^t(x) = P(\theta_i \leq \Lambda_i | x) = 1 - E[\exp(-\Lambda_i)|x] \tag{18.45}$$

can be written as

$$p_i^t(x) = 1 - E\left(\exp\left\{-G_i^{-1}\left[1 - \Phi\left(\rho_i x + \sqrt{1-\rho_i^2}\varepsilon_i\right)\right]\right\}\middle| x\right) \tag{18.46}$$

The expression for the unconditional marginal probability, Equation 18.38, is straightforward.

As for the joint unconditional default probabilities, they can still be written as Equation 18.39, as follows. From the fact that for the case at hand, the representation Equation 18.26 of the joint survival probability holds, it follows that the joint default is

$$F(t) = E\big\{[1 - \exp(-\Lambda_1)]\,[1 - \exp(-\Lambda_2)] \cdots [1 - \exp(-\Lambda_n)]\big\} \tag{18.47}$$

Conditioning it on $X = x$, we get

$$F(t) = \int_{\mathbb{R}} E\big\{[1 - \exp(-\Lambda_1)]\,[1 - \exp(-\Lambda_2)] \cdots [1 - \exp(-\Lambda_n)]|x\big\} f(x)\,\mathrm{d}x \tag{18.48}$$

Finally, recalling that the compensators are, by assumption, independent for given x, we have

$$F(t) = \int_{\mathbb{R}} \prod_{i=1}^{n} \big\{1 - E[\exp(-\Lambda_i)|x]\big\} f(x)\,\mathrm{d}x \tag{18.49}$$

which is nothing else than the Bernoulli representation of the default processes, Equation 18.39 above.

18.3.4 Stochastic Intensity Models with Dependent Thresholds

In this case, not only we need to perform the factorization for the multivariate Cox case at the univariate level, but we also need to impose a factor structure on the thresholds, to guarantee the conditional independence at the multivariate level.

At the univariate step, nothing changes with respect to the independence case just discussed, and the marginal conditional default probabilities in Equation 18.46 still hold.

At the multivariate level, Rogge and Schönbucher (2003) propose to assume that the copula of the uniform thresholds is an Archimedean one:

$$C^U(u_1, u_2, \ldots, u_n) = \varphi^{-1}[\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_n)] = \varphi^{-1}\left[\sum_{i=1}^{n} \varphi(u_i)\right] \tag{18.50}$$

where $\varphi(u) : [0,1] \rightarrow [0,\infty]$ is a continuous, strictly decreasing, and completely monotonic function, and its generalized inverse can be chosen to be the Laplace transform of a positive rv, Z.

We show in the appendix that, under this choice, a sufficient condition for obtaining a factor model is to assume that the variable Z is independent of the factor driving the intensities. As proven in Rogge and Schönbucher (2003), the previous copula choice is indeed equivalent to assuming a nonlinear factor structure for the thresholds:

$$U_i = \varphi^{-1}\left[-\frac{\ln \Phi(\zeta_i)}{Z}\right] \tag{18.51}$$

in which the ζ_i are independent standard normal, which are also independent of ε_i, X, and Z.

Summing up, both in structural and intensity-based models, factorization permits to substitute the original copulas with the product one, according to Equation 18.39, something which makes computations and analytical pricing results easier to obtain.

In particular, assuming, in addition to a unique risk factor, also equicorrelation of the single obligors with it, i.e., $\rho_i = \rho$ for all obligors i, calibration of default dependence boils down to finding an appropriate value for the parameter ρ: this is practically done by calibrating to the observed fees of liquid CDO (collateralized debt obligation) tranches. However, practical applications of the methodology lead to the so-called correlation skew phenomenon, which is inconsistent with the theoretical set up: for a survey and discussion see Andersen and Sidenius (2005). In turn, this has led to a number of calibration-oriented extensions, mainly related to stochastic correlation. Their full insertion in the intensity-based framework is certainly worth high consideration, while not being completed yet: however, it is far beyond the scope of this chapter.

18.4 STRUCTURAL MODEL WITH REMAPPING AND RISK-NEUTRAL CALIBRATION

The models reviewed in the previous sections present a number of limitations, as concerns either dynamic consistency or dependency calibration.

Intensity-based models with deterministic intensity in particular do not have a dynamic consistent structural representation, and an asset copula consistent over time, since intensities exist if and only if the time-to-default is not a totally predictable stopping time. This condition, as mentioned above, is violated by diffusion-based structural models.

When we enlarge our consideration to stochastic intensities, we encounter two problems. The limited range of dependency which can be reached via intensity-based models with stochastic intensity and independent thresholds renders their usefulness quite limited. Using stochastic intensities and allowing for dependent thresholds, one runs into the problem of unobservability of the thresholds themselves. These two difficulties justify the success of factor models, which span a wide range of dependency, while allowing easy risk-neutral calibration, at the price of equicorrelation and correlation skews. Or the use of a purely empirical approach to copula calibration, similar to the implied volatility one, as in Hull and White (2005).

Recently, it has been stressed that the lack of total predictability—and therefore, of intensity representation—of structural models can be overcome by assuming that the information flow on asset values is not continuous over time: at the opposite, since accounting data are revealed only at discrete tenors, the wedge between the filtration of investors and the actual one renders default times totally inaccessible. The seminal paper in the field, which is for the time being developed only at the single obligor level, is Duffie and Lando (2001).

As an alternative, the total predictability can be eliminated by assuming that asset values are not described by a diffusion process. To introduce this approach, we refer to a Merton-like default model, as originally introduced by Madan (2000), which can be extended to the case of several obligors. Let us consider a default mechanism as the one described in Section 18.1, and assume that asset returns, as described by the logarithm of the asset value, $\ln(V_i/V_0)$, follow a pure jump process instead of a diffusion one. The simplest way to introduce the pure jump process in this case is as a transform of a diffusion, such as a Brownian motion with drift, via a stochastic transform of time:

$$\ln\left[\frac{V_i(t)}{V_i(0)}\right] = mG(t) + \sigma W[G(t)] \tag{18.52}$$

where m is the (constant) drift of returns under the historical measure, σ is their instantaneous (constant) variance, W is a standard Brownian motion, while $G(t)$ is the value at time t a so-called subordinator, namely of a real increasing Lévy process, with no diffusion component and finite variation. The effect of such a time transform on time, via G, is that of making log returns belong to the realm of Lévy processes: in particular, if G is a gamma process, the corresponding returns are variance gamma (VG), while if the subordinator is generalized inverse Gaussian, we get a generalized hyperbolic model. The amount of time change can be easily interpreted as a stochastic clock, which in turn can be related to the amount of trade in a given market (Geman, 2005). The usual restriction is that the random time change is equal to actual time in expectation:

$$E[G(t)] = t \tag{18.53}$$

Let us denote with $p_i^t(x)$ the marginal default probability of obligor i, conditional on the time change being equal to x. Given that we have a Merton default mechanism, if $H_i^t(u|x)$ is the asset value distribution of obligor i at time t, conditional on the change x, we have

$$p_i^t(x) = H_i^t(K_i|x) \tag{18.54}$$

The remarkable property of describing individual assets as above is that, conditional on the time change, these values are independent and therefore their (conditional) copula is the product one. If we denote with f_t the density of the time change at t (with support $\mathbb{R}^+$), we have a representation analogous to the one in Section 18.3:

$$F_i(t) = \int_{\mathbb{R}^+} p_i^t(x) f_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^+} H_i^t(K_i|x) f_t(x)\,\mathrm{d}x \tag{18.55}$$

while the joint probability can still be represented through a (conditional) factor copula $C^{\perp}$:

$$F(t) = \int_{\mathbb{R}^+} \prod_{i=1}^{n} H_i^t(K_i|x) f_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^+} C^{\perp}\left[H_1^t(K_1|x), \ldots, H_n^t(K_n|x)\right] f_t(x)\,\mathrm{d}x \tag{18.56}$$

Luciano and Schoutens (2006) study the case in which the time change is gamma distributed, so that asset returns are VG: this process indeed has been repeatedly used in the equity literature. Under this choice, taking into consideration the restriction Equation 18.53, and focusing on time 1, we have

$$f_1(x) = \frac{\nu^{1/\nu}}{\Gamma(1/\nu)} x^{1/\nu - 1} \mathrm{e}^{-x/\nu} \tag{18.57}$$

where Γ is the usual gamma function. At the same time, since—conditionally on the time change—returns are normally distributed, we can easily derive the conditional probabilities $p_i^t(x)$. Focusing again on their value at time 1, $p_i^1(x)$

$$p_i^1(x) = H_i^1(K_i|x) = \Phi\left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}}\right) \tag{18.58}$$

where $\theta_i = r + w_i$ and the shift w_i is used so as to obtain a risk-neutral measure:

$$w_i = \frac{1}{\nu} \log\left(1 - \sigma_i^2 \nu/2 - m_i \nu\right) \tag{18.59}$$

In this framework, the correlation coefficient between the returns of obligors i and j is

$$\rho_{ij} = \frac{m_i m_j \nu}{\sqrt{\sigma_i^2 + m_i^2 \nu}\sqrt{\sigma_j^2 + m_j^2 \nu}} \tag{18.60}$$

This shows that there is no equicorrelation.

From conditional normality, it follows that the marginal and joint default probabilities at time 1 are, respectively,

$$F_i(1) = \int_{\mathbb{R}_+} \Phi\left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}}\right) \frac{\nu^{1/\nu} x^{1/\nu - 1} \mathrm{e}^{-x/\nu}}{\Gamma(1/\nu)} \mathrm{d}x \tag{18.61}$$

$$F(1) = \int_{\mathbb{R}_+} \prod_{i=1}^{n} \Phi\left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}}\right) \frac{\nu^{1/\nu} x^{1/\nu - 1} \mathrm{e}^{-x/\nu}}{\Gamma(1/\nu)} \mathrm{d}x \tag{18.62}$$

where the threshold $z_i = \ln[K_i/V_i(0)]$ is obtained from the inequality $V_i(1) < K_i$ (Luciano and Schoutens, 2006).

The copula representation of the last probability can be obtained only by numerical inversion; however, the model can be easily calibrated, under the risk-neutral measure, so as to be able to evaluate actual default probabilities. Luciano and Schoutens extract all the model parameters (single and joint) from CDS (credit default swap) quotes. For each obligor they get the standard VG parameters, namely m_i, σ_i; the third parameter (ν) is common to all of the obligors. With respect to the standard calibration of univariate VG processes, they impose a constraint on the last parameter: the fit they obtain at the marginal level is quite good. They have a risk-neutral joint default probability, since all the parameters are extracted from market prices, with no need for equity (historical) correlation.

The model just described therefore overcomes two main problems evidenced in the previous approaches: lack of structural models which can be used to map intensity-based ones and lack of risk-neutral calibration without equicorrelation.

Fiorani et al. (2007) provide an even more general model of joint default with VG margins and without equicorrelation. The model is based on an extended multivariate VG process in which each asset is subject to a common and an idiosyncratic time change. As a result, the constraint on the common gamma parameter (ν) can be eliminated. The calibration of this model to credit market data is performed in Fiorani et al. (2007), via equity correlation and without equicorrelation.

In Fiorani and Luciano (2006), it was shown that the univariate VG model was able to overcome the spread underprediction typical of the Merton approach and the prediction biases of more sophisticated diffusive structural models, since it worked well both on high yield and investment grade names. Since the multivariate models in Fiorani et al. (2007) build on the univariate calibration as in Fiorani and Luciano (2006), the marginal fit is guaranteed.

The explicit representation of the previous models as intensity ones is under investigation: for the time being, the existence result is guaranteed, and the representation is not needed to calibrate the model neither at the individual nor at the joint level.

18.5 SUMMARY AND CONCLUDING REMARKS

Default dependence, most of the times represented by default (linear) correlation, is an important feature of credit derivatives pricing and hedging. Copula functions have proven to be extremely useful in describing joint default and survival dependency, as well as probabilities, in credit risk applications.

In the first part of this chapter, we have overviewed the state of the art and pointed out some open modeling issues. We have discussed joint default modeling first in diffusion-based

structural models, then in intensity-based ones. We have focused on the possibility, and the dynamic inconsistency, of remapping a model of the second type into one of the first. For both types of models, we have discussed calibration issues under the risk-neutral measure, using the factor copula device.

The survey has led us to present a structural model, based on a nondiffusive asset value, which can be remapped in a dynamic consistent intensity-based one, and which can be calibrated under the risk-neutral measure without assuming equicorrelation. Its copula properties and calibration have been investigated in Luciano and Schoutens (2006). Fiorani et al. (2007) extend further both the multivariate credit model and the calibration. They fit default correlation via equity correlation. It turns out that the model fits actual data quite satisfactorily, when applied to the CDS spreads of both investment grade and high yield names. In particular, it overcomes the underprediction of the diffusion-based Merton case and the prediction biases of more sophisticated diffusive structural models.

Evidently, it is just one possible rigorous solution to the currently immature state of the research in the field. Models who are dynamically consistent in the sense adopted here and liable of calibration under the risk-neutral measure, for pricing and hedging purposes, are very welcome.

APPENDIX

In this appendix, we briefly review the factor model with stochastic intensities and nonindependent thresholds. Having assumed that the copula of the uniform thresholds is Archimedean, as in Equation 18.50, $\varphi^{-1}(u)$ can be chosen to be the Laplace transform of a positive rv, Z, i.e., $\varphi^{-1}(u) = E\,(\mathrm{e}^{-uZ})$, and the copula (assuming a density exists for Z) can be written as

$$C^U(u_1, u_2, \ldots, u_n) = E\left\{ \exp\left[-Z \sum_{i=1}^{n} \varphi(u_i) \right] \right\} = \int_{\mathbb{R}+} \exp\left[-z \sum_{i=1}^{n} \varphi(u_i) \right] h(z)\,\mathrm{d}z \quad (18.A1)$$

With the Archimedean assumption, the joint survival probability can then be manipulated as

$$S(t) = E\left\{ C^U[\exp(-\Lambda_1),\ \exp(-\Lambda_2), \ldots, \exp(-\Lambda_n)] | \mathcal{F}_0 \right\} = \quad (18.A2)$$

$$= E\left(\varphi^{-1}\left\{ \sum_{i=1}^{n} \varphi[\exp(-\Lambda_i)] \right\} \right) = \cdots = E\left(\int_{\mathbb{R}+} \exp\left\{ -z \sum_{i=1}^{n} \varphi[\exp(-\Lambda_i)] \right\} h(z)\,\mathrm{d}z \right) \quad (18.A3)$$

Let us change the order of integration and condition the inner expectation with respect to $X = x$, which is the factor used in the marginal factorization (the one for $W := \Phi^{-1}[1 - G_i(\Lambda_i)]$), and is assumed to be independent of Z:

$$S(t) = \int_{\mathbb{R}+} \int_{\mathbb{R}} E\left(\exp\left\{ -z \sum_{i=1}^{n} \varphi[\exp(-\Lambda_i)] \right\} | x \right) f(x)\,\mathrm{d}x h(z)\,\mathrm{d}z \quad (18.A4)$$

Recognizing that the compensators are independent, conditionally on $X = x$, we have

$$S(t) = \int_{\mathbb{R}+} \int_{\mathbb{R}} \prod_{i=1}^{n} E\left(\exp\left\{-z\varphi[\exp(-\Lambda_i)]\right\}|x\right) f(x)\,\mathrm{d}x h(z)\,\mathrm{d}z \tag{18.A5}$$

which is of the form corresponding to Equation 18.39, since

$$\begin{aligned} &E(\exp\{-z\varphi[\exp(-\Lambda_i)]\}|x) \\ &\quad = E\left\{\exp\left[-z\varphi\left(\exp\left\{-G_i^{-1}\left[1-\Phi(\rho_i x + \sqrt{1-\rho_i^2}\varepsilon_i)\right]\right\}\right)\right]\right\} \end{aligned} \tag{18.A6}$$

is the marginal survival probability conditional on x and z, $1 - p_i^t(x,z)$.

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CHAPTER 19

Correlated Default Processes: A Criterion-Based Copula Approach

Sanjiv R. Das and Gary Geng

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19.1 INTRODUCTION

Default risk at the level of an individual security has been extensively modeled using both structural and reduced-form models.* This chapter examines default risk at the portfolio level. Default dependences among issuers in a large portfolio play an important role in the quantification of a portfolio's credit-risk exposure for many reasons. Growing linkages in the financial markets have led to a greater degree of joint default. While the acutal portfolio loss due to the default of an individual obligor may be small unless the risk exposure is extremely large, the effects of simultaneous defaults of several issuers in a well diversified portfolio could be catastrophic. In order to efficiently manage and hedge the risk exposure to joint defaults, a model for default dependencies is called for. Further, innovations in the credit market have been growing at an unprecedented pace in recent years, and will likely persist in the near future. Many newly developed financial securities, such as collateralized debt obligations (CDOs), have payoffs depending on the joint default behavior of the underlying securities.† In order to accurately measure and price the risk exposure of these

* See the structural models of Merton (1974), Geske (1977), Leland (1994), Longstaff and Schwartz (1995), and reduced-form models of Duffie and Singleton (1999), Jarrow and Turnbull (1995), Das and Tufano (1996), Jarrow et al. (1997), Madan and Unal (1999), Das and Sundaram (2000), and Jarrow and Yu (2001).

† A CDO securitization comprises a pool of bonds (the "collateral") against which tranches of debt are issued with varying cashflow priority. Tranches vary in credit quality, from AAA to B, depending on subordination level. Seller's interest is typically maintained as a final equity tranche, carrying the highest risk. CDO collateral usually comprises from a 100 to over a 1000 bond issues. Tranche cashflows critically depend on credit events during the life of the CDO, requiring Monte Carlo simulation (see, for example, Duffie and Singleton 1999) of the joint default process for all issuers in the collateral. An excellent discussion of the motivation for CDOs and the analysis of CDO value is provided in Duffie and Garleanu (2001). For a parsimonious model of bond portfolio allocations with default risk, see Wise and Bhansali (2001).

securities, an understanding and accurate measurement of the default dependencies among the underlying securities is essential.* Accurate specification of default correlations is required in the pricing of basket default contracts. Correlation specifications are critical, as the baskets are not large enough to ensure diversification. Finally, the Basle Committee on Banking Supervision has identified poor portfolio risk management as an important source of risk for banking institutions. As a result, banks and other financial institutions have been required to measure their overall risk exposures on a routine basis. Default correlation is again an important ingredient to integrate multiple credit-risk exposures.

In this chapter, we use Moody's default database to develop a parsimonious numerical method of modeling and simulating correlated default processes for hundreds of issuers. Using copula functions, we capture the stylized facts of individual default probabilities as well as the dynamics of default correlations documented in Das et al. (2006) (DFGK, 2006). Our methodology is "criterion based," i.e., it uses a metric that compares alternative specifications of the joint default distribution using three criteria: (1) the *level* of default risk, (2) the *asymmetry* in default correlations, and (3) the *tail dependence* of joint defaults. We elaborate on these in what follows.

The base unit of joint credit risk is the probability of default (PD) of each issuer. Many popular approaches exist for computing PDs in the market place, developed by firms such as KMV Corporation, Moody's Risk Management Systems (MRMS), RiskMetrics, etc. In the spirit of reduced-form models, the default probability of issuer i at time t is usually expressed as a stochastic hazard rate, denoted as $\lambda_i(t)$, $i = 1, \ldots, N$. This chapter empirically examines the joint stochastic process for hazard rates, $\lambda_i(t)$, for N issuers.

We fit a joint dynamic system to the intensities (or hazard rates) of roughly 600 issuers, classified into six rating categories. Our approach involves capturing the correlation of PD levels across the six rating classes, as well as the correlation of firms' PDs within each rating class. We find (1) the appropriate copula function (for the joint distribution), (2) the stochastic process for rating-level default probabilities, and (3) the best set of marginal distributions for individual issuer default probabilities.

At the rating level, we choose from two classes of stochastic processes for the average hazard rate of the class: a jump-diffusion model, and a regime-switching one. At the issuer level, we choose from three distributional assumptions: a normal distribution, a Student-t distribution, and a skewed double-exponential distribution. We compare these distributions for each issuer using four goodness-of-fit criteria. At the copula level, we choose from four types: normal, Student-t, Gumbel, and Clayton. These inject varying amounts of correlation emanating from the joint occurrence of extreme observations.†

The various combinations of choices at the rating, issuer, and copula level result in 56 different systems of joint variation. To compare across these, we develop a *metric* to determine how well different multivariate distributions fit the observed covariation

* As demonstrated by Das et al. (2001), Gersbach and Lipponer (2000), both the individual default probability and default correlations have a significant impact on the value of a credit portfolio.

† Extreme value distributions allow for the fact that processes tend to evidence higher correlations when tail values are experienced. This leads to a choice of fatter-tailed distributions, with a reasonable degree of "tail dependence." The support of the distribution should also be such that the PD intensity $\lambda_i(t)$ lies in the range $[0,\infty)$.

in PDs. The metric accounts for the level of correlations, asymmetry of correlations, and the tail dependence in the data.

An important issue we look at here is "tail dependence." Tail dependence is the feature of the joint distribution that determines how much of the correlation between default intensity processes comes from extreme observations than from central observations. For example, imagine two different bivariate distributions, with the same standard normal marginal distributions. One distribution has a normal copula and the other a Student-*t* copula with a low degree of freedom. The former has lower tail dependence than the latter, which generates more joint tail observations. Hence, from a risk point of view, given the marginal distributions, the second joint distribution would be riskier for a large class of risk-averse investors. Our study sheds light on the tail dependence for default risk in a representative data set of U.S. firms.

Our study of joint default risk for U.S. corporates using copula techniques finds that the best choice of copula depends on the marginal distributions, and stochastic processes for rating-level PDs. The Student-*t* copula is best when the jump-diffusion model is chosen for the default intensity of each rating class. However, the normal or Clayton copula is more appropriate when the regime-switching model is chosen for rating-level intensities. In general, our metric prefers a regime-switching model to a jump-diffusion one (for rating-level intensities), irrespective of the choice of marginal distributions, and a perusal of the time series plot of the data (Figure 19.1) provides intuitive support for this result. While the regime-switching model for rating-level intensities is best, no clear winner emerges amongst the competing copulas. The interdependence between copula and marginal distributions justifies our wide-ranging search over 56 different specifications.

We also assess the impact of different copulas on risk-management measures. An examination of the tail loss distributions shows that substantial differences are possible amongst the 56 econometric specifications. Arbitrarily assigning high kurtosis distributions to a copula with tail dependence may even result in the unnecessary overstatement of joint default risk. Since different copulas inject varied levels of tail dependence, the metric developed in this chapter allows fine-tuning of the specification, which enhances the accuracy of credit VaR calculations.

The rest of the chapter proceeds as follows. In Section 19.2, we briefly describe the data on intensities. Section 19.3 provides the finance reader with a brief introduction to copulas, and Section 19.4 contains the estimation procedure and results. Section 19.5 presents the simulation model and the metric used to compare different copulas. Section 19.6 concludes the chapter.

19.2 DESCRIPTION OF THE DATA

Our data set comprises issuers tracked by Moody's from 1987 to 2000. For each issuer, we have PDs based on their econometric models for every month the issuer was tracked by Moody's. Moody's calibrates the PDs to match the level of realized defaults in the economy. Issuers are divided into seven rating classes. Rating classes one through six reflect progressively declining credit quality. Many more issuers fall into rating class seven,

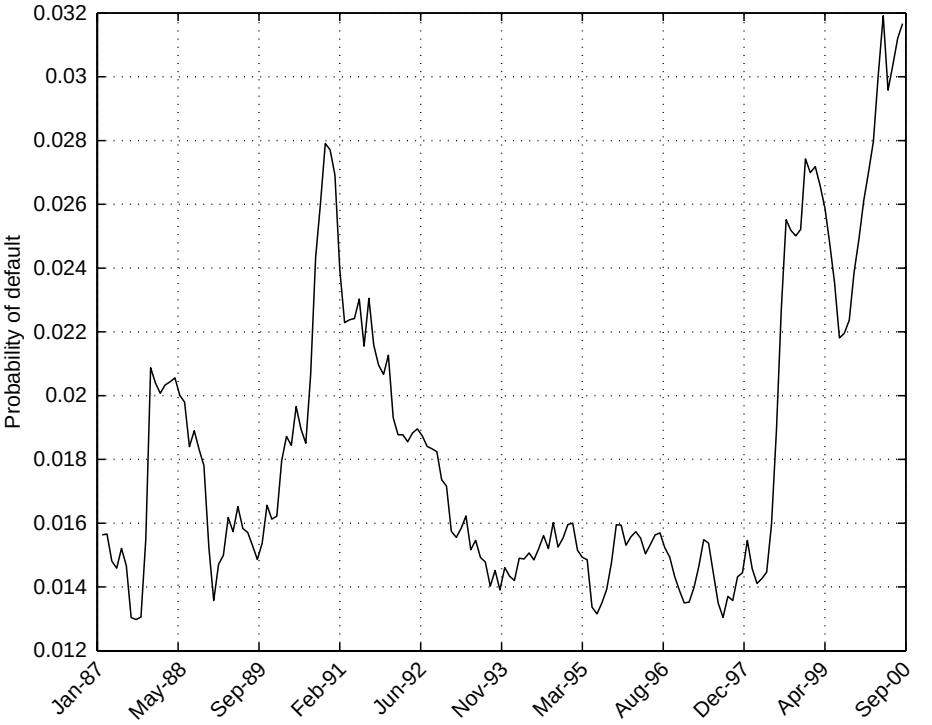


FIGURE 19.1 Time series of average PDs. This figure depicts the average level of default probabilities in the data set. The data show the presence of two regimes, one in which PDs were high, as in the early and latter periods of the data. In the other regime, PDs were much lower, less than half of those seen in the high PD regime. Complementary analysis of correlated default on the same data set by subperiod in this same time frame is presented in DFGK (2006) from where this figure is taken.

which comprises unrated issuers, and the PDs within this class range from high to low, resulting in an average PD that is close to the median PD of all the other rating classes. We do not consider PDs from rating class seven. We considered firms that had a continuous sample over the period. We obtained data on a total of 620 issuers classified into six rating classes. These 1 year default probabilities were converted into intensities using an assumption of constant hazard rates, i.e., $\lambda_{it} = -\ln(1 - \mathrm{PD}_{it})$. Table 19.1 provides summary data on Moody's rating categories.

The time series of average default probabilities is presented in Figure 19.1. Table 19.2 presents the descriptive statistics of our data from rating classes one through six. The mean increases from rating class one to six, as does the standard deviation. PD changes tend to be higher for lower grade debt. Table 19.3 presents the dependence between rating classes measured by Kendall's τ statistic. Higher grade debt evidences more rank correlation in PDs than low-grade debt. This highlights the fact that high-grade debt is more systematically related, and low-grade debt experiences greater idiosyncratic risk.

TABLE 19.1 Rating Classes

1	Aaa/Aa: Bonds rated Aaa/Aa carry the smallest degree of investment risk and are generally known as high-grade bonds
2	A: Bonds rated A possess many favorable investment attributes and are to be considered as upper-medium-grade obligations
3	Baa: Bonds rated Baa are considered as medium-grade obligations. Such bonds lack outstanding investment characteristics and in fact have speculative characteristics as well
4	Ba: Bonds rated Ba are judged to have speculative elements. Often the protection of interest and principal payments may be very moderate and thereby not well safeguarded during both good and bad times over the future
5	B: Bonds with B rating generally lack characteristics of the desirable investment. Assurance of interest and principal payments over any long period may be small
6	Caa/Ca/C: These are the bonds with high degree of speculation. Such issues may either be in default or elements of danger may be present with respect to principal or interest payments
7	Not rated

Note: The database comprises seven rating categories, in descending order of credit quality our data set does not contain data from firms in the financial sector.

19.3 COPULAS AND FEATURES OF THE DATA

19.3.1 Definition

We are interested in modeling an n-variate distribution. A random draw from this distribution comprises a vector $X \in R^n = \{X_1, \ldots, X_n\}$. Each of the n variates has its own marginal distribution, $F_i(X_i)$, $i = 1, \ldots, n$. The joint distribution is denoted as $F(X)$. The copula associated with $F(X)$ is a multivariate distribution function defined on the unit cube $[0, 1]^n$. When all the marginals, $F_i(X_i)$, $i = 1, \ldots, n$, are continuous, the associated copula is unique and can be written as

$$C(u_1, \ldots, u_n) = F\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right) \tag{19.1}$$

TABLE 19.2 Descriptive Statistics for PDs

Rating Category	PD in Levels		PD in Changes	
	Mean	Standard Deviation	Mean	Standard Deviation
1	0.0728	0.1051	0.0012	0.0441
2	0.1701	0.4884	0.0036	0.1468
3	0.6113	1.6808	0.0044	0.7317
4	1.4855	2.6415	0.0066	1.2694
5	3.8494	4.6401	0.0138	2.3576
6	6.5389	4.7731	0.0165	3.2075

Note: This table presents the summary of the time series of average PDs for each rating class. Each time series represents a diversified portfolio of issuers within each rating class, equally weighted. The probabilities are in percentages.

TABLE 19.3 Kendall's τ for Probabilities of Default

	Correlations					
Rating	**1**	**2**	**3**	**4**	**5**	**6**
1	1.0000	0.6508	0.2724	0.2868	0.2987	0.2762
2	0.4254	1.0000	0.4671	0.4569	0.4052	0.1794
3	0.1849	0.2312	1.0000	0.6416	0.4483	0.1080
4	0.1447	0.1393	0.0543	1.0000	0.5382	0.1708
5	0.0378	0.0877	0.0352	0.1245	1.0000	0.2911
6	0.0362	−0.0429	0.0238	−0.0276	0.1032	1.0000

Note: In this table we report Kendall's τ, which is a measure of the dependence between any two time series. The upper right triangle of the correlation matrix above reports the rank correlation for PD levels. The lower left triangle reports the rank correlation for changes in PDs.

Every joint distribution may be written as a copula. This is Sklar's theorem (Sklar 1959, 1973). For an extensive discussion of copulas, see Nelsen (1999).* Copulas allow the modeling of the marginal distributions separately from their dependence structure. This greatly simplifies the estimation problem of a joint stochastic process for a portfolio with many issuers. Instead of estimating all the distributional parameters simultaneously, we can estimate the marginal distributions separately from the joint distribution. Given the estimated marginal distribution for each issuer, we then use appropriate copulas to construct the joint distribution with a desired correlation structure. The best copula is determined by examining the statistical fit of different copulas to the data.†

Copula techniques lend themselves to two types of credit-risk analysis. First, given a copula, we can choose different marginal distributions for each individual issuer. By changing the types of marginal distributions and their parameters, we can examine how individual default affects the joint default behavior of many issuers in a credit portfolio. Second, given marginal distributions, we can vary the correlation structures by choosing different copulas, or the same copula with different parameter values.

19.3.2 Copulas Used in the Chapter

In this chapter, in order to capture the observed properties of the joint default process, we consider the following four types of copulas:

Normal Copula: The normal copula of the n-variate normal distribution with correlation matrix ρ, is defined as

$$C_\rho(u_1, \ldots, u_n) = \Phi_\rho^n\left(F^{-1}(u_1), \ldots, F^{-1}(u_n)\right)$$

where

$\Phi(u)$ is the normal cumulative distribution function (CDF)

$F^{-1}(u)$ is the inverse of the CDF

* There is a growing literature on copulas. The following references contain many useful papers on the topic: Embrechts et al. (1997), Embrechts et al. (1999a,b), Frees and Valdez (1998), Frey and McNeil (2001a), Li (1999), Nelsen (1999), Lindskog (2000), Schonbucher and Schubert (2001), Wang (2000).

† Durrleman et al. (2000) discuss several parametric and nonparametric copula estimation methods.

Student-t copula: Let $T_{\rho,\nu}$ be the standardized multivariate Student-t distribution with ν degrees of freedom and correlation matrix ρ. The multivariate Student-t copula is then defined as follows:

$$C(u_1, \ldots, u_n; \rho, \nu) = T_{\rho,\nu}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)\right)$$

where t_ν^{-1} is the inverse of the cumulative distribution function of a univariate Student-t distribution with ν degree of freedom.

Gumbel copula: This copula was first introduced by Gumbel (1960) and can be expressed as follows:

$$C(u_1, \ldots, u_n) = \exp\left[-\left(\sum_{i=1}^{n}(-\ln u_i)^\alpha\right)^{1/\alpha}\right] \tag{19.2}$$

where α is the parameter determining the tail dependence of the distribution.

Clayton copula: This copula, introduced by Clayton (1978), is as follows:

$$C(u_1, \ldots, u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-1/\alpha} \tag{19.3}$$

Again, $\alpha > 1$ is a tail dependence parameter.

The procedure for simulating an n-dimensional random vector from the above copulas can be found in Wang (2000), Bouye et al. (2000), and Embrechts et al. (2001).

19.3.3 Tail Dependence

An important feature of the use of copulas is that it permits varying degrees of *tail dependence*. Tail dependence refers to the extent to which the dependence (or correlation) between random variables arises from extreme observations.

Suppose (X_1, X_2) is a continuous random vector, with marginal distributions F_1 and F_2. The coeffcient of upper tail dependence is

$$\lambda_U = \lim_{z \to 1} Pr\left[X_2 > F_2^{-1}(z) | X_1 > F_1^{-1}(z)\right] \tag{19.4}$$

If $\lambda_U > 0$, then upper tail dependence exists. Intuitively, upper tail dependence is present when there is a positive probability of positive extreme observations occurring jointly.*

For example, the Gumbel copula has upper tail dependence with $\lambda_U = 2 - 2^{1/\alpha}$. The Clayton copula has lower tail dependence with $\lambda_L = 2^{-1/\alpha}$. The Student-$t$ has equal upper and lower tail dependence with $\lambda_U = 2\bar{t}_{\nu+1}\left(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$.

* Lower tail dependence is symmetrically defined. The coefficient of lower tail dependence is

$$\lambda_L = \lim_{z \to 0} Pr[X_2 < F_2^{-1}(z) | X_1 < F_1^{-1}(z)] \tag{19.5}$$

If $\lambda_L > 0$, then lower tail dependence exists.

Table 19.4 provides a snapshot of the tail dependence between rating classes. The results in the table present the correlations between rating pairs over observations in the tails of the bivariate distribution. Results are provided both, in levels and changes. There is strong

TABLE 19.4 Measures of Tail Dependence between the Different Rating Classes

Rating Pair		Lower Tail Dependence			Upper Tail Dependence		
Rating	**Rating**	**10%**	**20%**	**30%**	**70%**	**80%**	**90%**
Panel A: Tail Dependence in Levels							
1	2	0.4853	0.2879	0.2212	0.7518	0.7386	0.7206
1	3	0.0147	−0.3371	−0.1673	0.5837	0.5795	0.5441
1	4	−0.1471	0.0379	−0.0008	0.6376	0.4545	0.1471
1	5	−0.0882	−0.1439	−0.1314	0.6865	0.6061	0.2059
1	6	−0.2206	−0.1705	−0.0727	0.5298	0.4583	−0.0735
2	3	−0.1912	−0.2197	0.1004	0.5967	0.5985	0.5588
2	4	−0.2353	−0.1326	−0.0857	0.6000	0.4015	0.1324
2	5	−0.0147	−0.2045	−0.2065	0.6718	0.5189	0.0882
2	6	−0.1912	−0.3447	−0.1559	0.4171	0.3826	−0.1765
3	4	0.2500	0.2424	0.2539	0.5559	0.3902	0.1765
3	5	−0.0294	−0.1402	0.0694	0.5576	0.4205	0.2647
3	6	−0.1765	−0.1591	−0.0955	0.3535	0.4583	0.0147
4	5	0.0588	−0.1023	0.0155	0.7257	0.6439	0.4412
4	6	0.1471	−0.0833	−0.0318	0.4922	0.6326	0.4853
5	6	0.2206	0.0455	0.1020	0.4563	0.5152	0.2059
Panel B: Tail Dependence in Changes							
1	2	0.5000	0.4394	0.4106	0.4253	0.4015	0.3971
1	3	0.0441	0.0189	0.0367	0.0776	0.0038	0.2206
1	4	0.1029	0.1439	0.0188	0.1429	0.0644	0.1618
1	5	0.1029	−0.0455	−0.0269	0.1069	0.1439	0.2794
1	6	0.0588	−0.0455	−0.0580	−0.2033	−0.1856	0.0735
2	3	0.1618	0.0606	0.0873	0.0204	−0.0606	0.1324
2	4	−0.2647	−0.0076	−0.0580	0.1984	0.0606	0.2647
2	5	−0.0147	0.0189	0.0645	0.0302	0.2045	0.3382
2	6	0.1618	0.0455	0.0482	−0.1559	−0.1667	−0.2500
3	4	0.2941	0.1364	−0.0433	0.0122	−0.0720	−0.0441
3	5	−0.0441	−0.1212	−0.0367	0.2082	−0.0152	0.0294
3	6	0.0882	0.0076	0.1216	−0.0694	−0.1136	−0.3382
4	5	−0.0588	−0.0227	0.1265	0.2555	0.3864	0.3824
4	6	−0.0735	0.1515	0.0971	0.0041	0.1098	−0.1324
5	6	−0.0294	0.1553	0.0171	0.0547	0.0341	−0.0147

Note: This table presents correlations between hazard rates amongst the rating classes based on different tail percentiles. We examine correlation amongst observations in the bottom 10th, 20th, and 30th percentiles. We also present the correlations in the top 10th, 20th, and 30th percentiles, i.e., the cutoffs are the 70th, 80th, and 90th percentiles. The values in the table are the correlations when observations from the second rating in the pair of rating classes lies in the designated portion of the tail of the distribution. Results are presented for levels of hazard rates and for changes in hazard rates.

positive correlation in the upper tail, evidencing upper tail dependence. Correlations in the lower tail are often low and negative, and hence, there is not much evidence of lower tail dependence.

19.3.4 Empirical Features of Dependence in the Joint Distribution

In order to compare the statistical fitting of joint distributions associated with different copulas, we develop a metric, which captures three features of the dependence relationship in the joint distribution:

- *Correlation levels*: We wish to ensure that our copula permits the empirically observed correlation levels in conjunction with the other moments. This depends on the copula and the attributes of the data.
- *Correlation asymmetry*: Hazard rate correlations are level dependent, and are higher when PD levels are high—correlations increase when PD levels jump up, and decrease when PDs jump down.
- *Tail dependence*: It is important that we capture the correct degree of tail dependence in the data, i.e., the extent to which extreme values drive correlations.

Following Ang and Chen (2002), and Longin and Solnik (2001), we present the correlations for different rating classes in a correlation diagram (Figure 19.2). This plot is created as follows. We compute the total hazard rate (THR) across issuers (indexed by i) at each point in time (t), i.e., $\text{THR}_t = \sum_{i=1}^{N} \lambda_i(t)$, $\forall t$. We normalize the THR by subtracting the mean from each observation and dividing by the standard deviation. We then segment our data set based on exceedance levels (ξ) determined by the normalized THR. Our exceedance levels are drawn from the set $\xi = \{-1.5, -1.0, -0.5, 0, 0, 0.5, 1.0, 1.5\}$. (Note that zero appears twice because we account for both, left and right exceedance.) Ang and Chen (2002) developed a version of this for bivariate processes. Our procedure here is a modification for the multivariate case. For example, the exceedance correlation at level $-\xi$ is determined by extracting the time series of PDs for which the normalized THR is less than $-\xi$, and computing the correlation matrix therefrom. We then find the average value of all entries in this correlation matrix, to obtain a summary number for the correlation in the rating class at the given exceedance level. These are plotted in Figure 19.2. There is one line for each rating class. The line extends to both the left and right sides of the plot. To summarize, the exceedance graph shows (for each rating category) the correlation of default probabilities when the overall level of default risk exceeds different thresholds. As threshold levels become very small or very large, correlations decline, and the rate at which they decline indicates how much tail dependence there is.

The three features of the dependence relationship are evident from the graph. First, the height of the exceedance line indicates the level of correlation, and we can see that high-grade debt has greater correlation than lower grade debt [see Xiao (2003) for a similar result with credit spreads]. Second, there is clear evidence of correlation asymmetry, as correlation levels are much higher on the right side of the graph, i.e., when hazard rate changes are positive. Third, the amount of tail dependence is inferred from the slope of the

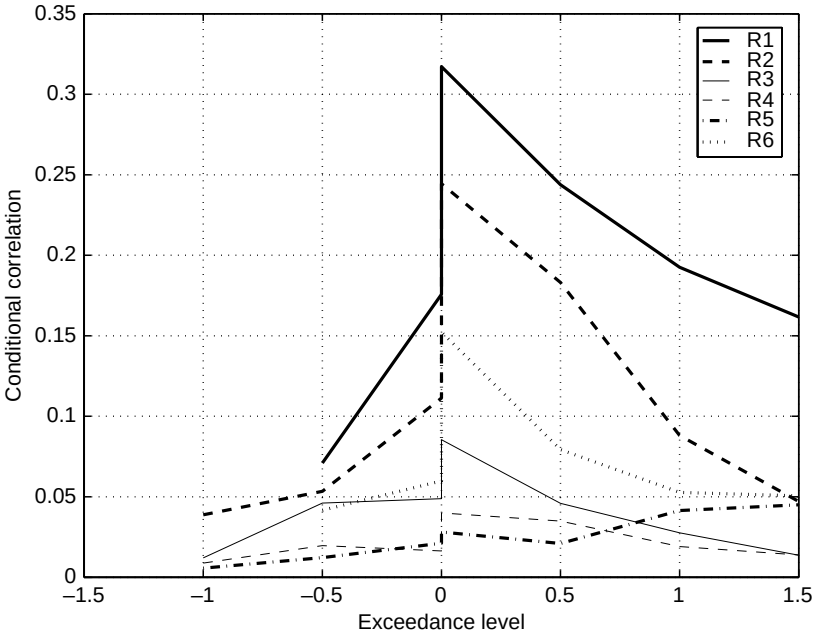


FIGURE 19.2 Asymmetric correlation of PDs This figure plots the exceedance correlations for the different rating classes in our study. The figure breaks this out into the individual ratings. The plots are consistent with the presence of asymmetric correlation. The degree of asymmetry appears to be higher for the better rating categories. Complementary analysis of correlated default by subperiod in this same time frame is presented in DFGK (2001a) where different analyses are conducted on the same data set. This figure corresponds to their paper.

correlation line as the exceedance level increases. The flatter the line the greater the amount of tail dependence. As absolute exceedance levels increase, we can see that correlation levels drop. However, they fall less slowly if there is greater tail dependence. We can see that lower grade debt appears to have more tail dependence than higher grade debt.

Overall, the exceedance plot shows that bonds within high quality ratings have greater default correlation than bonds within low quality ratings. However, tail dependence is higher for lower rated bonds. Both results are intuitive—high-grade debt tends to be issued by large firms that experience greater amounts of systematic risk, and low-grade firms evidence more idiosyncratic risk; however, when economy-wide default risk escalates, low-grade bonds are more likely to experience contagion, leading to greater tail dependence.

19.4 DETERMINING THE JOINT DEFAULT PROCESS

19.4.1 Overall Method

Our data comprise firms categorized into six rating classes. We average across firms within a rating class to obtain a time series of the average intensity (λ_k) for each rating class k. We assume that the stochastic processes for the six average intensities are drawn from a joint

distribution characterized by a copula, which establishes the joint dependence between rating classes. The correlation matrix for the mean intensities is calculated directly from the data. The copula is set to one of the four possible choices: normal, Student-*t*, Gumbel, or Clayton. The form of the stochastic process for each average hazard rate is taken to be either a jump-diffusion or a regime-switching one. This setup provides for the correlation *between* rating classes.

Next, the correlations *within* a rating class are obtained from the linkage of all firms in a rating class to the mean PD process of the rating category. We model individual firm PDs as following a non-negative stochastic process with reversion to the stochastic mean of each rating.

To conduct a Monte Carlo simulation of the system, we propagate the average rating PDs using either a jump-diffusion or regime-switching model, with correlations across ratings given by a copula. Then, we propogate the individual issuer intensities using stochastic processes that oscillate around the rating means. The entire system consists of a choice of (1) stochastic process for average intensities (jump-diffusion versus regime-switching), (2) copula (normal, Student-*t*, Gumbel, Clayton), and (3) marginal distribution for issuer-level stochastic process for intensities (one of seven different options). All told, therefore, we have 56 alternate correlated default system structures to choose from.

19.4.2 Estimation Phase

Estimation is undertaken in a two-stage manner: (a) the estimation of the stochastic process for the mean intensity of each rating class and (b) the estimation of the best candidate distribution for the stochastic process of each individual issuer's intensity. We chose two processes for step (a). Both choices were made to inject excess kurtosis into the intensity distribution. One, we used a normal-jump model. Two, we used a regime-switching model. Each is described in turn in the next two subsections.

19.4.2.1 Estimation of the Mean of Each Rating Class Using a Jump Model

For our six rating classes (indexed by k), and all issuers (indexed by j), we compute intensities from the probabilities of default (P_{kj}, $j = 1, \ldots, N_k$, $k = 1, \ldots, 6$) in our data set. The intensities are computed as: $\lambda_{kj} = -\ln(1 - P_{kj}) \geq 0$. We denote M as the total number of rating classes and N_k as the total number of issuers within the rating class for which data are available.

Let $\lambda_k(t)$ be the average intensity across issuers within rating class k. Therefore, $\lambda_k(t) = 1/N_k \sum_{j=1}^{N_k} \lambda_{kj}(t)$, $\forall t$. We assume that $\lambda_k(t)$ follows the stochastic process:

$$\begin{aligned}
\Delta\lambda_k(t) &= \kappa_k[\theta_k - \lambda_k(t)]\Delta t + x_k(t)\sqrt{\lambda_k(t)\Delta t} \\
x_k(t) &= \varepsilon_k(t) + J_k(t)L_k(q_k, t) \\
\epsilon_k(t) &\sim N[0, \sigma_k^2] \\
J_k(t) &\sim U[a_k, b_k] \\
L_k(q_k, t) &= \begin{cases} 1 & \text{w/prob} \quad q_k \\ 0 & \text{w/prob} \quad 1 - q_k \end{cases}
\end{aligned} \tag{19.6}$$

We assume that the jump size J_k follows a uniform distribution over $[a_k,b_k]$.* This regression accommodates a mean-reverting version of the stochastic process, and κ_k calibrates the persistence of the process. This process is estimated for each rating class k, and the parameters are used for subsequent simulation. Copulas are used to obtain the joint distribution of correlated default. The correlation matrix used in the copula comes from the residuals $x_k(t)$ computed in the regression above.

We use maximum-likelihood estimation to obtain the parameters of this process. Since there is a mixture of a normal and a jump component, we can decompose the unconditional density function for the residual term $x_k(t)$ into the following:

$$f[x_k(t)] = q_k f[x_k(t)|L_k = 1] + (1 - q_k) f[x_k(t)|L_k = 0] \tag{19.7}$$

The latter density, $f[x_k(t)|L_k = 0]$ is conditionally normal. The marginal distribution of $x_k(t)$ conditional on $L_k = 1$ can be derived as follows:

$$f[x_k(t)|L_k = 1] = \int_{a_k}^{b_k} \frac{1}{(b_k - a_k)\sqrt{2\pi\sigma_k^2}} \times \exp\left\{-\frac{(x_k(t) - J_k)^2}{2\sigma_k^2}\right\} \mathrm{d}J_k \tag{19.8}$$

Let $z = (J_k - x_k(t))/\sigma_k$. Then,

$$\begin{aligned} f[x_k(t)|L_k = 1] &= \int_{\frac{a_k - x_k(t)}{\sigma}}^{\frac{b_k - x_k(t)}{\sigma}} \frac{1}{b_k - a_k} \exp\left\{-\frac{z^2}{2}\right\} \frac{1}{\sqrt{2\pi}} \mathrm{d}z \\ &= \frac{1}{b_k - a_k}\left\{\Phi\left[\frac{b_k - x_k(t)}{\sigma_k}\right] - \Phi\left[\frac{a_k - x_k(t)}{\sigma_k}\right]\right\} \end{aligned}$$

In the case when $a_k = b_k$, the process has a constant jump size with probability 1 and the marginal distribution of $x_k(t)$ degenerates to a normal distribution with mean a_k and standard deviation σ_k. This can be seen by letting $b_k = a_k + c$. We, then, have

$$\begin{aligned} f[x_k(t)|L_k = 1] &= \lim_{c \to 0} \frac{1}{c}\left\{\Phi\left(\frac{a_k + c - x_k(t)}{\sigma_k}\right) - \Phi\left(\frac{a_k - x_k(t)}{\sigma_k}\right)\right\} \\ &= \lim_{c \to 0} \frac{1}{\sigma_k}\phi\left(\frac{a_k + c - x_k(t)}{\sigma_k}\right) \\ &= \frac{1}{\sigma_k}\phi\left(\frac{x_k(t) - a_k}{\sigma_k}\right) \end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution functions, respectively. Since jumps are permitted to be of any sign, we may expect that $b_k \geq 0$ and $a_k \leq 0$. Also, if $|b_k| > |a_k|$, it implies a greater probability of positive jumps. When a_k is close to b_k,

* This specification does permit the hazard rate to populate negative values. While this is not an issue during the estimation phase, the simulation phase is adjusted to truncate the shock if the negative support is accessed. However, we remark that this occurs in very rare cases, since $\lambda_k(t)$ is the average across all issuers within the rating class, and the averaging drives the probability of negative hazard rates to minuscule levels.

TABLE 19.5 Estimation of the Average Rating Process

	Rating Class					
Parameter	**1**	**2**	**3**	**4**	**5**	**6**
θ_k	0.0524	0.1336	0.3499	1.1550	2.3812	5.4848
	7.73	16.74	2.75	4.01	1.99	7.12
κ_k	0.4813	0.0018	0.6680	0.4811	0.4305	2.8350
	2.84	0.01	1.88	1.87	2.01	4.19
σ_k	0.0487	0.1050	0.2518	0.2633	0.2872	0.5743
	14.17	13.39	12.48	11.44	8.51	5.85
b_k	0.0839	0.1316	0.1241	0.2070	0.2684	1.1231
	12.39	5.64	1.94	1.58	1.52	5.50
a_k	−0.0707	−0.0644	0.1241	0.2069	0.2684	0.0771
	−6.33	−3.46	1.96	1.61	2.37	0.21
q_k	0.1016	0.0777	0.1514	0.0950	0.2461	0.4374
	3.87	3.18	2.94	2.07	2.08	2.07
Log-likelihood	−622.16	−462.95	−205.48	−133.56	−32.82	−35.03

Note: In this table we report the estimation results of the stochastic process for the mean of the intensities of each rating class. Numbers below the estimates are the *t*-statistics from the estimation, which is undertaken by maximum-likelihood.

or when they have the same sign, the process has a constant or one-direction jump, respectively.

The estimation results are presented in Table 19.5. The mean of the intensity process (θ_k) increases with rating class k as expected. The variance (σ_k) also increases with declining credit quality. Generally speaking, the probability of a jump (q_k) in the hazard rate is higher for lower quality issuers. We can see that rating classes 3–5 have a constant jump and rating 6 has only positive jumps. While the mean jump for all rating classes appears to be close to zero, that for the poorest rating class is much higher than zero. Hazard rates jump drastically when a firm approaches default.

After obtaining the parameters, we compute residuals for each rating class. The residuals include randomness from both the normal and jump terms. The covariance matrix of these residuals is stored for later use in Monte Carlo simulations.

19.4.2.2 Estimating the Mean Processes in a Regime-Switching Environment

From Figure 19.1, we note that there are periods in which PDs are low, interjected by smaller, sporadic regimes of spikes in the hazard rates. A natural approach to capture this behavior is to use a regime-switching model.*

In order to determine regimes, we first computed the average intensity (hazard rate, approximately) $\bar{\lambda} = \sum_{i=1}^{N} \lambda_i$ across all issuers in our database. Within each regime the intensity is assumed to follow a square-root model represented in discrete-time as follows:

* Ang and Chen (2002) found this type of model to be good at capturing the three stated features of asymmetric correlation in equity markets.

$$\Delta\bar{\lambda}_r(t) = \kappa_r[\theta_r - \bar{\lambda}_r(t)]\Delta t + \sigma_r\sqrt{\bar{\lambda}_r(t)\Delta t} \ \in \ (t), \quad r = \{HI, LO\} \tag{19.9}$$

The two regimes are indexed by r, which is either HI or LO. κ_r is the rate of mean-reversion. The mean value within the regime is θ_r and σ_r is the volatility parameter.

The probability of switching between regimes comes from a logit model based on a transition matrix:

$$\begin{bmatrix} p_{LO} & 1 - p_{LO} \\ 1 - p_{HI} & p_{HI} \end{bmatrix}$$

where $p_r = \exp(\alpha_r)/(1 + \exp\ (\alpha_r))$, $r \in \{LO, HI\}$. Estimation is undertaken using maximum-likelihood. We fixed the values of θ_r based on the historical PDs, with a high–low regime cutoff of 2%. θ_{LO} was found to be 1.68% and θ_{HI} to be 2.40%.* We then estimated the rest of the parameters in the above model. The estimation results are presented in Table 19.6. In the high PD regime, the higher level of hazard rates is matched by a higher level of volatility.

Our next step is to use the determined regimes to estimate the parameters of the mean process for each rating class, i.e., λ_k, $k = 1, \ldots, 6$. We fit regime-shifting models to each of the rating classes, using a stochastic process similar to that in Equation 19.9:

$$\Delta\bar{\lambda}_{k,r}(t) = \kappa_{k,r}[\theta_{k,r} - \bar{\lambda}_{k,r}(t)]\Delta t + \sigma_{k,r}\sqrt{\bar{\lambda}_{k,r}(t)\Delta t} \ \in \ (k, t),$$
$$r = \{HI, LO\}, \quad k = 1, \ldots, 6 \tag{19.10}$$

TABLE 19.6 Estimation of the Regime-Switching Model Across All Issuers

Parameter	Low-PD Regime	High-PD Regime
θ	1.68%	2.40%
κ	1.5043	1.7746
t-stat	2.4053	2.0558
σ	0.1681	0.2397
t-stat	14.2093	7.9849
α	4.1286	3.3644
t-stat	4.6649	2.5428
Log-likelihood	−195.5948	

Note: This table provides estimation results for regimes estimated on the average intensity process $\left(\bar{\lambda} = \sum_{i=1}^{N} \lambda_i\right)$ across all issuers in the dataset. We designated periods where $\bar{\lambda} \leq 2\%$ as the "low-PD" regime, and periods in which $\bar{\lambda} > 2\%$ as the "high-PD" regime. The regimes were bifurcated exogenously, as described, and we estimated the values of θ_r by simply averaging the hazard rate within each regime. Estimation is undertaken using maximum-likelihood.

* We found that the estimations of θ_r were very sensitive to the initial values used in the optimization. By fixing these two values, the estimation turned out to be more stable.

TABLE 19.7 Estimation of the Regime-Switching Model for the Mean of Each Rating Class

	Rating Class					
Parameter	**1**	**2**	**3**	**4**	**5**	**6**
θ_{LO}	0.0559	0.1625	0.5678	1.3588	3.6367	6.3138
t-stat	0.40	7.06	11.90	14.22	45.24	74.32
θ_{HI}	0.0991	0.3757	0.9818	2.2358	5.1216	7.8295
t-stat	0.36	6.55	4.91	19.29	18.74	8.07
κ_{LO}	1.3535	0.5094	1.5855	1.1317	2.2365	6.2456
t-stat	0.23	1.31	2.86	2.25	3.49	6.45
κ_{HI}	0.6152	0.1775	1.1717	2.8567	2.0833	1.8862
t-stat	0.35	1.74	0.95	2.52	1.67	1.81
σ_{LO}	0.0568	0.1224	0.3214	0.2986	0.3338	0.6798
t-stat	17.63	17.27	16.19	16.57	16.92	16.02
σ_{HI}	0.1103	0.2214	0.3275	0.3305	0.3681	0.8336
t-stat	8.41	7.83	7.84	7.93	8.15	8.24

Note: This table presents parameters for the regime-switching model applied to the mean process for each rating class, i.e. $\bar{\lambda}_k$, $k = 1, \ldots, 6$.

The results of the estimation are presented in Table 19.7. The parameters are as expected. The mean of the processes is higher in the high regime, as is the volatility. We now proceed to look at the choice of individual issuer marginal distributions.

19.4.2.3 Estimation of the Stochastic Process for Each Individual Issuer

Assume that the intensity for each individual issuer follows a square-root process:

$$\Delta\lambda_{kj}(t) = \kappa_{kj}\{\theta_{kj} + \gamma_{kj}\lambda_k - \lambda_{kj}(t)\} + \sigma_{kj}\sqrt{\lambda_{kj}(t)}\varepsilon_{kj}(t) \tag{19.11}$$

We normalize the residuals by dividing both sides by $\sqrt{\lambda_{kj}(t)\mathrm{d}t}$. We then use OLS to estimate parameters, which are stored for use in simulations. The resulting residuals are used to form the covariance matrix for each class.

The long-term mean is defined as the sum of a constant number θ_{kj}, and the group mean $\lambda_k(t)$ multiplied by a constant, γ_{kj}. This indexes the mean hazard rate for each issuer to the current value of the mean for the rating class. Consequently, there are four parameters for each regression, κ_{kj}, θ_{kj}, γ_{kj}, and σ_{kj}. Note that κ_{kj} should be positive, θ_{kj} and γ_{kj} can be positive or negative as long as they are not both negative at the same time. None of the issuers have negative θ_{kj} and γ_{kj} at the same time, although 20 issuers have negative κ_{kj}. We, therefore, deleted the 20 issuers with negative κ from our dataset for further analysis.*

* The estimation of Equation 19.11 relies on the assumption that the mean around which the individual hazard rate oscillates depends on the initial rating class of the issuer, even though this may change over time. Extending the model to map default probabilities to rating classes is a nontrivial problem, and would complicate the estimation exercise here beyond the scope of this paper. Indeed, this problem in isolation from other estimation issues is complicated enough to warrant separate treatment and has been addressed in a paper by Das et al. (2002).

19.4.2.4 Estimation of the Marginal Distributions

We fitted the residuals from the previous section to the normal, Student-*t*, and skewed double-exponential distributions. We used prepackaged functions in Matlab for maximum-likelihood estimation for the normal and Student-*t* distributions. The likelihood function for the skewed double-exponential distribution is as follows. Assume that the residual ε_{kj} has a normal distribution with mean γV and variance V. Further, the variance V is assumed to have an exponential distribution with the following density function:

$$\text{pdf}(V) = \frac{1}{V_0} \exp\left(-\frac{V}{V_0}\right) \tag{19.12}$$

Then, it can be shown that ε_{kj} has a skewed double-exponential distribution with the following density function:

$$\text{pdf}(\varepsilon) = \frac{\lambda}{V_0} \exp\left(-\frac{|\varepsilon_{kj}|}{\lambda} - \gamma\varepsilon_{kj}\right) \tag{19.13}$$

where

$$\lambda = \sqrt{\frac{V_0}{2 + \gamma^2 V_0}}$$

See the Appendix for a derivation of these results. The log-likelihood function L is given as follows:

$$L = -\frac{n}{2}\left\{\log(V_0) + \log(2 + \gamma^2 V_0)\right\} - \sum_{i=1}^{n} |\varepsilon_i| \sqrt{\frac{2 + \gamma^2 V_0}{V_0}} + \gamma \sum_{i=1}^{n} \varepsilon_i$$

19.4.2.5 Goodness of Fit of Marginal Distributions

Since we choose one of the three distributions for the residuals of each issuer, we used four criteria to decide the best one. These are as follows:

- Kolmogorov distance: This is defined as the supremum over the absolute differences between two cumulative density functions, the empirical one, $F_{\text{emp}}(x)$ and the estimated, fitted one, $F_{\text{est}}(x)$.
- Anderson and Darling statistic: This is given by

$$AD = \max_{x \in R} \frac{|F_{\text{emp}}(x) - F_{\text{est}}(x)|}{\sqrt{F_{\text{est}}(x)[1 - F_{\text{est}}(x)]}}$$

  The AD statistic puts more weight on the tails compared to the Kolmogorov statistic.
- L^1 distance: equal to the average of the absolute differences between the empirical and statistical distributions.
- L^2 distance: which is the root-mean-squared difference between the two distributions.

TABLE 19.8 Fitting Marginal Distributions to Individual Intensities

Statistic	Double-Exponential	Normal	Student-*t*
Kolmogorov	522	55	42
Anderson–Darling	409	7	203
L^2	529	53	37
L^1	118	500	1

Note: In this table we report the estimation results of the fit of individual intensity residuals to various distributions. The three distributions chosen were: double-exponential, normal, and the Student-*t*. We used four distance metrics to compare the empirical residuals to standardized distributions: the Kolmogorov distance, the Anderson–Darling statistic, and distances in the L^1 and L^2 norms. The table below presents the number of individual issuers that best fit each of the distributions under the different metrics. A total of 619 issuers were classified in this way.

We used these criteria to choose between the normal, Student-*t*, and skewed double-exponential distributions for each issuer. In each of our 56 simulated systems, the issuer-level distributions are chosen from the following seven cases:

1. All marginal distributions were chosen to be normal.
2. All marginal distributions are Student-*t*.
3. All marginals are from the skewed double-exponential family.
4. For each issuer, the best distribution is chosen based on the Kolmogorov criterion.
5. For each issuer, the best distribution is chosen based on the Anderson and Darling statistic.
6. Each marginal is chosen based on the best L^1 distance.
7. Each marginal is chosen based on the best L^2 distance.

For each criterion, we counted the number of times each distribution provided the best fit. The results are reported in Table 19.8, which provides interesting features. On the basis of the Kolmogorov, Anderson–Darling, and L^2 statistics, the skewed double-exponential distribution is the most likely to fit the marginals best. However, the L^1 statistic finds that the normal distribution fits most issuers better. The better fit of the skewed double-exponential as marginal distribution comes from its ability to match the excess kurtosis of the distribution.

19.4.3 Calibration

Our calibration procedure is as follows. First, using MLE, we determine the parameters of the various stochastic processes (rating-level averages and individual issuer PDs). Second, we choose one of the 56 possible correlation systems as described above, and simulate the entire system 100 times, to obtain an average simulated exceedance graph. Third, we compare the simulated exceedance graph with the empirical one (pointwise) and

determine the mean squared difference between the two graphs. This distance metric is used to determine a ranking of the 56 possible systems we work with.

19.5 SIMULATING CORRELATED DEFAULTS AND MODEL COMPARISONS

19.5.1 Overview

In this section, we discuss the simulation procedure. This step uses the estimated parameters from the previous section to generate correlated samples of hazard rates. The goal is for the simulation approach to deliver a model with the three properties described in the previous section. Therefore, the asymmetric correlation plot from simulated data should be similar to that in Figure 19.2.

We implemented the simulation model with an additional constraint, whereby we ensure that the intensities ($\lambda_k(t)$) are monotonically increasing in rating level k. This prevents the average PD for a rating class from being lower than the average PD for the next better rating category, i.e., $\lambda_k(t)$ should be such that for all time periods t, it must be such that if $i<j$, then $\lambda_i(t)<\lambda_j(t)$. This check was instituted during the simulation as follows. During the Monte Carlo step, if $\lambda_i(t)>\lambda_j(t)$ when $i<j$, then we set $\lambda_j(t)=\lambda_i(t)$. As an example, see Figure 19.3 where we plot the times series of λ_k, $k=1,\ldots,6$ for a random simulation of the sample path of hazard rates.

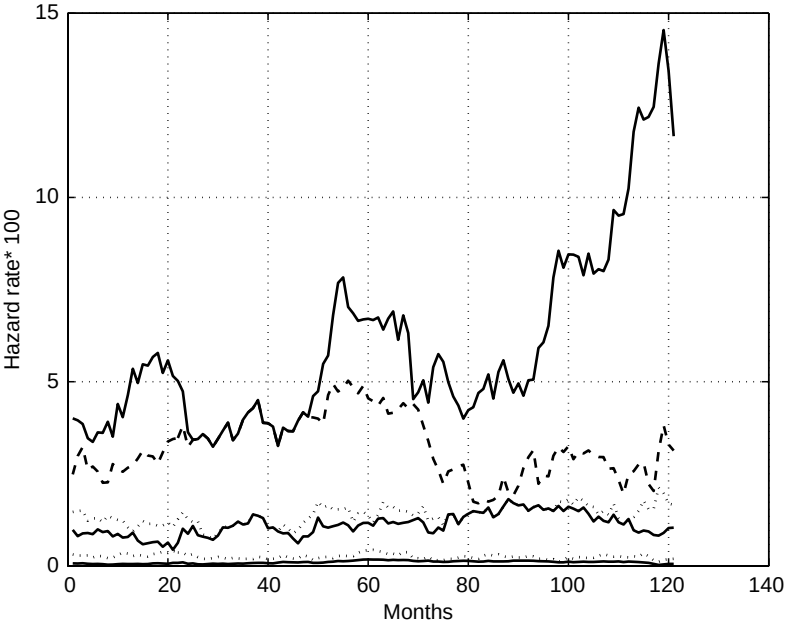


FIGURE 19.3 Simulated series of average PDs by rating class: this figure depicts the average level of default probabilities in the data set by rating class. The hazard rates are (from top to bottom of the graph) from rating class six to rating class one. The simulation ensures that $\lambda_i(t)<\lambda_j(t)$ if $i<j$.

19.5.2 Illustrative Monte Carlo Experiment

As a simple illustration that our approach arrives at a correlation plot fairly similar to that seen in the data, we ran a naive Monte Carlo experiment. In this exercise, we assumed that all error terms were normal, except under some conditions, when we assumed the Student-*t* distribution. We ran the Monte Carlo model 25 times and computed the average exceedance correlations for all rating classes. To obtain asymmetry in the correlations, we need to have higher correlations when hazard rates are high. To achieve this, the simulation uses the Student-*t* distribution with six degrees of freedom when the average level of hazard rates in the previous period in the simulation is above the empirical average of hazard rates. These features provided the results in Figure 19.4. The similarity between the empirical exceedance graph (Figure 19.2) and the simulated one (Figure 19.4) shows that the two-stage Monte Carlo model is able to achieve the three correlation properties of interest.

19.5.3 Determining Goodness of Fit

The dependence structure amongst the intensities is depicted in the asymmetric correlation plots. We develop a measure to assess different specifications of the joint distribution. This metric is the average squared point-wise difference between the empirical exceedance correlation plot and the simulated one. The points in each plot that are used are for the

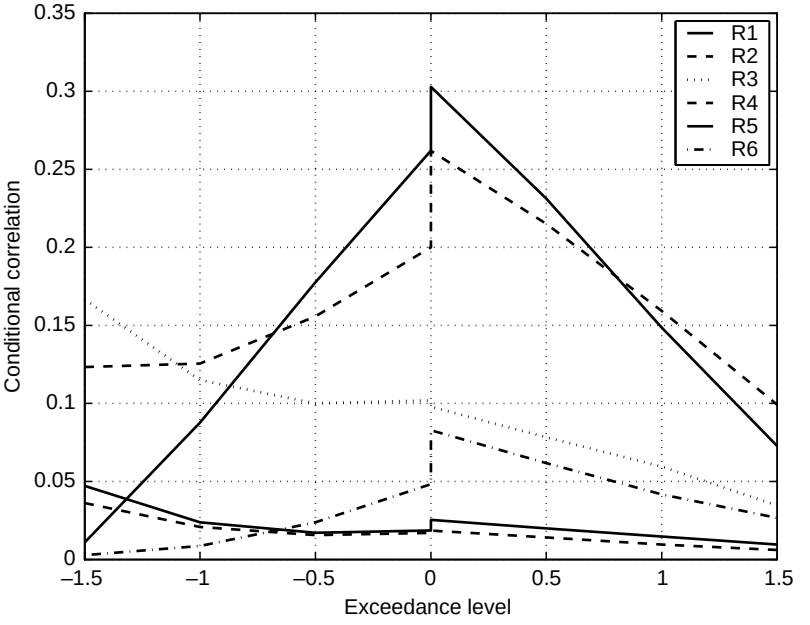


FIGURE 19.4 Asymmetric correlation of simulated PDs: this figure plots the exceedance correlations for the different rating classes in our study, based on simulated data. The figure breaks this out into the individual ratings. The plots are consistent with the presence of asymmetric correlation. The degree of asymmetry is higher for the better rating categories.

combination of rating class (from 1 to 6) and exceedance levels (from −1.5 to +1.5). Hence, there are a total of 48 points in each plot which are used for computing the metric. Define the points in the empirical plot as $h_{k,x}$, where k indexes the rating class and x indexes the exceedance levels. The corresponding points in the simulated plot are denoted by $h'_{k,x}$. The metric q (an RMSE statistic) is as follows:

$$q = \sqrt{\frac{1}{48}\sum_{k=1}^{6}\sum_{x=-1.5}^{1.5}\left(h_{k,x} - h'_{k,x}\right)^2} \tag{19.14}$$

A smaller value of q implies a better fit of the joint dependence relationship. In the following section, we use the simulation approach and this metric to compare various models of correlated default.

19.5.4 Calibration Results and Metric

It is important to note that the joint dependence across all issuer intensities depends on three aspects: (1) the marginal distributions used for individual issuers, (2) the stochastic process used for each rating class, and (3) the copula used to implement the correlations between rating classes. Optimizing the choice of individual issuer marginal distributions will not necessarily achieve the best dependence relationship, since the copula chosen must also be compatible. In this chapter, we focus on four copulas (normal, Student-*t*, Gumbel, and Clayton), and allow the rating class intensity stochastic process to be either a jump-diffusion model or a regime-shifting model.

Tables 19.9 and 19.10 provide a comparison of the jump and the regime-switching models across copulas. Twenty-eight systems use a jump-diffusion model for the average PDs in each rating, and 28 use a regime-switching model. We see that the values of the q metric are consistently smaller for the regime-switching model using all test statistics. This implies that the latter model provides a better representation of the stochastic properties amongst the issuer hazard rates. In particular, it fits the asymmetric correlations better.* We conclude that, for rating-level PDs, a regime-switching model performs better than a jump-diffusion model, irrespective of the choice of issuer process and copula.†

Tables 19.9 and 19.10 allow us to compare the fit provided by the four copulas. We find that when the jump-diffusion model is used, the Student-*t* copula performs best, no matter which marginal distribution is used. However, when the regime-switching model is used, sometimes the normal copula works best and at other times, the Clayton copula is better. The normal copula works best when the marginals are all normal or Student-*t*, and when the marginals are chosen using the Anderson–Darling criterion. The Clayton copula is best when the marginals are all skewed double-exponential or when the marginal criteria are Kolmogorov or L^2. The Student-*t* copula is best when the L^1 criterion is chosen.

* This result is consistent with that of Ang and Chen (2002), who undertake a similar exercise with equity returns.

† The fact that the Student-*t* marginal distribution works best for the jump models may indicate that the jump-model does not capture the dependences as well as the regime-switching model does, since the Student-*t* distribution is good at injecting excess kurtosis into the conditional distribution of the marginals.

TABLE 19.9 Metric for Best Correlated Default Model (Normal and Gumbel Copulas)

	Gaussian Copula		Gumbel Copula	
Model	**Jump-Diffusion**	**Regime-Shifting**	**Jump-Diffusion**	**Regime-Shifting**
Normal	0.0508	0.0276	0.0486	0.0462
Student-*t*	0.0375	0.0327	0.0416	0.0515
Double-exponential	0.0752	0.0355	0.0700	0.0381
Kolmogorov	0.0802	0.0387	0.0743	0.0364
Anderson–Darling	0.0447	0.0315	0.0452	0.0450
L^2	0.0764	0.0367	0.0710	0.0376
L^1	0.0667	0.0318	0.0625	0.0401

Note: This table presents the summary statistic for the asymmetric correlation metric to determine the best simulation model. For each model we generate the asymmetric correlation plot and then compute the distance metric. The asymmetric correlations (the metric q) are computed for the following seven cases: normal distribution, Student-*t* distribution, skewed double-exponential distribution, the combination of the best distributions based on Kolmogorov criterion, the combination of the best distributions based on the Anderson and Darling statistic, the combination based on the L^1 and L^2 norms. We report the results for both models, the jump-diffusion setup and the regime-switching one, and two copulas, the Gaussian and Gumbel Copulas.

These results bear some explanation. A comparison of the correlations from the skewed double-exponential distribution with those from the raw data reveals that the model overestimates the up-tail dependences a lot. The Clayton copula decreases the up-tail dependences and increases the low-tail dependences. As a result, it corrects some of the overestimation from the double-exponential marginals, resulting in better fitting. Hence, the Clayton copula seems to combine well with skewed double-exponential marginals. On the other hand, the

TABLE 19.10 Metric for Best Correlated Default Model (Clayton and Student-*t* Copulas)

	Clayton Copula		Student-*t* Copula	
Model	**Jump-Diffusion**	**Regime-Shifting**	**Jump-Diffusion**	**Regime-Shifting**
Normal	0.0484	0.0395	0.0446	0.0506
Student-*t*	0.0415	0.0473	0.0370	0.0564
Double-exponential	0.0694	0.0304	0.0672	0.0433
Kolmogorov	0.0738	0.0293	0.0714	0.0404
Anderson–Darling	0.0445	0.0397	0.0418	0.0514
L^2	0.0703	0.0296	0.0681	0.0423
L^1	0.0622	0.0324	0.0596	0.0452

Note: This table presents the summary statistic for the asymmetric correlation metric to determine the best simulation model. For each model we generate the asymmetric correlation plot and then compute the distance metric. The asymmetric correlations (the metric q) are computed for the following seven cases: normal distribution, Student-*t* distribution, skewed double-exponential distribution, the combination of the best distributions based on Kolmogorov criterion, the combination of the best distributions based on the Anderson and Darling statistic, the combination based on the L^1 and L^2 norms. We report the results for both models, the jump-diffusion setup and the regime-switching one, and two copulas, the Clayton and Student-*t* copulas.

q-metric favors the normal copula if the criterion chooses normal marginals (as with the normal, Student-t and L^1 criteria), since less balancing of tail dependences is required.

19.5.5 Empirical Implications of Different Copulas

The degree of tail dependence varies with the choice of copula. It is ultimately an empirical question as to whether the parameterized joint distribution does result in differing tail risk in credit portfolios. To explore this question, we simulated defaults using a portfolio comprising all the issuers in this study, under the regime-switching model. Remember that we fitted four copulas, and seven different choices of marginal distributions, resulting in 28 different models, each with its attendant parameter set. To compare copulas, we fix the marginal distribution, and then vary the copula.

As an illustration, we present Figure 19.5. This plots the tail loss distributions from the four copulas when all marginals are assumed to be normal. The line most to the right (the Gumbel copula) has the most tail dependence. There is roughly a 90% chance that the number of defaults will be less than 75, i.e. a 10% probability that the number of defaults will exceed 75. For the same level of 75 defaults, the leftmost line (from the normal copula), there is only a 7% probability of the number of losses being greater than 75. By examining all four panels of the figure, we see that the ranking of copulas by tail dependence is

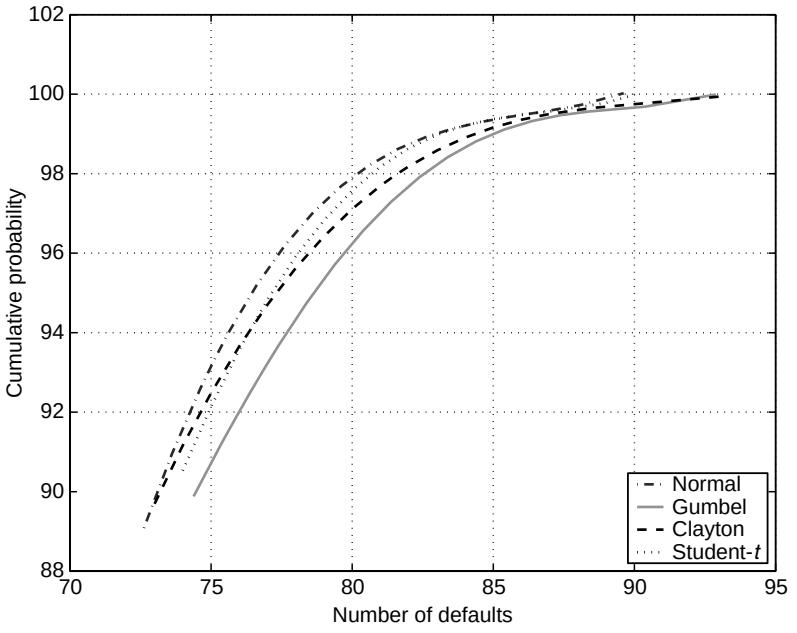


FIGURE 19.5 Comparing copula tail loss distributions. This figure presents plots of the tail loss distributions for four copulas, when the marginal distribution is normal. The x-axis shows the number of losses out of more than 600 issuers, and the y-axis depicts the percentiles of the loss distribution. The simulation runs over a horizon of 5 years and accounts for regime shifts as well. The copulas used are: normal, Gumbel, Clayton, Student-t.

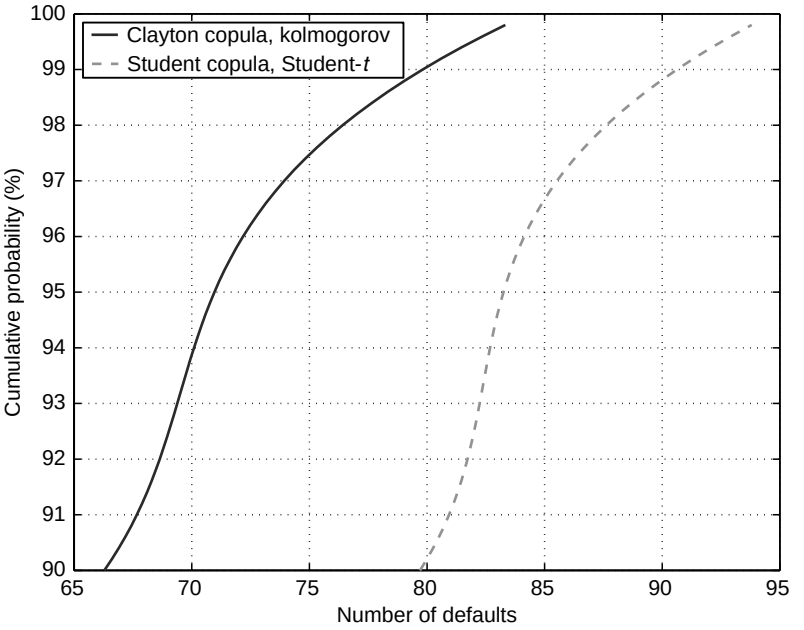


FIGURE 19.6 Comparing copula tail loss distributions in this figure we plot the tail loss distributions for two models, the best fitting one and the worst. The best fit model combines the Clayton copula, and marginal distributions based on the Kolmogorov criterion. The worst fit copula combines the Student-*t* copula with Student-*t* marginals. The simulation runs over a horizon of 5 years and accounts for regime shifts as well.

unaffected by the choice of marginal distribution, i.e., the normal copula is the leftmost plot, followed by the Student-*t*, Clayton, and Gumbel copulas.

In Figure 19.6 we plot the tail loss distributions for two models, the best fitting one and the worst. The best fit model combines the Clayton copula, and marginal distributions based on the Kolmogorov criterion. The worst fit copula combines the Student-*t* copula with Student-*t* marginals. A comparison of the two models shows that the worst fitted copula in fact grossly overstates the extent of tail loss. Therefore, while there is tail dependence in the data, careful choice of copula and marginals is needed to avoid either under- or overestimation of tail dependence.

19.6 DISCUSSION

This chapter develops a criterion-based methodology to assess alternative specifications of the joint distribution of default risk across hundreds of issuers in the U.S. corporate market. The study is based on a data set of default probabilities supplied by Moody's Risk Management Services. We undertake an empirical examination of the joint stochastic process of default risk over the period 1987–2000. Using copula functions, we separate the estimation of the marginal distributions from the estimation of the joint distribution.

Using a two-step Monte Carlo model, we determine the appropriate choice of multivariate distribution based on a new metric for the assessment of joint distributions.

We explored 56 different specifications for the joint distribution of default intensities. Our methodology uses two alternative specifications (jumps and regimes) for the means of default rates in rating classes. We consider three marginal distributions for individual issuer hazard rates, combined using four different copulas. An important extension to this model structure is the inclusion of rating changes. In our analysis, we centered firms as lying within the same rating for the period of the simulation, based on their most prevalent rating. The two-step simulation model would need to be enhanced to a three-step one, with an additional step for changes in ratings.*

Other than the myriad specifications, there are many useful features of the analysis for modelers of portfolio credit risk. First, we developed a simple metric to measure best fit of the joint default process. This metric accounts for different aspects of default correlation, namely level, asymmetry, and tail dependence or extreme behavior. Second, the simulation model, based on estimating the joint system of over 600 issuers, is able to replicate the empirical joint distribution of default. Third, a comparison of the jump model and the regime-switching model shows that the latter provides a better representation of the properties of correlated default. Fourth, the skewed double-exponential distribution is a suitable choice for the marginal distribution of each issuer hazard rate process, and combines well with the Clayton copula in the joint dependence relationship amongst issuers. Our simulation approach is fast and robust, allowing for rapid generation of scenarios to assess risk in credit portfolios. Finally, the results show that it is important to correctly capture the interdependence of marginal distributions and copula to achieve the best joint distribution depicting correlated default. Thus, this chapter delivers the empirical counterpart to the body of theoretical chapters advocating the usage of copulas in modeling correlated default.

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* We are grateful to Darrell Duffie for this suggestion.

APPENDIX: SKEWED DOUBLE-EXPONENTIAL DISTRIBUTION

Assume that a random variable X has a normal distribution with mean $\mu + \gamma V$, and variance V, where V has an exponential distribution with the following density function:

$$\text{pdf}(V) = \frac{1}{V_0} \exp\left(-\frac{V}{V_0}\right) \tag{19.15}$$

Then, X has a skewed double-exponential distribution.

A.1 Density Function

The density function is derived as follows:

$$\begin{aligned}
\text{pdf}(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi V}} \exp\left\{-\frac{(x-\mu-\gamma V)^2}{2V}\right\} \times \frac{1}{V_0} \exp\left(-\frac{V}{V_0}\right) \mathrm{d}V \\
&= \frac{1}{\sqrt{2\pi} V_0} \exp\{\gamma(x-\mu)\} \int_0^\infty \frac{1}{\sqrt{V}} \times \exp\left\{-\frac{(x-\mu)^2}{2V} - \frac{\gamma^2 V_0 + 2}{2V_0} V\right\} \mathrm{d}V \\
&= \frac{1}{\sqrt{2\pi} V_0} \exp\{\gamma(x-\mu)\} \sqrt{\frac{2\pi V_0}{2+\gamma^2 V_0}} \times \exp\left\{-|x-\mu| \sqrt{\frac{2+\gamma^2 V_0}{V_0}}\right\} \\
&= \frac{1}{V_0} \sqrt{\frac{V_0}{2+\gamma^2 V_0}} \times \exp\left\{-|x-\mu| \sqrt{\frac{2+\gamma^2 V_0}{V_0}} + \gamma(x-\mu)\right\} \\
&= \frac{\lambda}{V_0} \exp\left\{-\frac{|x-\mu|}{\lambda} + \gamma(x-\mu)\right\}
\end{aligned}$$

A.2 Maximizing the Log-Likelihood Function

This subsection deals with the technical details of the maximization of log-likelihoods for the skewed double-exponential model. The likelihood function is

$$L = -\frac{n}{2}\{\log(V_0) + \log(2+\gamma^2 V_0)\} - \sum_{i=1}^{n} |x_i - \mu| \sqrt{2+\gamma^2 V_0/V_0} + \gamma \sum_{i=1}^{n} (x_i - \mu)$$

where m is the number of observations where x_i is greater than μ, l is the number of observations otherwise, and $n = m + l$.

The first-order condition can be derived as follows:

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}\mu} &= \sum_{i=1}^{n} \frac{\mathrm{d}|x_i-\mu|}{\mathrm{d}\mu}\left(\sqrt{\frac{2+\gamma^2 V_0}{V_0}}\right) + \gamma\sum_{i=1}^{n}(-1) \\
&= -n\gamma - \sqrt{\frac{2+\gamma^2 V_0}{V_0}}\sum_{i=1}^{n}\frac{\mathrm{d}|x_i-\mu|}{\mathrm{d}\mu} = 0 \\
\frac{\mathrm{d}L}{\mathrm{d}\gamma} &= -\frac{n}{2}\left(\frac{2\gamma V_0}{2+\gamma^2 V_0}\right) - \sum_{i=1}^{n}|x_i-\mu|\left(\frac{\gamma}{\sqrt{(2+\gamma^2 V_0)/V_0}}\right) + \sum_{i=1}^{n}(x_i-\mu) \\
&= -\frac{n\gamma V_0}{2+\gamma^2 V_0} - \sum_{i=1}^{n}\gamma|x_i-\mu|\sqrt{\frac{V_0}{2+\gamma^2 V_0}} + \sum_{i=1}^{n}(x_i-\mu) = 0 \\
\frac{\mathrm{d}L}{\mathrm{d}V_0} &= -\frac{n}{2}\left(\frac{1}{V_0}+\frac{\gamma^2}{2+\gamma^2 V_0}\right) - \sum_{i=1}^{n}|x_i-\mu|\left(\frac{(\gamma^2 V_0-2-\gamma^2 V_0)/V_0^2}{2\sqrt{(2+\gamma^2 V_0)/V_0}}\right) \\
&= -\frac{n(1+\gamma^2 V_0)}{V_0(2+\gamma^2 V_0)} + \sum_{i=1}^{n}\frac{|x_i-\mu|}{V_0\sqrt{V_0(2+\gamma^2 V_0)}} = 0
\end{aligned}$$

Solving the above first-order conditions gives

$$\mu = \frac{m^2\sum_{x\in A1}x_i + l^2\sum_{x\in A2}x_i}{nml}$$

$$V_0 = \frac{2ml\left(\frac{1}{n}\sum_{i=1}^{n}x_i-\mu\right)^2}{(l-m)^2}$$

$$\gamma = \frac{(l-m)^2}{2ml\left(\frac{1}{n}\sum_{i=1}^{n}x_i-\mu\right)}$$

where

$A1 = \{x : x_i - \mu \geq 0\}$
$A2 = \{x : x_i - \mu < 0\}$

Given a dataset, we may not be able to find a value of μ to satisfy the above equations. However, by choosing μ to be as close as given by the first-order conditions, we can fit the data well with the skewed double-exponential distribution.

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CHAPTER 20

Systematic Credit Risk: CDX Index Correlation and Extreme Dependence

Sofiane Aboura and Niklas Wagner

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20.1 INTRODUCTION

Dependence, including correlation, is an important issue in credit risk portfolio modeling and pricing. Credit portfolio derivatives such as collateralized debt obligations and index tranches cannot be priced without a precise characterization of dependence, which allows for a derivation of portfolio loss distributions. Among the dependence approaches to credit risk that are documented in the literature,* copula and factor approaches are two dominant concepts. First, the statistical copula approach as suggested for credit risk dependence modeling for example by Li (1999) allows for general representations of dependence, offering a highly flexible method, while the choice of a suitable copula function often seems crucial, see also e.g., Bluhm and Overbeck (2007). An empirical study on copula functions in modeling default dependence is by Das and Geng (2004). Second, the statistical factor approach as applied in credit risk modeling, for example, by Vasicek (1987) is a simplified dependence approach that allows for parsimonious and robust implementations, while it may suffer from oversimplification. Although several authors argue that factor approaches may not be able to match observed dependence, Yu (2003) shows that factor approaches that take common risk factors of firm-specific credit risks into account may not be subject to this shortcoming.

In this chapter we address systematic credit risk, which is probably the most striking manifestation of credit risk dependence. Systematic credit risk was previously studied in a different setting by Pedrosa and Roll (1998). We discuss a straightforward common factor model of credit risk dependence. The model follows from the class of intensity models of credit risk, which model dependence in the driving intensity process. This is motivated in part by factor approaches as in Duffie and Singleton (1998), Yu (2003), and van der Voort (2004), for example.

A number of previous studies dealt with the phenomenon of dependent default events in credit portfolios. To this aim, structural credit risk models typically model dependence in latent underlying asset values (often proxied by equity values) of the obligors. While Lucas (1995) uses historical data whose availability is limited, Zhou (2001) analyses dependence in a bivariate structural Merton model. De Servigny and Renault (2002) study both equity-based and historical-default correlations. They conclude that there does not seem to be an obvious link between the two. Das, Freed, Geng, and Kapadia (2006) point out that in the Merton model, correlation and time variation in single firm equity values and equity volatilities will cause variation in default probabilities. Accordingly, they find that joint default probabilities are related to varying debt ratios, but are especially driven by varying correlated volatilities. Das, Duffie, Kapadia, and Saita (2007) provide evidence that the majority of joint default risk is due to covariation in firm-specific default probabilities. As their results suggest, observable macroeconomic

* For overviews on default dependence modeling see, for example, Duffie and Singleton (2003), Chapter 10, and Schlögl (2002). Using market variables, we may distinguish (1) rating and default correlations, (2) spread correlations, (3) asset or equity correlations, and (4) implied correlations. Since default events are rare events by definition, historical data are sparse. Therefore, latent credit risk dependence modeling approaches dominate.

variables (including e.g., a business cycle variable) and particularly latent common variables play a major role in explaining correlated defaults.

We empirically study credit risk dependence under the risk-neutral measure, i.e., we consider derivatives prices and study their dependence implications. As pointed out in the preface to this volume, the market for credit default swap (CDS) contracts has grown enormously during the last couple of years. Hence, data for a study of credit risk dependence have recently become available through the derivatives markets. A study of credit risk dependence under the risk-neutral measure appears relevant as it reveals how credit risk is priced and traded in the market and it also forms the building block for the risk-neutral distribution of credit portfolios, which serve as an underlying for other derivatives.

We study the dependence between index and component spread changes in our common factor model using liquid quotes from the U.S. CDS market during the years 2004–2006. In the first step, we argue that CDS spread changes show distinct time-series dynamics, which should be taken into account in advance of studying dependence. As such, we then consider factor dependence of unpredictable spread changes. We find that the CDX factor is significant for the chosen sample of large-cap U.S. obligors but has low explanatory power. Additionally, we shed light on a heavily time-varying nature of factor sensitivities. Various intensity models of joint defaults (see Section 20.2.2.4) predict positive jumps—i.e., large positive changes—in credit spreads conditional on the default of one or several obligors. In our setting, this translates into upper tail extreme factor dependence, a feature which is observable for most of our obligors.

The remainder of this chapter is organized as follows. Section 20.2 presents our methodology, which includes the derivation of the factor model, its implications, and the methods used for inference. Section 20.3 outlines our empirical findings. The contribution ends with a brief conclusion in Section 20.4.

20.2 METHODOLOGY

Starting with the intensity-based (or reduced-form) approach to credit risk modeling, this section outlines a straightforward empirical model specification of common factor systematic credit risk under a risk-neutral measure. For a comprehensive summary of the model background and further literature see, for example, Duffie and Singleton (2003).

20.2.1 Factor Intensity Approach

To model default probabilities under risk-neutrality, where $\mathbb{Q}$ denotes the equivalent martingale measure, intensity models introduce a latent intensity process $(\lambda_{i,t})_{0\le t\le T}$ of the ith obligor. Under standard assumptions, it follows that the conditional probability of default between time t and T is given by

$$p^{\mathbb{Q}}_{i,t,T} = 1 - \mathbb{E}_t \exp\left(-\int_t^T \lambda_{i,s}\, \mathrm{d}s \right) \tag{20.1}$$

A typical simplifying assumption is that the risk-neutral expected loss given default rate, loss given default in short, is constant and denoted by l_i. Then, the credit spread is simply given as

$$S_{i,t} = l_i \lambda_{i,t} \tag{20.2}$$

To introduce credit risk dependence of the obligors, we apply a common factor specification. A one-factor intensity model of credit risk dependence, in parts related, e.g., to Duffie and Singleton (1998), Yu (2003), and van der Voort (2004), is given by

$$\lambda_{i,t} = \lambda_{F,t} + \lambda_{i,t}^{\varepsilon} \tag{20.3}$$

where $\lambda_{F,t}$ and $\lambda_{i,t}^{\varepsilon}$ are two independent intensity processes. By assigning any time-variation in the spread (Equation 20.2) to the time-varying intensity $\lambda_{i,t}$ only,* a simple model of the credit spread dynamics follows. Fixing $S_{i,0}$ and $(\lambda_{F,0}, \lambda_{i,0})$, we can define the spread process $(S_{i,t})_{0 \le t \le T}$ via the equation

$$dS_{i,t} = l_i d\lambda_{F,t} + l_i \, d\lambda_{i,t}^{\varepsilon} \tag{20.4}$$

Fixing also $S_{F,0}$, we obtain for the factor spread, $dS_{F,t} = l_F \, d\lambda_{F,t}$, and therefore

$$dS_{i,t} = l_i / l_F \; dS_{F,t} + l_i \, d\lambda_{i,t}^{\varepsilon} \tag{20.5}$$

We know from a broad body of empirical literature that observable spread changes are driven by various additional factors that are not related to default risk, (Collin-Dufresne et al. 2001). However, Equation 20.5 forms a fundamental model, which (1) shows how spread changes are driven by changes in intensity, (2) models default dependence via a common factor intensity $\lambda_{F,t}$, and, finally, (3) can easily be extended by taking other credit spread determinants into account.

20.2.2 Factor Model Analysis

In a discrete-time empirical setting, Equation 20.5 translates into a regression model of the form

$$\Delta S_{i,t} = a_i + b_i \Delta S_{F,t} + \varepsilon_{i,t}, \quad t = 1, \ldots, T \tag{20.6}$$

where the usual assumptions apply and the innovations are independent with identical distribution, $\varepsilon_{i,t} \sim (0; \sigma_i^2)$.

In this setting, we see from Equations 20.5 and 20.6 that systematic credit risk as measured via b_i relates to individual and factor loss given default l_i and l_F, with $b_i = l_i / l_F$. Hence, systematic credit risk is related to the ith obligor's relative expected loss given default. There are various econometric approaches whose application can be motivated by the simple model in Equation 20.6. Of course, a straightforward application is an ordinary least squares estimation of the model. In the following, we discuss how we use the model as a starting point for our analysis of cross-sectional credit risk dependence.

Following Equation 20.6, our focus is on the dependence between changes in the factor and the single obligor spreads. However, univariate analysis is a necessary and important

* Here, this assumption can be made without loss of generality since the intensity process is latent.

step in going beyond a standard regression application and also necessary for a cross-sectional dependence study. Our following analysis considers four areas:

- Univariate time-series dependence in spread changes
- Univariate extreme behavior of spread changes
- Time-varying factor sensitivity of spread changes
- Extreme factor dependence of spread changes

20.2.2.1 Time-Series Dependence

As univariate spread changes, ΔS_t, are driven by various factors including liquidity effects,* it appears promising to account for liquidity effects at least within a linear time-series model. Additionally, there is evidence that nonlinear effects of volatility clustering are observable in spread changes as well (Wagner et al. 2005). To allow for lagged responses to previous innovations, we may assume a model of the ARMA(1,1)-form

$$\Delta S_t = \mu + \varphi \Delta S_{t-1} + \phi U_{t-1} + U_t, \quad t = 1, \ldots, T \tag{20.7}$$

Furthermore, the unpredictable spread changes, U_t, may have a time-varying conditional variance as given by a GARCH(1,1)-specification

$$\begin{aligned} U_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega_0 + \omega_1 U_{t-1}^2 + \omega_2 \sigma_{t-1}^2, \quad \omega_0 > 0,\ \omega_1,\ \omega_2 \geq 0 \end{aligned} \tag{20.8}$$

which is based on given start random variables (σ_0^2, Z_0) (Bollerslev et al. 1992). The Z_t's are standardized iid random variables with common distribution function F_Z, where $Z_t \sim (0;\ 1)$. They represent standardized unpredictable spread changes.

20.2.2.2 Extreme Behavior

To characterize the extreme behavior of the univariate spread changes ΔS_t, we may assume that the distribution function F_Z of the innovations Z_t in Equation 20.8 is fat-tailed. Extreme value theory (EVT) (e.g., Embrechts et al. 1997 or Coles 2001) shows that this is equivalent to assuming that F_Z has Pareto-like upper and lower tail

$$\begin{aligned} 1 - F_Z(z) &\sim c_{\mathrm{U}} z^{-1/\xi_U}, \quad c_{\mathrm{U}} > 0, \quad \xi_{\mathrm{U}} > 0, \\ F_Z(-z) &\sim c_{\mathrm{L}} z^{-1/\xi_L}, \quad c_{\mathrm{L}} > 0, \quad \xi_{\mathrm{L}} > 0, \quad \text{as} \quad z \to \infty \end{aligned} \tag{20.9}$$

The parameters ξ_{U} and ξ_{L} denote the tail index of the upper tail and the lower tail, respectively.

We specify Pareto tail approximations for excesses of a sufficiently high positive threshold, for example, via the generalized Pareto distribution (GPD). Choosing two

* Note that liquidity effects typically relate to liquidity differentials between the treasury bond, the corporate bond, and the credit derivatives market.

thresholds in our case, namely $\upsilon_U > 0$ and $\upsilon_L > 0$, we specify GPDs for a random number of positive excesses of the upper threshold, $Y_U = Z - \upsilon_U > 0$, and the lower threshold, $Y_L = -Z - \upsilon_L > 0$, respectively:

$$1 - F_Z(z - \upsilon_U) = \left(1 + \xi_U \frac{z - \upsilon_U}{\beta_U}\right)^{-1/\xi_U}, \quad \beta_U > 0 \tag{20.10}$$

$$F_Z(-z - \upsilon_L) = \left(1 - \xi_L \frac{z - \upsilon_L}{\beta_L}\right)^{-1/\xi_L}, \quad \beta_L > 0 \tag{20.11}$$

A characterization of the extreme behavior of spread changes allows for the modeling of so-called gapping risk (or jump risk) in credit spreads. This denotes the risk of large sudden spread changes, for example, a sudden widening in spreads of more than 20 basis points for an index or more than 100 basis points for a single obligor (Duffie and Singleton 2003, p. 138, Bluhm and Overbeck 2007, p. 301). Previous studies that apply EVT in credit risk modeling include Phoa (1999), Campbell and Huisman (2002), and Lucas, Klaassen, Spreij, and Straetmans (2001).

20.2.2.3 Time-Varying Factor Sensitivity

Our model approach in Equation 20.6 assumes for simplicity that systematic credit risk, $b_i = l_i / l_F$, is constant. One extension to the model, which may prove suitable in modeling credit risk, is to allow for a time-varying factor sensitivity, $b_{i,t}$ of the unpredictable spread changes $U_{t,i}$.

Allowing again for time-varying marginal spread change volatilities, $\sigma_{i,t}$ and $\sigma_{F,t}$, as in Equation 20.8, multivariate autoregressive conditional heteroskedasticity offers a convenient model approach. We choose the diagonal BEKK-specification by Engle and Kroner (1995). In a bivariate setting, we can model individual and factor spread changes simultaneously, where a generalization of the standard relation for the correlation coefficient holds, yielding

$$b_{i,t} = \rho_{i,F,t} \frac{\sigma_{i,t}}{\sigma_{F,t}} \tag{20.12}$$

Hence, the time-varying conditional factor sensitivity $b_{i,t}$ is defined by the time-varying conditional correlation as well as the conditional spread change volatilities. Supporting the above specification, the empirical credit risk literature finds evidence of time-varying credit risk correlations. See, for example, the studies by Zhou (2001), de Servigny and Renault (2002), and Das, Freed, Geng, and Kapadia (2006).

20.2.2.4 Extreme Factor Dependence

To characterize the joint extreme behavior of the spread changes $\Delta S_{t,i}$ and $\Delta S_{t,F}$, we may assume that the extreme behavior of the joint distribution function F of the innovations $Z_{t,i}$ and $Z_{t,F}$ can be characterized via the Pickands EVT-copula approach (Coles 2001).*

* For applications of the Pickands EVT-copula approach in finance see, e.g., Longin and Solnik (2001) and Marsh and Wagner (2000).

The Pickands representation theorem shows that a potential limiting distribution H, satisfying the required max-stability condition, has to be of the form

$$H(z_i,z_F) = \exp\left[-\left(\frac{1}{z_i}+\frac{1}{z_F}\right)A\left(\frac{z_i}{z_i+z_F}\right)\right], \quad z_i > \upsilon_{U,i}, \quad z_F > \upsilon_{U,F} \tag{20.13}$$

Here, H has a unique EVT-copula, which is defined via the dependence function $A(w)$: $[0,1] \mapsto [0,1]$, where $w = z_i/(z_i + z_F)$. $A(w)$ characterizes bivariate extreme dependence. If $A(w) = 1$, the tails of the joint distribution are independent, whereas $A(w) = \max(w, 1 - w)$ indicates perfect dependence. We choose three different parametric models for $A(w)$. A standard model is the symmetric logistic model, which relates to the Gumbel copula, where

$$A_\alpha(w) = [(1-w)^{1/\alpha} + w^{1/\alpha}]^\alpha \tag{20.14}$$

with $0 < \alpha \leq 1$. Here, independence is reached when $\alpha = 1$ and perfect dependence when $\alpha \to 0$. A generalization of the Gumbel copula allows for asymmetric dependence with

$$A_{\alpha,\theta_1,\theta_2}(w) = (1-\theta_1)(1-w) + (1-\theta_2)w + \left[(1-w)^{1/\alpha}\theta_1^{1/\alpha} + w^{1/\alpha}\theta_2^{1/\alpha}\right]^\alpha \tag{20.15}$$

where $0 < \alpha \leq 1$ and $\theta_1 \geq 0$, $\theta_2 \leq 1$. Under this asymmetric model, symmetry is obtained under $\theta_1 = \theta_2 = 1$ and independence is reached when $\alpha = 1$, $\theta_1 = 0$, or $\theta_2 = 0$. The asymmetric negative logistic model defines the Galambos–Joe copula and is given by

$$A_{\alpha,\theta_1,\theta_2}(w) = 1 - \left[\left(\frac{1-w}{\theta}\right)^{-\alpha} + (w/\theta_2)^{-\alpha}\right]^{-1/\alpha} \tag{20.16}$$

with $\alpha > 0$ and $\theta_1 > 0, \theta_2 \leq 1$. Independence is reached when either $\alpha \to 0$ or $\theta_1 \to 0, \theta_2 \to 0$.

For each of the above candidate dependence functions, it is possible to derive the limiting conditional exceedance probabilities

$$\chi_U = \lim_{\upsilon_U \to \infty} P(Z_i > \upsilon_U | Z_F > \upsilon_U)$$
$$\chi_L = \lim_{\upsilon_L \to \infty} P(-Z_i > \upsilon_L | -Z_F > \upsilon_U)$$

These limiting conditional probabilities, χ_U and χ_L, indicate the presence of upper and lower factor tail dependence, respectively. For example, we may interpret χ_U as the probability of a significant jump in the spread of the ith obligor given that the factor spread shows a significant jump, where both jump sizes exceed a very high threshold.

On the theory side, various models, including Davis and Lo (2001), Jarrow and Yu (2001), Schönbucher and Schubert (2000), and Duffie, Eckner, Horel, and Saita (2006), predict that the default of one obligor may cause other obligors' spreads to jump. Empirical evidence supporting this behavior is by Collin-Dufresne, Goldstein, and Helwege (2003) and Zhang (2004), for example. The risk of individual spreads widening sharply in a state where spreads generally increase results in the presence of spread jump dependence

between our common factor and single spreads. For increasingly large jumps, this would predict upper factor tail dependence in our setting.

20.3 EMPIRICAL ANALYSIS

20.3.1 Credit Default Swap Data Set

We study the common factor model of Section 20.2 based on a sample of CDS data. The data come from the Open Bloomberg system. The CDS obligors that we pick are large-cap U.S. corporations taken from the sixth revision of the Dow Jones CDX Investment Grade (DJ CDX.NA.IG) index. The latter index universe consists of 125 U.S. corporations where CDSs are actively quoted. Hence, our sample assures a high level of market liquidity.

Our sample contains 625 CDS quotes for the index and for each corporation during the period January 9, 2004 to July 7, 2006. The quoted CDX index spread serves as a proxy for our common factor (CDX). The 11 individual corporations in our sample are Altria Group Inc. (MO), American International Group Inc. (AIG), Boeing Co. (BA), Caterpillar Inc. (CAT), Dow Chemical Co. (DOW), Honeywell International Inc. (HON), Hewlett–Packard Co. (HPQ), International Business Machines Corp. (IBM), Mariott International Inc. (MAR), Motorola Inc. (MOT), and Walt Disney Co. (DIS). The CDS quotes are averages between quoted bid and ask prices as actual transaction prices are not publicly available. All quote series with the exception of BA and HPQ were obtained as complete series. In the case of BA and HPQ, there were two missing quotes, respectively. These missing data were linearly interpolated.

Spread changes are modeled as relative spread changes, $\Delta S_t = \ln CDS_t - \ln CDS_{t-1}$, due to resulting unbounded support and other statistical properties (see also, e.g., a note in Wagner et al. 2005). A preliminary analysis of the spread change series gives summary statistics results as presented in Table 20.1. Mean values indicate that spreads were decreasing on average in the sample period. Spread change standard deviations are in a

TABLE 20.1 Summary Statistics for Daily Logarithmic Credit Default Swap Spread Changes ΔS_t

Name	Mean	Std	Skew	Kurt	$\text{Corr}_{t;t-1}$	$\text{Corr}_{t;t-2}$
CDX	0.000	0.023	0.402	10.124	−0.018	−0.011
AIG	0.000	0.034	3.360	42.899	0.217	0.014
BA	−0.002	0.030	−0.304	5.817	−0.190	0.111
CAT	0.000	0.028	0.381	6.867	−0.194	−0.024
DIS	−0.001	0.032	0.044	16.606	−0.111	0.160
DOW	−0.001	0.030	0.455	8.995	0.022	0.057
HON	−0.001	0.028	0.992	11.002	−0.150	−0.029
HPQ	0.000	0.032	0.158	12.965	−0.196	0.119
IBM	−0.001	0.027	2.277	26.671	−0.162	0.072
MAR	0.000	0.021	0.684	10.031	−0.084	0.078
MO	−0.002	0.027	1.206	21.594	0.021	0.076
MOT	−0.001	0.025	−0.429	11.846	0.134	0.123

range between 0.021 for MAR and 0.034 for AIG, with 0.023 for the CDX index, indicating a remarkable annualized index volatility of 36.4%. Generally speaking, distributions tend to be positively skewed with excess kurtosis and the spread change observations show negative autocorrelation.

20.3.2 Results

The findings as summarized in Table 20.1 indicate that standard approaches to CDS spread-risk and factor-dependence modeling are prone to failure and that a more detailed time-series, tail, and tail dependence analysis should apply. The presentation of our estimation results refers to the methods of Section 20.2.2.

In the first step, we estimate the time-series model of Section 20.2.2.1 for our 11 obligors and the CDX via the standard maximum-likelihood (ML) method. It can be noted that all GARCH parameter estimates (ω_1,ω_2) turn out to be significant at the 99% level for all series. At least one of the ARMA parameter estimates (φ,ϕ) is significant at the 90% level for all but CAT, MO, and CDX index spread changes.* This finding generally underlines the distinct time-series properties of CDS spread changes. We study unpredictable spread changes in the sequel.

In the second step, we postpone extreme behavior and consider the factor model (Equation 20.6). We estimate least squares constant factor sensitivities and ML-BEKK time-varying factor sensitivities as in Section 20.2.2.3. A summary of our results is given in Table 20.2.

As we can see from Table 20.2, CDS spread changes are significantly related to CDX index spread changes with all coefficient estimates positive and significant at the 95% level. As our sample consists of high-quality investment grade obligors, the coefficients'

TABLE 20.2 Point Estimates of Factor Sensitivities b_i Together with t-Values and Coefficients of Determination, Statistics on the BEKK Time-Varying Factor Sensitivities $b_{i,t}$

Name	b_i	t-Value	R^2	$\min(b_{i,t})$	$\max(b_{i,t})$	$\text{Std}(b_{i,t})$
AIG	0.121	3.05	0.015	−0.066	0.204	0.019
BA	0.113	2.84	0.013	−0.102	0.188	0.025
CAT	0.146	3.67	0.021	0.112	0.262	0.010
DIS	0.205	5.21	0.042	−0.134	0.707	0.068
DOW	0.159	4.03	0.025	−0.109	0.252	0.033
HON	0.088	2.19	0.008	0.061	0.084	0.003
HPQ	0.135	3.41	0.018	−0.098	0.138	0.013
IBM	0.082	2.04	0.007	−0.108	0.079	0.013
MAR	0.133	3.33	0.018	−0.152	0.251	0.037
MO	0.115	2.89	0.013	0.109	0.117	0.001
MOT	0.144	3.61	0.020	−0.048	0.323	0.024

* We therefore leave the results unreported and deliver them upon request.

absolute values are relatively low and in a range between 0.082 (for IBM) and 0.205 (for DIS). Still, we have to attest that the results document weak overall explanatory power of our simple one-factor model: R-squared statistics are in a range between 0.7% (for IBM) and 4.2% (for DIS). This finding illustrates that the model lacks additional covariates that explain default risk dependence (Das et al. 2007). To some extent, the finding may also be explainable by a nonlinear impact of the factor on firm-specific credit risk: while common factor changes do hardly affect single obligor spreads most of the time, they become a driving force in some states of the economy. Time-varying factor sensitivities and the analysis of extreme dependence can shed more light on such a scenario.

Table 20.2 also reports results on the ML-BEKK time-varying factor sensitivities. The results show that factor sensitivities remarkably fluctuate for most of our sample corporations, including DIS, MAR, DOW, BA, and MOT. In contrast, MO and HON show stable sensitivities. Figure 20.1 plots time-varying factor sensitivities for two of our sample companies, namely AIG and DIS. As can be seen in the plot, factor sensitivities heavily fluctuate around their mean values. As shown by the results in Table 20.2, DIS has a higher average factor sensitivity than AIG. For short periods of time, DIS sensitivity regularly jumps to values as high as 0.35–0.70 while AIG sensitivity always remains below 0.21.

In the third step, we consider extreme spread change behavior. We use ARMA-GARCH standardized residuals and then perform a simultaneous ML-estimation of the bivariate GPD model as described in Sections 20.2.2.2 and 20.2.2.4. This method yields the marginal parameter estimates (see Section 20.2.2.2) as well as the dependence (or copula) parameter estimates (see Section 20.2.2.4) (cf. Coles 2001). On the basis of mean residual life plots (or mean excess function plots, see, e.g., Embrechts et al. 1997), we fix the marginal thresholds $\upsilon_U > 0$ and $\upsilon_L > 0$ for each series.

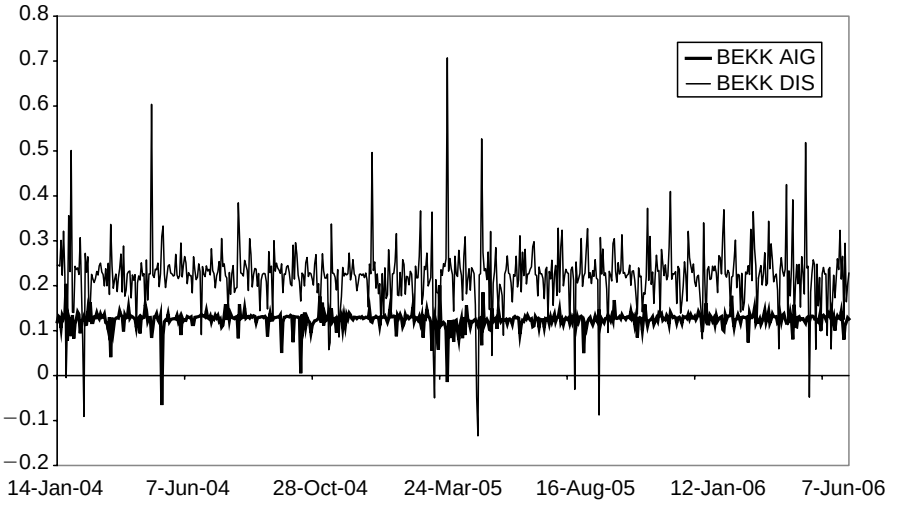


FIGURE 20.1 BEKK conditional CDX factor correlation for American International Group Inc. (AIG) and Walt Disney Co. (DIS).

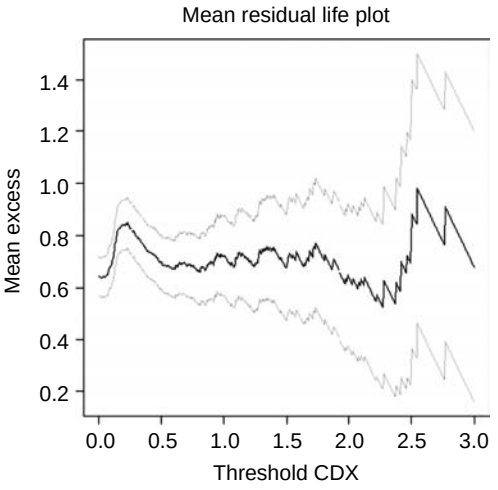


FIGURE 20.2 Mean excess function for upper tail CDX standardized unpredictable spread changes.

The GPD marginal parameter estimates yield estimated upper tail indices in the range of −0.081 (DOW) and 0.334 (MOT). Lower tail indices are between −0.233 (CAT) and 0.551 (DIS). These point estimate results do not obviously support fatter upper than lower tails. For the overall market, as represented by the CDX index, we document moderate tail behavior (i.e., neither fat tails nor thin tails) for the lower as well as upper tail. This is also documented by the mean residual life plots in Figures 20.2 and 20.3. Hence, our results

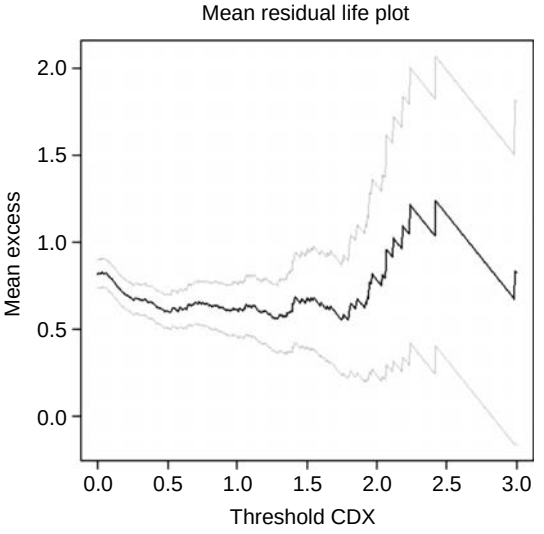


FIGURE 20.3 Mean excess function for lower tail CDX standardized unpredictable spread changes.

TABLE 20.3 Upper Tail ML-Estimation Results for the Bivariate GPD EVT-Copula Model

		Gumbel		Asymmetric Gumbel			Galambos		
Name	ν	α	χ%	α	θ_1	θ_2	α	θ_1	θ_2
CDX	1.00								
AIG	0.95	0.906[a]	12.6	0.911[a]	0.999[a]	0.748	0.489[a]	0.999[a]	0.313
BA	1.05	0.979[a]	2.9	0.999[a]	0.745	0.787	15.00[a]	0.0316	0.260
CAT	1.25	0.939[a]	8.3	0.956[a]	0.734	1.000[a]	7.25	0.0679	0.355
DIS	1.25	0.910[a]	12.1	0.908[a]	1.000[a]	0.741[a]	15.00[a]	0.192[a]	0.0730
DOW	1.00	0.939[a]	8.3	0.961[a]	0.999[a]	0.762	13.50	0.489[a]	0.0335
HON	1.25	0.999[a]	0.1	0.999[a]	0.775	0.987	0.200[a]	0.454	0.487
HPQ	1.50	0.945[a]	7.5	0.550	0.149	0.340	1.174	0.144	0.332
IBM	1.50	1.000[a]	0.1	0.999[a]	0.763	0.984	0.201[a]	0.454	0.631
MAR	1.50	0.911[a]	12.0	0.457	0.264[a]	0.161	1.496	0.254[a]	0.168
MO	0.85	0.961[a]	5.3	0.986[a]	0.734	1.000[a]	1.572	0.0607	0.679[a]
MOT	1.25	0.929[a]	9.6	0.947[a]	0.734	1.000[a]	0.790	0.177	0.414

[a] Parameter different from zero at the 95% level.

support the hypothesis that the GARCH model captures nonnormality in market spread changes. However, for some individual sample corporations, including AIG, DIS, HON, HPQ, IBM, and MOT, we find upper tail index point estimates above 0.20, i.e., extreme risk for upward (adverse) spread residuals.

Our copula parameter estimation results for the upper tail are given in Table 20.3 and those for the lower tail are given in Table 20.4. We additionally report the conditional asymptotic probabilities χ_U and χ_L for the Gumbel copula model 20.14.

The results from Table 20.3 allow for the following conclusions. With the exception of HON and IBM, where the estimates of the dependence parameter α approach the limiting

TABLE 20.4 Lower Tail ML-Estimation Results for the Bivariate GPD EVT-Copula Model

		Gumbel		Asymmetric Gumbel			Galambos		
Name	ν	α	χ%	α	θ_1	θ_2	α	θ_1	θ_2
CDX	1.50								
AIG	1.50	1.000[a]	0.0	1.000[a]	0.789	0.782	0.201[a]	0.561	0.553
BA	1.45	1.000[a]	0.0	0.999[a]	0.944	0.797	0.349	0.0169	1.000[a]
CAT	1.50	1.000[a]	0.0	0.999[a]	0.811	0.801	0.200[a]	0.595	0.606
DIS	1.95	0.964[a]	5.0	0.687[a]	0.053	0.999[a]	0.366	0.0985	1.000[a]
DOW	0.25	0.988[a]	1.7	0.962[a]	0.001	0.941	0.200[a]	0.0350	0.552
HON	1.50	0.969[a]	4.3	0.961[a]	0.712	1.000[a]	9.865	0.0572	0.119
HPQ	1.55	0.999[a]	0.1	0.999[a]	0.706	0.955	0.200[a]	0.610	0.705
IBM	1.00	0.999[a]	0.1	1.000[a]	0.826	0.883	0.200[a]	0.375	0.529
MAR	0.75	0.985[a]	2.1	0.989[a]	0.999[a]	0.728	0.285	0.999[a]	0.0937
MO	1.50	0.999[a]	0.1	1.000[a]	0.858	0.848	0.200[a]	0.569	0.565
MOT	1.50	0.999[a]	0.1	1.000[a]	0.887	0.801	0.201[a]	0.646	0.627

[a] Parameter different from zero at the 95% level.

value of 1, upper tail extreme dependence in CDS spread changes is present for the obligors. The corresponding conditional asymptotic probabilities χ_U can be interpreted as default probabilities given a major credit spread shock to the market: we see that AIG, DIS, and MAR carry the largest extreme systematic credit risk with χ_U estimated to be larger than 10%. Evidence of an asymmetric dependence structure within the upper tail is weak. Table 20.3 shows that the asymmetric Gumbel copula (Equation 20.15) yields θ-parameter estimates close or equal to one in most cases and mostly estimates of similar magnitude. Only the cases of HPQ and MO may indicate some asymmetry. The Galambos–Joe copula (Equation 20.16) estimation results principally confirm our findings from the Gumbel copula estimations while the results show remarkably less stability.

The results from Table 20.4 show that the dependence parameter α closely approaches the value of 1 for all of the obligors. Hence, lower tail dependence is negligible and estimated factor dependence in large spread decreases for DIS and HON is minor. We may generally assume that spread changes are asymptotically independent (as, e.g., under the bivariate normal model) and that the corresponding conditional asymptotic probabilities χ_L are zero. Again, evidence of an asymmetric dependence structure within the individual tails is not given.

Comparing the Gumbel copula estimation results in Tables 20.3 and 20.4 (upper tail and lower tail, respectively) suggests that extreme dependence is asymmetric when considering the upper- versus the lower-tail behavior. Extreme dependence appears as an issue with upward shocks to CDX spreads.

20.4 CONCLUSION

Systematic credit risk is driven by credit risk dependence. In this chapter, we show that a common factor model of credit risk dependence has several interesting features. While common factor changes may hardly affect single obligors' spreads most of the time, they may become a driving force in some rare states of the economy. Time-varying factor sensitivities and the analysis of extreme factor dependence show that obligors will have varying exposures to the market not only cross-sectionally (as, e.g., given by their rating class) but also during time. We emphasize that a time-varying nature of systematic credit risk goes along with asymmetric extreme dependence. These findings are important for portfolio diversification and for future research into asset pricing models of credit risk premia.

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Part VI

Options, Portfolios, and Pricing Loss Distribution Tranches

CHAPTER 21

CDS Options through Candidate Market Models and the CDS-Calibrated CIR++ Stochastic Intensity Model

Damiano Brigo

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21.1 INTRODUCTION

We consider some alternative expressions for CDS (credit default swap) payoffs, stemming from different conventions on the payment flows and on the protection leg for these contracts. We consider standard running CDS (RCDS), postponed payments running CDS (PPCDS), and briefly up-front CDS (UCDS). Each different RCDS definition implies a different definition of forward CDS rate, which we consider with some detail.

We introduce defaultable floating-rate notes (DFRNs). We point out which kind of CDS payoff produces a forward CDS rate that is equal to the fair spread in the considered DFRN. An approximated equivalence between CDSs and DFRNs is established, which allows to view CDS options as structurally similar to the optional component in defaultable callable notes. Equivalence of CDS and DFRNs has been known for a while in the market (Schönbucher 1998), where the simpler case with continuous flows of payments is considered. Here we consider a discrete set of flows, as in real market contracts, and find that the equivalence holds only after postponing or anticipating some relevant default indicators or discount factors.

We briefly investigate the possibility to express forward CDS rates in terms of some basic rates and discuss a possible analogy with the LIBOR (London Interbank Offered Rate) and swap default-free models. We then discuss the change of numeraire approach to deriving a Black-like formula for CDS options, allowing us to quote CDS options through their implied volatilities. This work is in the same field as Schönbucher (2000, 2004) and especially Jamshidian (2002). Interesting considerations are also in Hull and White (2003). Here, using Jamshidian's approach as a guide, and based on a result by Jeanblanc and

Rutkowski (2000), we derive CDS option prices for CDS payoffs given in the market and for the new approximated CDS payoffs. We do so by means of a rigorous change of numeraire technique. We consider the standard market model for CDS options resulting from this approach. In doing so, we point out some analogies with the default-free LIBOR and swap market models. This approach allows also for writing a dynamics for CDS forward rates leading to a CDS options volatility smile.

In the final part of the chapter, we introduce a formula for CDS option pricing under the CDS-calibrated (Cox–Ingersoll–Ross) CIR++ stochastic intensity model. We give patterns of implied volatilities as functions of the CIR++ model parameters.

The chapter is structured as follows: Section 21.2 introduces notation, different kind of CDS discounted payoffs, and the main definition of CDS forward rate. The notion of CDS-implied hazard function and its possible use as quoting mechanism are recalled. UCDSs are hinted at.

Section 21.3 examines some possible variant definitions of CDS rates. Furthermore, we examine the relationship between CDS rates on different periods and point out some parallels with the default-free LIBOR and swap market rates.

Section 21.4 introduces DFRNs and explores their relationship with CDS payoffs, finding equivalence under some payment schedules.

Section 21.5 describes the payoffs and structural analogies between CDS options and callable DFRN.

Section 21.6 introduces the market model for CDS options and the related market formula for CDS options and callable DFRN. Hints on modeling of volatility smile for CDS options are given, even though the smile in such market has not appeared yet.

Finally, Section 21.7 introduces a formula for CDS option pricing under the CDS-calibrated CIR++ stochastic intensity model. The formula is based on Jamshidian's decomposition. We investigate patterns of implied volatilities as functions of the CIR++ model parameters.

An earlier version of this chapter is in Brigo (2004), while a reduced related version has appeared in Brigo (2005).

21.2 CREDIT DEFAULT SWAPS: DIFFERENT FORMULATIONS

21.2.1 CDS Payoffs

We recall briefly some basic definitions for CDSs. Consider a CDS where we exchange protection payment rates R at times $T_{a+1}, \ldots, T_b$ (the premium leg) in exchange for a single protection payment LGD (loss given default, the protection leg) at the default time τ of a reference entity C, provided that $T_a < \tau \leq T_b$. This is called a running CDS discounted payoff. Formally, we may write the RCDS discounted value at time t as

$$\Pi \text{RCDS}_{a,b}(t) := D(t,\tau)[\tau - T_{\beta(\tau)-1}]R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\text{LGD} \tag{21.1}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$ i.e., $T_{\beta(t)}$ is the first date among the T_i's that follows t, and where α_i is the year fraction between T_{i-1} and T_i. The stochastic discount factor at time t for maturity T is denoted by $D(t,T) = B(t)/B(T)$, where $B(t) = \exp\left(\int_0^t r_u \, du\right)$ denotes the bank-account numeraire, r being the instantaneous short interest rate.

We explicitly point out that we are assuming the offered protection amount LGD to be deterministic and, in particular, not to depend on the CDS rate but only on the reference entity and on the payment dates. Typically, LGD = 1 − REC, where the recovery rate, REC, is assumed to be deterministic and the notional is set to one.

Sometimes a slightly different payoff is considered for RCDS contracts. Instead of considering the exact default time τ, the protection payment LGD is postponed to the first time T_i following default, i.e., to $T_{\beta(\tau)}$. If the grid is 3 or 6 months spaced, this postponement consists in a few months at worst. With this formulation, the CDS discounted payoff can be written as

$$\Pi\text{PRCDS}_{a,b}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1} < \tau < T_i\}} D(t,T_i)\text{LGD} \tag{21.2}$$

which we term "postponed payoffs running CDS" (PRCDS) discounted payoff. Compare with the earlier discounted payout Equation 21.1 where the protection payment occurs exactly at τ. The advantage of the postponed protection payment is that no accrued-interest term in $[\tau - T_{\beta(\tau)-1}]$ is necessary and also that all payments occur at the canonical grid of the T_i's. The postponed payout is better for deriving market models of CDS rates dynamics, as we shall see shortly. Yet, unless explicitly specified, in the following, we consider the first payout Equation 21.1, since this is the formulation most resembling market practice. When we write simply CDS we refer to the RCDS case.

A slightly different postponed discounted payoff would be more appropriate. Indeed, if we consider

$$\Pi\text{PR2CDS}_{a,b}(t) := \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R \mathbf{1}_{\{\tau > T_{i-1}\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1} < \tau < T_i\}} D(t,T_i)\text{LGD} \tag{21.3}$$

(notice the T_{i-1} in the indicators of the first summation), we see that we are including one more R-payment with respect to the earlier postponed case. This is appropriate, since by pretending default is occurring at $T_{\beta(\tau)}$ instead of τ we are in fact introducing one more whole interval we have to account for in the premium leg.

From a different point of view, and since the protection leg, even if postponed, is discounted with the appropriate discount factor taking into account postponement, notice that in cases where τ is slightly larger than T_i then the first postponed payoff Equation 21.2 is a better approximation of the actual one. Instead, in cases where τ is slightly smaller than T_i, the postponed payoff Equation 21.3 represents a better approximation. We will see the different implications of these two payoffs.

At times there is interest in UCDS contracts. In this version, the present value of the protection leg is paid up front by the party that is buying protection. In other terms, instead of exchanging a possible protection payment for some coupons, one exchanges it with an up-front payment.

The discounted payoff of the protection leg is simply

$$\Pi\text{UCDS}_{a,b}(t) := \mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ \text{LGD} = \sum_{i=a+1}^{b}\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}D(t,\tau)\text{LGD} \tag{21.4}$$

Alternatively, one can approximate this leg by a postponed payment version, where we postpone the protection payment until the first T_i following default τ:

$$\Pi\text{UPCDS}_{a,b}(t) := \sum_{i=a+1}^{b}\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}D(t,T_i)\text{LGD} \tag{21.5}$$

21.2.2 CDS Pricing and Cox Processes

We denote by CDS(t, $[T_{a+1},\ldots,T_b]$, T_a, T_b, R, LGD) the price at time t of the above standard running CDS. At times some terms are omitted, such as for example the list of payment dates $[T_{a+1},\ldots,T_b]$. We add the prefixes PR1 or PR2 to denote, respectively, the analogous prices for the postponed payoffs Equations 21.2 and 21.3. We add the prefix "U" (up front) to denote the present value at t of the protection leg Equation 21.4 of the CDS and "UP" (up front postponed) in case we are considering the present value of Equation 21.5.

The pricing formulas for these payoffs depend on the assumptions on interest-rate dynamics and on the default time τ. Here we place ourselves in a stochastic intensity framework, where the intensity is an $\mathcal{F}_t$-adapted continuous positive process, $\mathcal{F}_t$ denoting the basic filtration without default, typically representing the information flow of interest rates, intensities, and possibly other default-free market quantities. Default is modeled as the first jump time of a Cox process with the given intensity process. In the Cox process setting, we have $\tau = \Lambda^{-1}(\xi)$, where Λ is the stochastic hazard function which we assume to be $\mathcal{F}_t$-adapted, absolutely continuous, and strictly increasing, and ξ is exponentially distributed with parameter 1 and independent of $\mathcal{F}$. These assumptions imply the existence of a positive-adapted process λ, which we assume also to be right continuous and limited on the left, such that $\Lambda(t) = \int_0^t \lambda_s\,ds$ for all t. We will not model the intensity directly in this chapter, except in Section 21.7. Rather, we model some market quantities embedding the impact of the relevant intensity model that is consistent with them. In general, we can compute the CDS price according to risk-neutral valuation (Bielecki and Rutkowski 2001):

$$\text{CDS}(t,T_a,T_b,R,\text{LGD}) = \mathbb{E}\{\Pi\text{RCDS}_{a,b}(t)|\mathcal{G}_t\} \tag{21.6}$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\},\ u \leq t)$ and $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ. We will denote by $\mathbb{E}_t$ the expectation conditional on the sigma field $\mathcal{F}_t$.

This expected value can also be written, under very general assumptions, as

$$\mathrm{CDS}(t, T_a, T_b, R, \mathrm{LGD}) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\mathbb{E}\{\Pi\mathrm{RCDS}_{a,b}(t)|\mathcal{F}_t\} \tag{21.7}$$

(see again Bielecki and Rutkowski (2001), Formula 5.1, p. 143; or more in particular Jeanblanc and Rutkowski (2000) for the most general result of this kind).

This second expression, and the analogous definitions with postponed payoffs, will be fundamental for introducing the market model for CDS options in a rigorous way.

For the time being, let us deal with the definition of (running) CDS forward rate $R_{a,b}(t)$. This can be defined as R that makes the CDS value equal to zero at time t, so that

$$\mathrm{CDS}(t, T_a, T_b, R_{a,b}(t), \mathrm{LGD}) = 0$$

The idea is then solving this equation in $R_{a,b}(t)$. In doing this, one has to be careful. It is best to use Equation 21.7 rather than Equation 21.6. Equate this expression to zero and derive R correspondingly. Strictly speaking, the resulting R would be defined on $\{\tau > t\}$ only, since elsewhere the equation is satisfied automatically, thanks to the indicator in front of the expression, regardless of R. Since the value of R does not matter when $\{\tau < t\}$, the equation being satisfied automatically, we need not worry about $\{\tau < t\}$ and may define, in general,

$$R_{a,b}(t) = \frac{\mathrm{LGD}\mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\le T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i) + \mathbb{E}\{D(t,\tau)[\tau - T_{\beta(\tau)-1}]\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t\}} \tag{21.8}$$

where $\bar{P}(t,T) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)$ is the "no survival-indicator" part of the defaultable T-maturity (no recovery) zero-coupon bond, i.e.,

$$\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T)$$

is the price at time t of a defaultable zero-coupon bond maturing at time T. Notice that replacing $\{\tau > T\}$ by $\{\tau \ge T\}$, as we implicitly do in Equation 21.8, does not change anything since we are assuming continuous processes for the short rate and the stochastic intensity. We will denote by $P(t,T)$ the default-free zero-coupon bond at time t for maturity T.

This approach to define $R_{a,b}$ amounts to equating to zero only the expected value in Equation 21.7 and in a sense is a way of privileging $\mathcal{F}_t$ expected values to $\mathcal{G}_t$ ones. The technical tool allowing us to do this is the above-mentioned Jeanblanc and Rutkowski (2000) result and this is the spirit of part of the work in Jamshidian (2002).

Finally, we chose Cox processes as the underlying framework, but this is not necessary. More generally, we may have market models where one directly models market-related rates without presuming existence of a default intensity (Jamshidian 2002 and Brigo and Morini 2005).

21.2.3 Market Quoting Mechanism and Implied Hazard Functions

Now we explain shortly how the market quotes RCDS and UCDS prices. First, we notice that typically the T's are 3 month spaced. Let us begin with RCDSs. Usually at time $t = 0$,

provided default has not yet occurred, the market sets R to a value $R^{\text{MID}}_{a,b}(0)$ that makes the CDS fair at time zero, i.e., such that $\text{CDS}(0, T_a, T_b, R^{\text{MID}}_{a,b}(0), \text{LGD}) = 0$. In fact, in the market RCDSs used to be quoted at a time zero through a bid and an ask value for this fair $R^{\text{MID}}_{a,b}(0)$, for CDSs with $T_a = 0$, and with T_b spanning a set of canonical final maturities, $T_b = 1$ year up to $T_b = 10$ years. As time moves on to say $t = 1$ day, the market shifts the T's of t, setting $T_a = 0 + t, \ldots, T_b = 10 \text{ years} + t$, and then quotes $R^{\text{MID}}_{a,b}(t)$ satisfying $\text{CDS}(t, T_a, T_b, R^{\text{MID}}_{a,b}(t), \text{LGD}) = 0$. This means that as time moves on, the maturities increase and the times to maturity remain constant.

Recently, the quoting mechanism has changed and has become more similar to the mechanism of the futures markets. Let zero be the current time. Maturities $T_a, \ldots, T_b$ are fixed at the original time zero to some values such as 1 year, 2 years, 3 years, etc. and then, as time moves for example to $t = 1$ day, the CDS maturities are not shifted correspondingly of 1 day as before but remain 1 year, 2 years, etc. from the original time zero. This means that the times to maturity of the quoted CDSs decrease as time passes. When the quoting time approaches maturity, a new set of maturities are fixed and so on. A detail concerning the constant maturities paradigm is that when the first maturity T_a is less than 1 month away from the quoting time (say zero), the payoff two terms

$$T_a D(0, T_a) R \mathbf{1}_{\{\tau > T_a\}} + (T_{a+1} - T_a) D(0, T_{a+1}) R \mathbf{1}_{\{\tau > T_{a+1}\}}$$

are replaced by

$$T_{a+1} D(0, T_{a+1}) R \mathbf{1}_{\{\tau > T_{a+1}\}}$$

in determining the fair R. If we neglect this last convention, once we fix the quoting time (say to zero) the method to strip implied hazard functions is the same under the two quoting paradigms. For example, Brigo and Alfonsi (2003) present a more detailed section on the constant time-to-maturity paradigm, and illustrate the notion of implied deterministic intensity (hazard function), satisfying

$$\mathbb{Q}\{s < \tau < t\} = \mathrm{e}^{-\Gamma(s)} - \mathrm{e}^{-\Gamma(t)}$$

The market Γ's are obtained by inverting a pricing formula based on the assumption that τ is the first jump time of a Poisson process with deterministic intensity $\gamma(t) = \mathrm{d}\Gamma(t)/\mathrm{d}t$. In this case, one can derive a formula for CDS prices based on integrals of γ, and on the initial interest-rate curve, resulting from the above expectation:

$$\begin{aligned}\text{CDS}(t, T_a, T_b, R, \text{LGD}; \Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}} \Bigg(& R \int_{T_a}^{T_b} P(t,u)[T_{\beta(u)-1} - u] d\left\{\mathrm{e}^{-[\Gamma(u)-\Gamma(t)]}\right\} \\ & + \sum_{i=a+1}^{n} P(t, T_i) R \alpha_i \mathrm{e}^{\Gamma(t)-\Gamma(T_i)} + \text{LGD} \int_{T_a}^{T_b} P(t,u) d\left\{\mathrm{e}^{-[\Gamma(u)-\Gamma(t)]}\right\} \Bigg)\end{aligned} \tag{21.9}$$

By equating to zero the above expression in γ for $t = 0$, $T_a = 0$, after plugging in the relevant market quotes for R, one can extract the γ's corresponding to CDS market quotes

for increasing maturities T_b and obtain market implied γ^{mkt}'s and Γ^{mkt}'s. It is important to point out that usually the actual model one assumes for τ is more complex and may involve stochastic intensity either directly or through stochastic modeling of the future R dynamics itself. Even so, the γ^{mkt}'s are retained as a mere quoting mechanism for CDS rate market quotes and may be taken as inputs in the calibration of more complex models.

UCDS are simply quoted through the present value of the protection leg. Under deterministic hazard rates γ, we have

$$\text{UCDS}(t, T_a, T_b, R, \text{LGD}; \Gamma(\cdot)) = \mathbf{1}_{\{t<\tau\}}\text{LGD}\int_{T_a}^{T_b} P(t,u)d\left\{e^{-[\Gamma(u)-\Gamma(t)]}\right\}$$

As before, by equating to the corresponding up-front market quote the above expression in γ, one can extract the γ's corresponding to UCDS market quotes for increasing maturities and obtain again market implied γ^{mkt}'s and Γ^{mkt}'s.

Once the implied γ are estimated, it is easy to switch from the RCDS quote R to the UCDS quote UCDS, or vice versa. Indeed, we see that, without postponed payments, the two quotes are linked by

$$\begin{aligned}\text{UCDS}\left(t, T_a, T_b, R, \text{LGD}; \Gamma^{\text{mkt}}(\cdot)\right) = R_{a,b}(t)\Bigg(&\int_{T_a}^{T_b} P(t,u)[T_{\beta(u)-1} - u]d\left\{e^{-[\Gamma^{\text{mkt}}(u)-\Gamma^{\text{mkt}}(t)]}\right\} \\ &+ \sum_{i=a+1}^{n} P(t,T_i)\alpha_i e^{\Gamma^{\text{mkt}}(t)-\Gamma^{\text{mkt}}(T_i)}\Bigg)\end{aligned}$$

21.3 DIFFERENT DEFINITIONS OF CDS FORWARD RATES AND ANALOGIES WITH THE LIBOR AND SWAP MODELS

The procedure of equating to zero the current price of a contract to derive a sensible definition of forward rate is rather common. For example, the default-free forward LIBOR rate $F(t, S, T)$ is obtained as the rate at time t that makes the time t price of a forward rate agreement (FRA) contract vanish. This FRA contract locks in the interest rate between time S and T. An analogous definition of forward swap rate at time t is obtained as the rate in the fixed leg of the swap that makes the swap value at time t equal to zero. For a discussion on both the default-free FRA and swap cases see for example Brigo and Mercurio (2006), Chapter 1.

In the current context, we can set a CDS price to zero to derive a forward CDS rate. Clearly, the obtained rate changes according to the different RCDS payoff we consider. For example, by equating to zero Equation 21.7 and solving in R, we have the standard RCDS forward rate given in Equation 21.8. We may wonder about what we would have obtained as definition of forward CDS rates when considering CDS payoffs PRCDS with postponed protection payments (Equation 21.2) or even PR2CDS (Equation 21.3). By straightforwardly adapting the above derivation, we would have obtained a CDS forward rate defined as

$$R^{\text{PR}}_{a,b}(t) = \frac{\text{LGD}\sum_{i=a+1}^{b}\mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t]} = \frac{\text{LGD}\sum_{i=a+1}^{b}\mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_i)}$$

and

$$R^{\mathrm{PR2}}_{a,b}(t) = \frac{\mathrm{LGD}\sum_{i=a+1}^{b}\mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t]}$$

where PR and PR2 stand for postponed-running payoffs of the first and second kind, respectively.

Can we use the forward CDS rate definition, limited to a one-period interval, to introduce defaultable one-period forward rates? A straightforward generalization of the definition of forward LIBOR rates to the defaultable case is given for example in Schönbucher (2000). This definition mimics the definition in the default-free case, in that from zero-coupon bonds one builds a defaultable forward LIBOR rate

$$\bar{F}(t;T_{j-1},T_j) := (1/\alpha_j)[\bar{P}(t,T_{j-1})/\bar{P}(t,T_j) - 1]$$

on $\tau > t$. However, as noticed earlier, the default-free F is obtained as the fair rate at time t for a FRA contract. Can we see $\bar{F}$ as the fair rate for a sort of defaultable FRA? Since the most liquid credit instruments are CDSs, consider a running postponed CDS on a one-period interval, with $T_a = T_{j-1}$ and $T_b = T_j$. We obtain (take LGD = 1)

$$R^{\mathrm{PR}}_j(t) := \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau<T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)} = \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)} \tag{21.10}$$

where we have set $R^{\mathrm{PR}}_j := R^{\mathrm{PR}}_{j-1,j}$. The analogous part of $\bar{F}_j = \bar{F}(\cdot, T_{j-1}, T_j)$ would be, after adjusting the conditioning to $\mathcal{F}_t(\hat{F}_j(t) = \bar{F}_j(t)$ on $\tau > t$ but $\hat{F}$ is defined also on $\tau \le t)$

$$\hat{F}_j(t) = \frac{\mathbb{E}[D(t,T_{j-1})\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

The difference is that in R^{PR}_j's numerator we are taking expectation of a quantity that vanishes in all paths where $\tau > T_j$, whereas in $\bar{F}$ the corresponding quantity does not vanish necessarily in paths with $\tau > T_j$. Moreover, while R_j comes from a financial contract, $\bar{F}$ remains an abstraction not directly linked to a financial payoff.

Schönbucher (2000) defines the discrete tenor credit spread, in general, to be

$$H_j(t) = \frac{1}{\alpha_j}\left[\frac{\bar{P}(t,T_{j-1})/P(t,T_{j-1})}{\bar{P}(t,T_j)/P(t,T_j)} - 1\right]$$

(in $\tau > t$), and it is easy to see that we get

$$H_j(t) = R^{\mathrm{PR}}_j(t)$$

but under independence of the default intensity and the interest rates, and not in general. Again, in general, R_j comes from imposing a one-period CDS to be fair, whereas H_j does not.

A last remark concerns an analogy with the default-free swap market model, where we have a formula linking swap rates to forward rates through a weighted average: $S_{a,b}(t) = \sum_{i=a+1}^{b} \left\{ \alpha_i P(t,T_i) / \left[\sum_{k=a+1}^{b} \alpha_k P(t,T_k) \right] \right\} F_i(t) = \sum_{i=a+1}^{b} w_i(t;F(t)) F_i(t)$. This is useful since it leads to an approximated formula for swaptions in the LIBOR model (Brigo and Mercurio 2006, Chapter 6). A similar approach can be obtained for CDS forward rates. It is easy to check that

$$R^{\mathrm{PR}}_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \alpha_i R^{\mathrm{PR}}_i(t) \bar{P}(t,T_i)}{\sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i)} = \sum_{i=a+1}^{b} \bar{w}_i(t) R^{\mathrm{PR}}_i(t) \approx \sum_{i=a+1}^{b} \bar{w}_i(0) R^{\mathrm{PR}}_i(t) \tag{21.11}$$

A similar relationship for $R^{\mathrm{PR2}}_{a,b}$ involving a weighted average of one-period rates is obtained when resorting to the second type of postponed payoff.

A possible lack of analogy with the swap rates is that the $\bar{w}$'s cannot be expressed as functions of the R_i's only, unless we make some particular assumptions on the correlation between default intensities and interest rates. However, if we freeze the $\bar{w}$'s to time zero, which we have seen to work in the default-free LIBOR model, we obtain easily a useful approximate expression for $R_{a,b}$ and its volatility in terms of R_i's and their volatilities/correlations.

More generally, when not freezing, the presence of stochastic intensities besides stochastic interest rates adds degrees of freedom. Now the $\bar{P}$'s (and thus the $\bar{w}$'s) can be determined as functions for example of one- and two-period rates. Indeed, it is easy to show that

$$\bar{P}(t,T_i) = \bar{P}(t,T_{i-1}) \frac{\alpha_{i-1} \left[R^{\mathrm{PR}}_{i-1}(t) - R^{\mathrm{PR}}_{i-2,i}(t) \right]}{\alpha_i \left[R^{\mathrm{PR}}_{i-2,i}(t) - R^{\mathrm{PR}}_{i}(t) \right]} \tag{21.12}$$

We have to assume $R^{\mathrm{PR}}_{i-2,i}(t) - R^{\mathrm{PR}}_{i}(t) \neq 0$. Actually, if we assume this to hold for the initial conditions, i.e., $R^{\mathrm{PR}}_{i-2,i}(0) - R^{\mathrm{PR}}_{i}(0) \neq 0$, and then take a diffusion dynamics for the two rates, the probability of our condition to be violated at future times is the probability of a continuous random variable to be zero, i.e., it is zero in general. We will see later how this formula will help us in obtaining a market model for CDS rates. For the time being let us keep in mind that the exact weights $\bar{w}(t)$ in Equation 21.11 are completely specified in terms of $R_i(t)$'s and $R_{i-2,i}(t)$'s, so that if we include these two rates in our dynamics the system is closed in that we also know all the relevant $\bar{P}$'s. The difference with the LIBOR/swap model is that here to close the system we need also two-period rates.

21.4 DEFAULTABLE FLOATER AND CDS

Consider a prototypical DFRN.

Definition 21.1 *A prototypical defaultable floating-rate note is a contract ensuring the payment at future times $T_{a+1}, \ldots, T_b$ of the LIBOR rates that reset at the previous instants $T_a, \ldots, T_{b-1}$ plus a spread X, each payment conditional on the issuer having not defaulted before the relevant previous instant. Moreover, the note pays a last cash flow consisting of the reimbursement of the notional value of the note at final time T_b if the issuer has not*

defaulted earlier. We assume a deterministic recovery value REC to be paid at the first T_i following default if default occurs before T_b. The note is said to quote at par if its value is equivalent to the value of the notional paid at the first reset time T_a in case default has not occurred before T_a.

Recall that if no default is considered, then the fair spread making the FRN quote at par is zero (Brigo and Mercurio 2006).

When in presence of default, the note discounted payoff, including the initial cash flow of 1 paid in T_a, is

$$\begin{aligned}\Pi\text{DFRN}_{a,b} = &- D(t,T_a)1_{\{\tau>T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)[L(T_{i-1},T_i)+X]\mathbf{1}_{\{\tau>T_i\}} \\ &+ D(t,T_b)1_{\{\tau>T_b\}} + \text{REC}\sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}\end{aligned}$$

where REC is the recovery rate, i.e., the percentage of the notional that is paid in replacement of the notional in the case of default, and it is paid at the first instant among $T_{a+1},\ldots,T_b$ following default. This is the correct definition of DFRN, consistent with market practice. The problem with such definition is that it has no equivalent in terms of approximated CDS payoff. This is due to the fact that, in a Cox process setting, it is difficult to disentangle the LIBOR rate L from the indicator and stochastic discount factor in such a way to obtain expectations of pure stochastic discount factors times default indicators. This becomes possible if we replace $1_{\{\tau>T_i\}}$ in the first summation with $1_{\{\tau>T_{i-1}\}}$, as one can see from computations (Equation 21.13) below. The same computations, in case we keep T_i in the default indicator, even in the simplified case where LGD = 1 and interest rates are independent of default intensities, would lead us to a corresponding definition of CDS forward rate where protection is paid at the last instant T_i before default, which is not natural since one should anticipate default.

21.4.1 First Approximated DFRN Payoff

We thus consider two alternative definitions of DFRN. The first one is obtained by moving the default indicator of $L(T_{i-1}, T_i) + X$ from T_i to T_{i-1}. The related FRN discounted payoff is defined as follows:

$$\begin{aligned}\Pi\text{DFRN2}_{a,b} = &-D(t,T_a)1_{\{\tau>T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)[L(T_{i-1},T_i)+X]\mathbf{1}_{\{\tau>T_{i-1}\}} \\ &+ D(t,T_b)1_{\{\tau>T_b\}} + \text{REC}\sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1}\le\tau<T_i\}}\end{aligned}$$

Recall that, in the CDS payoff, LGD = 1 − REC. We may now value the above discounted payoff at time t and derive the value of X that makes it zero. Define

$$\text{DFRN2}_{a,b}(t,X,\text{REC}) = \mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{F}_t\}/\mathbb{Q}(\tau>t|\mathcal{F}_t)$$

and solve $\mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{F}_t\} = 0$ in X. The only nontrivial part is computing

$$\alpha_i\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] = \alpha_i\mathbb{E}\{\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t\} = \cdots$$

Under our Cox process setting for τ we can write

$$\begin{aligned}
\cdots &= \alpha_i\mathbb{E}\{\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\xi>\Lambda(T_{i-1})\}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t\}\\
&= \alpha_i\mathbb{E}\{D(t,T_{i-1})L(T_{i-1},T_i)\exp[-\Lambda(T_{i-1})]\mathbb{E}[D(T_{i-1},T_i)|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_t\}\\
&= \alpha_i\mathbb{E}\{\exp[-\Lambda(T_{i-1})]D(t,T_{i-1})L(T_{i-1},T_i)P(T_{i-1},T_i)|\mathcal{F}_t\}\\
&= \mathbb{E}\{\exp[-\Lambda(T_{i-1})]D(t,T_{i-1})[1-P(T_{i-1},T_i)]|\mathcal{F}_t\}\\
&= \mathbb{E}\{D(t,T_{i-1})[1-P(T_{i-1},T_i)]\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t\}\\
&= \mathbb{E}[D(t,T_{i-1})\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t] \qquad (21.13)
\end{aligned}$$

Now the LIBOR flow has vanished from the above payoff and we have expressed everything in terms of pure discount factor and default indicators. Some of these computations could have been performed more simply by means of standard and model independent arguments, but we carried them out in the explicit intensity case so that the reader may try them when not replacing $1_{\{\tau>T_i\}}$ to see what goes wrong.

We may write also

$$\begin{aligned}
\text{DFRN2}_{a,b}(t,X,\text{REC}) = [1_{\{\tau>t\}}/\mathbb{Q}(\tau>t|\mathcal{F}_t)]\Bigg(&-\mathbb{E}_t[D(t,T_a)1_{\{\tau>T_a\}}]+\mathbb{E}_t[D(t,T_b)1_{\{\tau>T_b\}}]\\
&-\sum_{i=a+1}^{b}\mathbb{E}_t\{[D(t,T_i)-D(t,T_{i-1})]\mathbf{1}_{\{\tau>T_{i-1}\}}\}\\
&+X\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]+\text{REC}\sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}]\Bigg)
\end{aligned}$$

We may simplify terms in the summations and obtain

$$\begin{aligned}
\text{DFRN2}_{a,b}(t,X,\text{REC}) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\Bigg\{&-\text{LGD}\sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}}]\\
&+X\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]\Bigg\}
\end{aligned}$$

from which we notice en passant that

$$\text{DFRN2}_{a,b}(t,X,\text{REC}) = \text{PR2CDS}(t,T_a,T_b,X,1-\text{REC}) \qquad (21.14)$$

By taking into account this result, the expression for X that makes the DFRN quote at par is clearly the running "postponed of the second kind" CDS forward rate

$$X^{(2)}_{a,b}(t) = R^{\mathrm{PR2}}_{a,b}(t)$$

i.e., the fair spread in a DFRN is equal to the postponed RCDS forward rate.

21.4.2 Second Approximated DFRN Payoff

The second alternative definition of DFRN, leading to a useful relationship with approximated CDS payoffs, is obtained by moving the default indicator of $L(T_{i-1}, T_i) + X$ from T_i to T_{i-1} but only for the LIBOR flow, not for the spread X. This payoff is closer to the original ΠDFRN payoff than the approximated ΠDFRN2 payoff considered above. Set

$$\begin{aligned}\Pi\mathrm{DFRN1}_{a,b} = &- D(t,T_a)1_{\{\tau>T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)[L(T_{i-1},T_i)\mathbf{1}_{\{\tau>T_{i-1}\}} + X\mathbf{1}_{\{\tau>T_i\}}] \\ &+ D(t,T_b)1_{\{\tau>T_b\}} + \mathrm{REC}\sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}\end{aligned}$$

By calling $\mathrm{DFRN1}_{a,b}(t, X, \mathrm{REC})$ the t-value of the above payoff and by going through the computations we can see easily that this time

$$\mathrm{DFRN1}_{a,b}(t,X,\mathrm{REC}) = \mathrm{PRCDS}(t,T_a,T_b,X,1-\mathrm{REC}) \tag{21.15}$$

and that, as far as fair spreads are concerned,

$$X^{(1)}_{a,b}(t) = R^{\mathrm{PR}}_{a,b}(t)$$

21.5 CDS OPTIONS AND CALLABLE DEFAULTABLE FLOATERS

Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, paying a fixed rate K at times $T_{a+1}, \ldots, T_b$ or until default, in exchange for protection against possible default in $[T_a, T_b]$. If default occurs a protection payment LGD is received. By noticing that the market CDS rate $R_{a,b}(T_a)$ will set the CDS value in T_a to zero, the payoff can be written as the discounted difference between said CDS and the corresponding CDS with rate K. We will see below that this is equivalent to a call option on the future CDS fair rate $R_{a,b}(T_a)$. The discounted CDS option payoff reads, at time t,

$$\Pi\mathrm{CallCDS}_{a,b}(t;K) = D(t,T_a)\{\mathrm{CDS}(T_a,T_a,T_b,R_{a,b}(T_a),\mathrm{LGD}) - \mathrm{CDS}(T_a,T_a,T_b,K,\mathrm{LGD})\}^+ \tag{21.16}$$

leading to two possible expressions, depending on whether we explicit the CDS values, given respectively by

$$\Pi\text{CallCDS}_{a,b}(t;K) = \frac{1_{\{\tau>T_a\}}}{\mathbb{Q}(\tau > T_a|\mathcal{F}_{T_a})} D(t,T_a)\Bigg(\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > T_a|\mathcal{F}_{T_a})\bar{P}(T_a,T_i) + \mathbb{E}\{D(T_a,\tau)[\tau - T_{\beta(\tau)-1}]\mathbf{1}_{\{\tau<T_b\}}|\mathcal{F}_{T_a}\}\Bigg)[R_{a,b}(T_a) - K]^+ \tag{21.17}$$

or, by remembering that by definition CDS $(T_a, T_a, T_b, R_{a,b}(T_a), \text{LGD}) = 0$, as

$$\Pi\text{CallCDS}_{a,b}(t;K) = D(t,T_a)[-\text{CDS}(T_a,T_a,T_b,K,\text{LGD})]^+ \tag{21.18}$$

This last expression points out that in holding this CDS option we would be interested in the option to buy protection LGD at time T_a against default in $[T_a,T_b]$ in exchange for a premium leg characterized by the strike-rate K. These options can be introduced for every type of underlying CDS. We have illustrated the standard CDS case, and we will consider the postponed CDS cases below.

The quantity inside round brackets in Equation 21.17 will play a key role in the following. We will often neglect the accrued-interest term in $[\tau - T_{\beta(\tau)-1}]$ and consider the related simplified payoff: in such a case the quantity between round brackets is denoted by $\hat{C}_{a,b}(T_a)$ and is called (no survival indicator) defaultable present value per basis point (DPVBP) numeraire. Actually the real DPVBP would have a $1_{\{\tau>\cdot\}}$ term in front of the summation, but by a slight abuse of language we call DPVBP the expression without indicator. More generally, at time t, we set

$$\hat{C}_{a,b}(t) := \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{C}_{a,b}(t), \quad \bar{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i)$$

When including as a factor the indicator $1_{\{\tau>t\}}$, this quantity is the price, at time t, of a portfolio of defaultable zero-coupon bonds with zero recovery and with different maturities, and as such it is the price of a tradable asset. The original work of Schönbucher (2000, 2004) is in this spirit, in that the numeraire is taken with the indicator, so that it may vanish and the measure it defines is not equivalent to the risk-neutral measure. If we keep the indicator away, following in spirit part of the work in Jamshidian (2002), this quantity maintains a link with said price and is always strictly positive, so that we are allowed to take it as numeraire.

The related probability measure, equivalent to the risk-neutral measure, is denoted by $\hat{Q}^{a,b}$ and the related expectation by $\hat{\mathbb{E}}^{a,b}$.

Neglecting the accrued-interest term, the option discounted payoff simplifies to

$$1_{\{\tau>T_a\}} D(t,T_a)\left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a,T_i)\right][R_{a,b}(T_a) - K]^+ \tag{21.19}$$

but this is only an approximated payoff and not the exact one.

21.5.1 First Equivalence: PRCDS and DFRN1

Let us follow the same derivation under the postponed CDS payoff of the first kind. Consider thus

$$\begin{aligned}\Pi\text{CallPRCDS}_{a,b}(t;K) = D(t,T_a)\{&\text{PRCDS}\left(T_a,T_a,T_b,R^{\text{PR}}_{a,b}(T_a),\text{LGD}\right)\\ &-\text{PRCDS}(T_a,T_a,T_b,K,\text{LGD})\}^+\end{aligned} \tag{21.20}$$

or, since $\text{PRCDS}\left(T_a,T_a,T_b,R^{\text{PR}}_{a,b}(T_a),\text{LGD}\right)=0$,

$$\Pi\text{CallPRCDS}_{a,b}(t;K) = D(t,T_a)[-\text{PRCDS}(T_a,T_a,T_b,K,\text{LGD})]^+$$

which, by Equation 21.15, is equivalent to

$$D(t,T_a)[-\text{DFRN1}_{a,b}(T_a,K,\text{REC})]^+ \tag{21.21}$$

with $\text{LGD}=1-\text{REC}$, or since $\text{DFRN1}_{a,b}(T_a, X_{a,b}(T_a), \text{REC})=0$, is equivalent to

$$D(t,T_a)\{\text{DFRN1}_{a,b}\left(T_a,X_{a,b}(T_a),\text{REC}\right)-\text{DFRN1}_{a,b}(T_a,K,\text{REC})\}^+$$

By expanding the expression of PRCDS we obtain as exact discounted payoff the quantity

$$\Pi\text{CallPRCDS}_{a,b}(t,K) = 1_{\{\tau>T_a\}}D(t,T_a)\left[\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)\right]\left[R^{\text{PR}}_{a,b}(T_a)-K\right]^+ \tag{21.22}$$

which is structurally identical to the approximated payoff (Equation 21.19) for the standard CDS case. Thus we have created a link between the first postponed CDS payoff and an option on an approximated DFRN. Notice in particular that the quantity in front of the optional part is the same as in the earlier standard approximated discounted payoff, i.e., the DPVBP.

21.5.2 Second Equivalence: PR2CDS and DFRN2

We may also consider the postponed RCDS of the second kind. The related discounted CDS option payoff reads, at time t,

$$\begin{aligned}\Pi\text{CallPR2CDS}_{a,b}(t,K) = D(t,T_a)\{&\text{PR2CDS}\left(T_a,T_b,R^{\text{PR2}}_{a,b}(T_a),\text{LGD}\right)\\ &-\text{PR2CDS}(T_a,T_b,K,\text{LGD})\}^+\end{aligned} \tag{21.23}$$

and given Equation 21.14, this is equivalent to

$$D(t,T_a)\{\text{DFRN2}_{a,b}\left(T_a,X_{a,b}(T_a),\text{REC}\right)-\text{DFRN2}_{a,b}(T_a,K,\text{REC})\}^+ \tag{21.24}$$

with $\text{REC}=1-\text{LGD}$, or by expanding the expression for PR2CDS, we get

$$[1_{\{\tau>T_a\}}/\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})]D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_{T_a}[D(T_a,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]\left[R^{\text{PR2}}_{a,b}(T_a)-K\right]^+ \tag{21.25}$$

Again we have equivalence between CDS options and options on the defaultable floater.

21.5.3 Callable Defaultable Floaters

The option on the floater can be seen as the optional component of a callable DFRN. A DFRN with final maturity T_b is issued at time zero with a fair rate $X_{0,b}(0)$ in such a way that $\text{DFRN}_{0,b}(0, X_{0,b}(0), \text{REC}) = 0$. Suppose that this FRN includes a callability feature: at time T_a the issuer has the right to take back the subsequent FRN flows and replace them with the notional 1. The issuer will do so only if the present value in T_a of the subsequent FRN flows is larger than 1 in T_a. This is equivalent, for the note holder, to receive $1_{\{\tau>T_a\}} + \min\{\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC}), 0\} = 1_{\{\tau>T_a\}} + \text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC}) - \{\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC})\}^+$ at time T_a if no default has occurred by then (recall that in our notation $\text{DFRN}_{a,b}$ includes a negative cash flow of 1 at time T_a).

The component $1_{\{\tau>T_a\}} + \text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC})$ when valued at time zero is simply the residual part of the original DFRN without callability features from T_a on, so that when added to the previous payments in $0 - T_a$ its present value is zero. This happens because $X_{a,b}(0)$ is the fair rate for the total DFRN at time zero. The component

$$\{\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC})\}^+ = \{\text{DFRN}_{a,b}(T_a, X_{0,b}(0), \text{REC}) - \text{DFRN}_{a,b}(T_a, X_{a,b}(T_a), \text{REC})\}^+$$

is structurally equivalent to a CDS option, provided we approximate its payoff with DFRN1 or DFRN2, as we have seen earlier, and we may value it if we have a model for CDS options. We are deriving a market model for such options in the next section, so that we will be implicitly deriving a model for callable defaultable floaters.

21.6 MARKET MODEL FOR CDS OPTIONS AND CALLABLE DEFAULTABLE FLOATERS

As usual, one may wish to introduce a notion of implied volatility for CDS options. This would be a volatility associated to the relevant underlying CDS rate R. To do so rigorously, one has to come up with an appropriate dynamics for $R_{a,b}$ directly, rather than modeling instantaneous default intensities explicitly. This somehow parallels what we find in the default-free interest-rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions. In a one-factor short-rate model, the dynamics of the forward swap rate is a by-product of the short-rate dynamics itself, through Ito's formula. On the contrary, the market model for swaptions directly postulates, under the relevant numeraire a (lognormal) dynamics for the forward swap rate.

21.6.1 Market Models for CDS Options

In the case of CDS options, the market model is derived as follows. First, let us ignore the accruing term in $[\tau - T_{\beta(\tau)-1}]$, by replacing it with zero. It can be seen that typically the order of magnitude of this term is negligible with respect to the remaining terms in the payoff. Failing this negligibility, one may reformulate the payoff by postponing the default payment to the first date among the T_i's following τ, i.e., to $T_{\beta(\tau)}$. This amounts to consider as underlying a payoff corresponding to Equation 21.2 and eliminates the accruing term altogether, even though it slightly modifies the option payoff. Take as numeraire the DPVBP $\hat{C}_{a,b}$, so that

$$R_{a,b}(t) = \frac{\text{LGD}\mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\le T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_i)} = \frac{\text{LGD}\mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau\le T_b\}}|\mathcal{F}_t]}{\hat{C}_{a,b}(t)}, \quad t \le T_a \tag{21.26}$$

having as numerator the price of an UCDS, can be interpreted as the ratio between a tradable asset and our numeraire. As such, it is a martingale under this numeraire's measure and can be modeled as a Black–Scholes driftless geometric Brownian motion, leading to a Black and Scholes formula for CDS options. Notice that, when $R_{a,b}(0)$ is not quoted directly by the market, we may infer it by the market implied γ^{mkt} according to

$$R_{a,b}(0) = \frac{-\text{LGD}\int_{T_a}^{T_b} P(0,u)d[\mathrm{e}^{-\Gamma^{\text{mkt}}(u)}]}{\sum_{i=a+1}^{b}\alpha_i P(0,T_i)\mathrm{e}^{-\Gamma^{\text{mkt}}(T_i)}}$$

Indeed, by resorting to the change of numeraire starting from Equation 21.19 (thus ignoring the accruing term or working with the postponed payoff) we see that

$$\begin{aligned}
&\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\left[\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_{T_a}\}|\mathcal{F}_t\right] \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\left[D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+\mathbb{E}\{1_{\{\tau>T_a\}}|\mathcal{F}_{T_a}\}|\mathcal{F}_t\right] \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\left[D(t,T_a)\sum_{i=a+1}^{b}\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\right] \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\left[D(t,T_a)\hat{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\right] \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\hat{C}_{a,b}(t)\hat{\mathbb{E}}^{a,b}\left[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\right] \\
&= 1_{\{\tau>t\}}\bar{C}_{a,b}(t)\hat{\mathbb{E}}^{a,b}\left[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\right]
\end{aligned}$$

and we may take

$$dR_{a,b}(t) = \sigma_{a,b}R_{a,b}(t)dW^{a,b}(t) \tag{21.27}$$

where $W^{a,b}$ is a Brownian motion under $\hat{\mathbb{Q}}^{a,b}$, leading to a market formula for the CDS option. We have

$$\begin{aligned}
&\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)[R_{a,b}(T_a)-K]^+|\mathcal{G}_t\} \\
&\quad = 1_{\{\tau>t\}}\bar{C}_{a,b}(t)\{R_{a,b}(t)N(d_1(t)) - KN(d_2(t))\}
\end{aligned} \tag{21.28}$$

$$d_{1,2} = \left\{\ln(R_{a,b}(t)/K) \pm (T_a-t)\sigma_{a,b}^2/2\right\} \Big/ \left(\sigma_{a,b}\sqrt{T_a-t}\right)$$

As happens in most markets, this formula could be used as a quoting mechanism rather than as a real model formula. That is, the market price can be converted into its implied volatility matching the given price when substituted in the above formula.

To completely specify the market model we need to show how the dynamics of $R_{a,b}$ changes when changing numeraire. We describe the essential steps briefly in two important cases, and we refer to the postponed running (PR) payoff.

21.6.2 One- and Two-Period CDS Rates Market Models

The first case we address is a family of one-period rates. This is to say that we are trying to build a sort of forward LIBOR model for CDS rates. As the LIBOR model is based on one-period forward rates, our first choice of a market model for CDS options will be based on one-period rates. The fundamental components of our numeraires $\hat{C}$ are the $\bar{P}$'s. The $\bar{P}$'s, through Equation 21.12, can be reduced to a function of a common initial $\bar{P}$ (that cancels when considering the relevant ratios) and of one- and two-period rates R_k and $R_{k-2,k}$ in the relevant range. We start then by writing the (martingale) dynamics of one- and two-period rates R_k and $R_{k-2,k}$ each under its canonical numeraire $\hat{\mathbb{Q}}^{k-1,k}$ and $\hat{\mathbb{Q}}^{k-2,k}$, respectively. At this point, we use the change of numeraire technique on each of this rates to write their dynamics under a single preferred $\hat{\mathbb{Q}}^{\cdot,\cdot}$. This is possible in terms of quadratic covariations between the one- and two-period rates being modeled and the analogous rates concurring to form the $\bar{P}$'s entering the relevant $\hat{C}$. Now we have the dynamics for all the relevant R_k's and $R_{k-2,k}$'s under a common measure $\hat{\mathbb{Q}}^{\cdot,\cdot}$. Since $R_{a,b}$ is completely specified in terms of one-and two-period rates through Equations 21.11 and 21.12, we have indirectly also $R_{a,b}$'s dynamics.

Notice that if first we assign the dynamics of one-period rates, then the dynamics of the two-period rates has to be selected carefully. For example, two-period rates will have to be selected into a range determined by one-period rates to avoid $\bar{P}(t,T_k)/\bar{P}(t,T_{k-1})$ to be negative or larger than one. The use of suitable martingale dynamics for each $R_{k-2,k}$ under $\hat{\mathbb{Q}}^{k-2,k}$ ensuring this property is currently under investigation.

If we are concerned about lognormality of R's, leading to Black-like formulas for CDS options, one of the possible choices is to impose one-period rates R_k to have a lognormal distribution under their canonical measures. It suffices to postulate a driftless geometric Brownian motion dynamics for each such rate under its associated measure. The resulting $R_{a,b}$ will only be approximately lognormal, especially under the freezing approximation for the weights $\bar{w}$, but this is the case also with LIBOR versus Swap models, since lognormal one-period swap rates (i.e., forward LIBOR rates) and multi-period swap rates cannot be all lognormal (each under its canonical measure). The important difference with the LIBOR model is that here we need also two-period rates to close the system. The need for two-period rates stems from the additional degrees of freedom coming from stochastic intensity whose maturities, in rates like R's, are not always temporally aligned with the stochastic interest rates maturities. More precisely, the fact that in the numerator of the last term in Equation 21.10 we have not only $\bar{P}$ (second term in the numerator), but also a term in $D(t,T_j)\mathbf{1}_{\{\tau>T_{j-1}\}}$ (first term in the numerator, notice the misaligned T_{j-1} and T_j) adds degrees of freedom that are accounted for by considering two-period rates.

A final remark is that the freezing approximation is typically questionable when volatilities are very large. Since, as we will see below, at the moment implied volatilities in the CDS option market are rather large, the freezing approximation has to be considered with care.

21.6.3 Coterminal and One-Period CDS Rates Market Models

Our second choice is based on coterminal CDS rates. Indeed, let us take a family of CDS rates $R_{a,b}$, $R_{a+1,b}$, and $R_{b-1,b}$. Keep in mind that we are always referring to PR rates. Can we write the dynamics of all such rates under (say) $\hat{\mathbb{Q}}^{a,b}$? The answer is affirmative if we take into account the following equality, which is not difficult to prove with some basic algebra:

$$\hat{C}_{i,b}(t) = \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_b)\prod_{k=i+1}^{b-1}\frac{R_{k,b}(t)-R_k(t)}{R_{k-1,b}(t)-R_k(t)}, \quad i=a,\ldots,b-2 \tag{21.29}$$

Notice that the term in front of the product is just $\hat{C}_{b-1,b}(t)$. Notice also that the one-period rates canonical numeraires $\bar{P}$ can be obtained from the above numeraires via $\hat{C}_{i,b}-\hat{C}_{i-1,b}$'s. Take into account that we need to assume $R_{k-1,b}(t) \neq R_k(t)$. Analogously to what seen previously for the one- and two-period rates case, we can assume this to hold at time zero and then the probability that this condition is violated at future times will be zero in general under a diffusion dynamics for the relevant rates.

As before, the set of rates and of different numeraires ratios are not a closed system. To close the system we need to include one-period rates $R_{a+1},\ldots,R_b$. In this framework, we may derive the joint dynamics of $R_{a,b}$, $R_{a+1,b}$, $R_{a+2,b},\ldots,R_{b-1,b}$; R_{a+1}, R_{a+2}, R_{b-1} under a common measure (say $\hat{\mathbb{Q}}^{a,b}$) as follows. First assume a lognormal driftless geometric Brownian motion dynamics for $R_{a,b}$ under $\hat{\mathbb{Q}}^{a,b}$ and suitable martingale dynamics for every other rate under its canonical measure. These different dynamics have to be chosen so as to enforce the needed constraints on the $\hat{C}_{k,b}(t)$, such as for example $\hat{C}_{k-1,b}(t) > \hat{C}_{k,b}(t)$ and similar inequalities implying the correct behavior of the embedded $\bar{P}$'s. Take then a generic rate in the family and write its dynamics under $\hat{\mathbb{Q}}^{a,b}$ with the following method. Thanks to the change of numeraire technique, the drift of this rate dynamics under $\hat{\mathbb{Q}}^{a,b}$ will be a function of the quadratic covariation between the rate being modeled and the ratio of $\hat{C}_{a,b}(t)$ with the canonical numeraire of the selected rate itself. Thanks to Equation 21.29, this ratio is a function of the rates in the family and therefore the relevant quadratic covariation can be expressed simply as a suitable function of the volatilities and correlation (diffusion coefficients and instantaneous Brownian covariations are more precise terms) of the one- and multi-period rates in our family. As before no inconsistency is introduced, thanks to the additional degrees of freedom stemming from stochastic intensity. Under this second coterminal formulation, we can obtain the Black-like market formula above for the $T_a - T_b$ tenor in the context of a consistent and closed market model. The definition of suitable martingale dynamics for CDS rates with different tenor is under investigation.

21.6.4 Self-Contained Approximated CDS Rates Dynamics

Consider now the one-period CDS rates $R_j(t) = R_{j-1,j}(t)$. Go through the following approximation:

$$R_j(t) := R_{j-1,j}(t) = \text{LGD}\frac{\mathbb{E}[D(t,T_j)1_{\{T_{j-1}<\tau<T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

$$= \text{LGD}\frac{\mathbb{E}[D(t,T_j)1_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)1_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

$$= \text{LGD}\frac{\mathbb{E}\{D(t,T_{j-1})1_{\{\tau>T_{j-1}\}}\,D(t,T_j)/D(t,T_{j-1})|\mathcal{F}_t\} - \mathbb{E}[D(t,T_j)1_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)} = \cdots$$

At this point, we approximate the ratio of stochastic discount factors with the related zero-coupon bonds, obtaining

$$\cdots \approx \text{LGD}\frac{\mathbb{E}[D(t,T_{j-1})1_{\{\tau>T_{j-1}\}}|\mathcal{F}_t]P(t,T_j)/P(t,T_{j-1}) - \mathbb{E}[D(t,T_j)1_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j\mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

$$= \text{LGD}\frac{\bar{P}(t,T_{j-1})P(t,T_j)/P(t,T_{j-1}) - \bar{P}(t,T_j)}{\alpha_j\bar{P}(t,T_j)}$$

$$= \frac{\text{LGD}}{\alpha_j}\left\{\frac{\bar{P}(t,T_{j-1})}{[1+\alpha_jF_j(t)]\bar{P}(t,T_j)} - 1\right\} = \cdots$$

At this point, by inserting a LIBOR model for the forward LIBOR rates F_j we could proceed toward a joint dynamics for CDS rates and interest rates, both stochastic. However, let us simplify further the rates by freezing F_j to time zero:

$$\cdots \approx \frac{\text{LGD}}{\alpha_j}\left\{\frac{\bar{P}(t,T_{j-1})}{[1+\alpha_jF_j(0)]\bar{P}(t,T_j)} - 1\right\} := \tilde{R}_j(t)$$

This last definition can be inverted so as to have

$$\frac{\bar{P}(t,T_{j-1})}{\bar{P}(t,T_j)} = \left(\frac{\alpha_j}{\text{LGD}}\tilde{R}_j + 1\right)[1+\alpha_jF_j(0)] > 1 \tag{21.30}$$

as long as $\tilde{R} > 0$, provided that $F_j(0) > 0$ as should be. This means that we are free to select any martingale dynamics for $\tilde{R}_j$ under $\hat{\mathbb{Q}}^{j-1,j}$, as long as $\tilde{R}_j$ remains positive. Choose then such a family of $\tilde{R}$ as building blocks

$$d\tilde{R}_i(t) = \sigma_i(t)\tilde{R}_i(t)dZ_i^i(t) \quad \text{for all } i$$

where Z^i is a vector Brownian motion (with possibly correlated components having correlation matrix ρ) under the measure $\hat{\mathbb{Q}}^{i-1,i}$. Now one can define $\bar{P}$ by using Equation 21.30 to obtain inductively $\bar{P}(t,T_j)$ from $\bar{P}(t,T_{j-1})$ and from $\tilde{R}_j$. This way, the numeraires $\bar{P}$

become functions only of the $\tilde{R}$'s, so that now the system is closed and all one has to model is the one-period rates $\tilde{R}$ vector. No need to model auxiliary rates to be able to close the system.

In this context, the change of numeraire yields [see Brigo (2006) for details]

$$dZ_k^j = dZ_k^i - \sum_{h=j+1}^{i} \rho_{k,h} \frac{\sigma_h(t)\tilde{R}_h}{\tilde{R}_h + \frac{\text{LGD}}{\alpha_h}} \mathrm{d}t$$

from which we have the dynamics of $\tilde{R}_i$ under $\hat{\mathbb{Q}}^j$:

$$d\tilde{R}_i = \sigma_i \tilde{R}_i dZ_i^i = \sigma_i \tilde{R}_i \left(dZ_i^j + \sum_{h=j+1}^{i} \rho_{j,h} \frac{\sigma_h \tilde{R}_h}{\tilde{R}_h + \frac{\text{LGD}}{\alpha_h}} \mathrm{d}t \right) \tag{21.31}$$

At this point, a Monte Carlo simulation of the vector process R based on a discretization scheme for the above vector SDE (stochastic differential equation) is possible. One only needs to know the initial CDS rates $\tilde{R}(0)$, which if not directly available one can build by suitably stripping spot CDS rates. Given the volatilities and correlations, one can easily simulate the scheme by means of standard Gaussian shocks.

A final remark is in order: although our approximated family of rates $\tilde{R}$ is based on partial freezing of the F's, this does not mean that we are assuming deterministic interest rates (or independence between the interest rates and the default time) altogether. Indeed, the building blocks for our $\tilde{R}$ are the $\bar{P}$'s that embed possibly correlated stochastic interest rates and default events.

In Brigo (2006), this framework is applied to constant maturity CDS valuation, obtaining a closed form formula involving a convexity adjustment involving volatilities and correlations. See also Brigo and Mercurio (2006).

21.6.5 Market Model for Callable DFRN

Since we are also interested in the parallel with DFRNs, let us derive the analogous market model formula under PRCDSs of the first kind. The derivation goes through as above and we obtain easily the same model as in Equations 21.27 and 21.28 with R^{PR} replacing R everywhere.

If we consider the second kind of approximation for FRNs, the option price is obtained as the price of a CDS option, where the CDS is a postponed CDS of the second kind. Compute then

$$\begin{aligned}
&\mathbb{E}\big(D(t,T_a)\{\text{PR2CDS}\big(T_a,T_a,T_b,R^{\text{PR2}}_{a,b}(T_a),\text{LGD}\big) - \text{PR2CDS}(T_a,T_a,T_b,K,\text{LGD})\}^+|\mathcal{G}_t\big) \\
&= \mathbb{E}\left\{ \frac{1_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} D(t,T_a) \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_{T_a}\big[D(T_a,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}\big]\big[R^{\text{PR2}}_{a,b}(T_a)-K\big]^+|\mathcal{G}_t \right\} \\
&= \mathbb{E}\left\{ 1_{\{\tau>T_a\}} D(t,T_a) \sum_{i=a+1}^{b} \alpha_i \frac{\mathbb{E}_{T_a}[D(T_a,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\big[R^{\text{PR2}}_{a,b}(T_a)-K\big]^+|\mathcal{G}_t \right\} = \cdots
\end{aligned}$$

This time let us take as numeraire

$$\check{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}] \quad \left(\text{notice } \hat{C}_{a,b}(t) = \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}]\right)$$

This quantity is positive, and when including the indicator $1_{\{\tau>t\}}$ this is, not surprisingly, a multiple of the premium leg of a PR2CDS at time t. We may also view it as a (no survival indicator) portfolio of defaultable bonds where the default maturity is one-period displaced with respect to the payment maturity. Thus this quantity is only approximately a numeraire. Compute

$$\begin{aligned}
\cdots &= \mathbb{E}\left\{1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+ | \mathcal{G}_t\right\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+\right\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left(\mathbb{E}_{T_a}\left\{1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+\right\}\right) \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+ \mathbb{E}_{T_a}(1_{\{\tau>T_a\}})\right\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \mathbb{E}_t\left\{D(t,T_a)\check{C}_{a,b}(T_a)\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+\right\} \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)} \check{C}_{a,b}(t) \check{\mathbb{E}}^{a,b}_t\left\{\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+\right\}
\end{aligned}$$

Now notice that R^{PR2} can also be written as

$$R^{\mathrm{PR2}}_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\check{C}_{a,b}(t)}$$

so that it is a martingale under $\check{\mathbb{Q}}^{a,b}$. As such, we may model it as

$$dR^{\mathrm{PR2}}_{a,b}(t) = \sigma_{a,b} R^{\mathrm{PR2}}_{a,b}(t) d\check{W}^{a,b}(t) \tag{21.32}$$

and compute the above expectation accordingly. We obtain, as price of the option,

$$\begin{aligned}
&\mathbb{E}\{1_{\{\tau>T_a\}} D(t,T_a) \frac{\check{C}_{a,b}(T_a)}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})}\left[R^{\mathrm{PR2}}_{a,b}(T_a)-K\right]^+ | \mathcal{G}_t\} \\
&\quad = 1_{\{\tau>t\}} \frac{\check{C}_{a,b}(t)}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\left[R^{\mathrm{PR2}}_{a,b}(t)N(d_1) - KN(d_2)\right]
\end{aligned} \tag{21.33}$$

where d_1 and d_2 are defined as usual in terms of $R^{\mathrm{PR2}}_{a,b}(t)$, K, and σ.

Which model should one use between DFRN1 and DFRN2 when dealing with DFRN options? DFRN1 has the advantage of better approximating the real DFRN; further, the

TABLE 21.1 CDS Forward Rates and Implied Volatilites on Three Companies on March 26, 2004. Rates Are in Basis Points and Volatilities Are Percentages

Company	Option: bid	mid	ask	$R_{0,b}(0)$	$R^{PR}_{a,b}(0)$	$R^{PR2}_{a,b}(0)$	K	$\sigma^{PR}_{a,b}$	$\sigma^{PR(2)}_{a,b}$
C1(T_a)	14	24	34	60	61.497	61.495	60	50.31	50.18
C2	32	39	46	94.5	97.326	97.319	94	54.68	54.48
C3	18	25	32	61	62.697	62.694	61	52.01	51.88
C1(T'_a)	28	35	42	60	65.352	65.344	61	51.45	51.32

related market model is derived under a numeraire. DFRN2 is derived only under an approximated numeraire and is a worse approximation of the real DFRN, but the related CDS payoff PR2CDS is in some cases a better approximation of a real CDS than PRCDS.

We present now some CDS options implied volatilities obtained with the postponed payoff of the first and second kind. We consider three companies C1, C2, and C3 on the euro market and the related CDS options quotes as of March 26, 2004; the recovery is REC = 0.4; C1 and C3 are in the telephonic sector, whereas C2 is a car industry; LGD = $1 - 0.4 = 0.6$; T_0 = March 26, 2004 (0). We consider two possible maturities T_a = June 20, 2004 (86 days $\approx$ 3 months) and T'_a = Dec 20, 2004 (269 days $\approx$ 9 months); T_b = June 20, 2009 (5 years and 87 days); we consider receiver option quotes (puts on R) in basis points (i.e., 1E-4 units on a notional of 1). We obtain the results presented in Table 21.1.

Implied volatilities are rather high when compared with typical interest-rate default-free swaption volatilities. However, the values we find have the same order of magnitude as some of the values found by Hull and White (2003) via historical estimation. Further, we see that while the option prices differ considerably, the related implied volatilities are rather similar. This shows the usefulness of a rigorous model for implied volatilities. The mere price quotes could have left one uncertain on whether the credit spread variabilities implicit in the different companies were quite different from each other or similar.

We analyze also the implied volatilities and CDS forward rates under different payoff formulations and under stress. Table 21.1 shows that the impact of changing postponement from PR to PR2 (maintaining the same $R_{0,b}(0)$'s and re-stripping Γ's) leaves both CDS forward rates and implied volatilities almost unchanged.

In Table 21.2, we check the impact of the recovery rate on implied volatilities and CDS forward rates. Every time we change the recovery we recalibrate the Γ's, since the only direct market quotes are the $R_{0,b}(0)$'s, which we cannot change, and our uncertainty is on the recovery rate that might change. As we can see from the table, the impact of the recovery rate is rather small, but we have to keep in mind that the CDS option payoff is built in such a way that the recovery direct flow in LGD cancels and the recovery remains only implicitly inside the initial condition $R_{a,b}(0)$ for the dynamics of $R_{a,b}$, as one can see for example from Equation 21.22, where LGD does not appear explicitly. It is $R_{a,b}(0)$ that depends on the stripped Γ's which, in turn, depend on the recovery.

TABLE 21.2 Impact of Recovery Rates on the Implied Volatility and on the CDS Forward Rates for the PR Payoff. Volatilities Are Expressed as Percentages and Rates as Basis Points

Company	REC = 20%	REC = 30%	REC = 40%	REC = 50%	REC = 60%
$\sigma^{PR}_{a,b}$:					
C1(T_a)	50.02	50.14	50.31	50.54	50.90
C2	54.22	54.42	54.68	55.05	55.62
C3	51.71	51.83	52.01	52.25	52.61
C1(T'_a)	51.13	51.27	51.45	51.71	52.10
$R^{PR}_{a,b}$:					
C1(T_a)	61.488	61.492	61.497	61.504	61.514
C2	97.303	97.313	97.326	97.346	97.374
C3	62.687	62.691	62.697	62.704	62.716
C1(T'_a)	65.320	65.334	65.352	65.377	65.415

In Table 21.3, we check the impact of a shift in the simply compounded rates of the zero-coupon interest-rate curve on CDS forward rates and implied volatilities. Every time we shift the curve we recalibrate the Γ's, while maintaining the same $R_{0,b}(0)$'s. We see that the shift has a more relevant impact than the recovery rate, an impact that remains small.

We also include the zero-coupon curve we used in Table 21.4 and the CDS market quotes we used in Table 21.5.

Finally, we consider the possibility of including a volatility smile in our CDS options model. Since the derivation is general, we may replace the dynamics Equation 21.27 or 21.32 by a different local volatility dynamics

$$dR^{PR}_{a,b}(t) = \nu_{a,b}\left[t, R^{PR}_{a,b}(t)\right] R^{PR}_{a,b}(t) d\check{W}a,b(t)$$

with ν as a suitable deterministic function of time and state. We might choose the constant elasticity of variance (CEV) dynamics, a displaced diffusion dynamics, an hyperbolic sine densities mixture dynamics, or a lognormal mixture dynamics. Several tractable choices are possible already in the local volatility diffusion setup, and one may select a smile dynamics for the LIBOR or swap model and use it to model R. There are several possible choices. For example, one may select $\nu_{a,b}$ from Brigo and Mercurio (2003) or Brigo et al. (2003).

TABLE 21.3 Implied Volatilities $\sigma_{a,b}$ (Left, as Percentages) and Forward CDS Rates $R^{PR}_{a,b}$ (Right, as Basis Points) as the Simply Compounded Rates Are Shifted Uniformly for All Maturities

	Implied Volatilities			Forward CDS		
Company	Shift −0.5%	0	+0.5%	Shift −0.5%	0	+0.5%
C1(T_a)	49.68	50.31	50.93	61.480	61.497	61.514
C2	54.02	54.68	55.34	97.294	97.326	97.358
C3	51.36	52.01	52.65	62.677	62.697	62.716

TABLE 21.4 Euro Curve for Zero-Coupon Bonds $P(0,T)$ as of March 26, 2004

Date	Discount	Date	Discount	Date	Discount
26-Mar-04	1	30-Dec-04	0.985454616	28-Mar-13	0.701853679
29-Mar-04	0.999829196	30-Mar-06	0.956335676	31-Mar-14	0.665778313
31-Mar-04	0.9997158	30-Mar-07	0.9261161	30-Mar-15	0.630686684
06-Apr-04	0.999372341	31-Mar-08	0.891575268	30-Mar-16	0.597987523
30-Apr-04	0.99806645	30-Mar-09	0.85486229	30-Mar-17	0.566052224
31-May-04	0.996398755	30-Mar-10	0.816705705	29-Mar-18	0.535085529
30-Jun-04	0.994847843	30-Mar-11	0.777867013	29-Mar-19	0.505632535
30-Sep-04	0.99014189	30-Mar-12	0.739273058		

21.7 CDS OPTION PRICING WITH THE SSRD STOCHASTIC INTENSITY MODEL

In this final section, we move to explicit modeling of the stochastic intensity process λ driving the Cox process whose first jump-time represents the default time τ. We consider the shifted square root diffusion (SSRD) model introduced in Brigo and Alfonsi (2003).

21.7.1 SSRD Intensity and Interest Rates Models

We now describe our assumptions on the short-rate process r and on the intensity λ dynamics. For more details on shifted r diffusion dynamics see Brigo and Mercurio (2001, 2006).

21.7.1.1 CIR++ Interest-Rate Model

We write the short-rate r_t as the sum of a deterministic function φ and of a Markovian process x_t^α (Brigo and Mercurio 2001):

$$r_t = x_t^\alpha + \varphi(t;\alpha), \quad t \geq 0 \tag{21.34}$$

where φ depends on the parameter vector α (which includes x_0^α) and is integrable on closed intervals. Notice that x_0^α is indeed one more parameter at our disposal: we are free to select its value as long as $\varphi(0;\alpha) = r_0 - x_0$. We take as reference model for x the CIR process:

$$\mathrm{d}x_t^\alpha = k(\theta - x_t^\alpha)\mathrm{d}t + \sigma\sqrt{x_t^\alpha}dW_t$$

TABLE 21.5 Quoted CDS Rates for the Three Companies in Basis Points as of March 26, 2004

Maturity T_b (year)	$R_{0,b}$ (C1)	$R_{0,b}$ (C2)	$R_{0,b}$ (C3)
1	30	38.5	27
3	49	72.5	49
5	60	94.5	61
7	69	104.5	73

where W is an $\mathcal{F}_t$-Brownian motion, and the parameter vector is $\alpha = (k, \theta, \sigma, x_0^\alpha)$, with k, θ, σ, and x_0^α as positive deterministic constants. The condition $2k\theta > \sigma^2$ ensures that the origin is inaccessible to the reference model, so that the process x^α is well defined and remains positive. As is well known, this process x^α features a noncentral chi-square distribution, and yields an affine term-structure of interest rates. Denote by f instantaneous forward rates, i.e., $f(t, T) = -\partial \ln P(t, T)/\partial T$. The initial market zero-coupon interest-rate curve $T \mapsto P^M(0, T)$ is automatically calibrated by our model if we set $\varphi(t; \alpha) = \varphi^{\text{CIR}}(t; \alpha)$ where $\varphi^{\text{CIR}}(t; \alpha) = f^M(0, t) - f^{\text{CIR}}(0, t; \alpha)$,

$$f^{\text{CIR}}(0, t; \alpha) = \frac{2k\theta(\mathrm{e}^{th} - 1)}{2h + (k + h)(\mathrm{e}^{th} - 1)} + x_0 \frac{4h^2 \mathrm{e}^{th}}{[2h + (k + h)(\mathrm{e}^{th} - 1)]^2}$$

with $h = \sqrt{k^2 + 2\sigma^2}$. For restrictions on the α's that keep r positive see Brigo and Mercurio (2001, 2006). Moreover, the price at time t of a zero-coupon bond maturing at time T is

$$P(t, T) = \frac{P^M(0, T)A(0, t; \alpha) \exp[-B(0, t; \alpha)x_0]}{P^M(0, t)A(0, T; \alpha) \exp[-B(0, T; \alpha)x_0]} P^{\text{CIR}}\left(t, T, r_t - \varphi^{\text{CIR}}(t; \alpha); \alpha\right) \tag{21.35}$$

where $P^{\text{CIR}}(t, T, x_t; \alpha) = \mathbb{E}\{\exp[-\int_t^T x^\alpha(u)\,\mathrm{d}u] | \mathcal{F}_t\} = A(t, T; \alpha) \exp[-B(t, T; \alpha)x_t]$ is the bond price formula for the basic CIR model with the classical expressions for A and B given for example in Brigo and Mercurio (2006). From P's the spot LIBOR rate $L(t,T)$ at t for maturity T, the forward LIBOR rates $F(t, T, S)$ at t for maturity T and expiry S, and all other rates can be computed as explicit functions of r_t.

The cap price formula for the CIR++ model can be derived in closed form from the corresponding formula for the basic CIR model. This formula is a function of the parameters α. One may calibrate the parameters α to cap prices, by inverting the analytical CIR++ formula, so that the interest-rate model is calibrated to the initial zero-coupon curve through φ and to the cap market through α, as in Brigo and Mercurio (2001, 2006).

21.7.1.2 CIR++ Intensity Model

For the intensity model, we adopt a similar approach, in that we set (Brigo and Alfonsi 2003)

$$\lambda_t = y_t^\beta + \psi(t), \quad t \geq 0 \tag{21.36}$$

where ψ is a positive deterministic function that is integrable on closed intervals. As before, the parameter vector is $\beta = (\kappa, \mu, \nu, y_0^\beta)$, with κ, μ, ν, and y_0^β as positive deterministic constants such that $2\kappa\mu > \nu^2$, and we take y again of the form

$$\mathrm{d}y_t^\beta = \kappa(\mu - y_t^\beta)\,\mathrm{d}t + \nu\sqrt{y_t^\beta}\,\mathrm{d}Z_t$$

where the process Z is an $\mathcal{F}_t$-Brownian motion. This ensures that λ be strictly positive, as should be for an intensity process. Notice incidentally that this basically forces the choice of a tractable y to the CIR model among all one-factor short-rate diffusion

models. Dependence of ψ on β and possibly on other parameters will be specified later when dealing with CDS calibration. We will often use the integrated process, that is $\Lambda(t) = \int_0^t \lambda_s \, ds$, and also $Y^\beta(t) = \int_0^t y_s^\beta \, ds$ and $\Psi(t) = \int_0^t \psi(s) \, ds$. We assume the short rate r and the intensity λ processes to be correlated, by assuming the driving Brownian motions W and Z to be instantaneously correlated according to $dW_t dZ_t = \rho \, dt$.

21.7.2 Joint SSRD Model Calibration to CDS: Separability

The SSRD model is characterized by the terms $\mathcal{P} = (\alpha, \varphi, \beta, \psi, \rho)$ and can be seen as an extension of the CIR++ model for interest rates. As we explained before, φ is chosen to fit exactly the default-free zero-coupon bonds and α is then selected to have the better approximation of the cap prices. This procedure for the CIR++ interest-rate part of the model r still remains valid in presence of a correlated λ since the products used for that calibration do not depend on the dynamics of λ and of (β,ρ,ψ).

Once α and φ are fixed, we would like to fit the three remaining terms to the credit derivatives market. To do so, we need first to calculate the price of CDSs in the SSRD model. We find easily, through iterated expectations and the definition of τ, that (see Brigo and Alfonsi (2003) for the details)

$$\begin{aligned}\mathrm{CDS}(0, T_a, T_b, R, \mathrm{LGD}; \mathcal{P}) = {} & R \int_{T_a}^{T_b} \mathbb{E}\left\{ \exp\left[- \int_0^u (r_s + \lambda_s) \, ds \right] \lambda_u \right\} [u - T_{\beta(u)-1}] \, du \\ & + \sum_{i=a+1}^{b} \alpha_i R \mathbb{E}\left\{ \exp\left[- \int_0^{T_i} (r_s + \lambda_s) \, ds \right] \right\} \\ & - \mathrm{LGD} \int_{T_a}^{T_b} \mathbb{E}\left\{ \exp\left[- \int_0^u (r_s + \lambda_s) \, ds \right] \lambda_u \right\} du \end{aligned} \tag{21.37}$$

We plan to use ψ to calibrate exactly the market CDS quotes (given for $T_a = 0$ and T_b spanning a set of increasing final maturities). More precisely, we want to find, for each (β,ρ), a function $\psi_\alpha(\cdot;\beta,\rho)$ that makes the CDS present values null, $\mathrm{CDS}\left(0, 0, T_b, R_{0,b}^{\mathrm{MID}}(0), \mathrm{LGD}, \mathcal{P}\right) = 0$. This could be done if we were able to calculate analytically the above expectations in general, taking as in the deterministic case a specific shape for ψ_α. Since these expectations are known only when $\rho = 0$, we first restrict ourselves to calibrate the subclass of models with $\rho = 0$. Interest rates and default intensities are independent with $\rho = 0$. By switching expectation and differentiation with respect to u and Fubini's theorem, it is easy to see that the price of the CDS satisfies the deterministic case Equation 21.9 when replacing terms such as $\exp(\Gamma(t) - \Gamma(u))$ by $\mathbb{E}\{\exp(\Lambda(t) - \Lambda(u))\}$ (with $u \geq t$). Therefore, at time $t = 0$, for any β we can calibrate automatically our model to the CDS by choosing ψ such that

$$e^{-\Gamma^{\mathrm{mkt}}(u)} = \mathbb{E}\left[e^{-\Lambda(u)}\right] = e^{-\Psi(u)} \mathbb{E}\left[e^{-Y^\beta(u)}\right] = e^{-\Psi(u)} P^{\mathrm{CIR}}(0, u, y_0; \beta)$$

The remarkable point is that ψ does not depend on α (the zero-coupon bonds have been calibrated exactly earlier), so that this calibration to CDS can be done independently of the

interest-rate calibration. This separability is of practical interest. We thus denote by $\psi(.;\beta)$ the obtained ψ function, given by

$$\psi(u;\beta) = \gamma^{\text{mkt}}(u) + \frac{\mathrm{d}}{\mathrm{d}u}\ln\left(\mathbb{E}\left[\mathrm{e}^{-Y^{\beta}(u)}\right]\right) = \gamma^{\text{mkt}}(u) + \frac{\mathrm{d}}{\mathrm{d}u}\ln\left(P^{\text{CIR}}(0,u,y_0;\beta)\right) \tag{21.38}$$

The shape of ψ is partly implicitly specified by our choice for γ^{mkt} (piecewise linear or otherwise).

So far we have described an analytical and exact calibration of the SSRD model in case $\rho = 0$. However, numerical tests in Brigo and Alfonsi (2003) show that ρ has practically a negligible impact on CDS prices computed under the SSRD model. Therefore, we may assume $\rho = 0$, even if this is not true, and calibrate the model with the above procedure. The error induced by this approximation will be negligible.

Now that CDSs are automatically calibrated, we would like to calibrate the parameters β to some options on the credit derivatives market in the same way as α is used to fit cap prices. To do this, we need a way to compute CDS options prices in the SSRD model. We will see a formula where this is possible under deterministic r (and stochastic CIR++ CDS-calibrated λ), and provide some hints on possible solutions in presence of stochastic r and nonzero ρ as well.

21.7.3 CDS Options Pricing with the Calibrated CIR++ λ Model

We developed this formula by an initial hint of Ouyang (2003). Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, receiving protection LGD against default up to time T_b, in exchange for a fixed rate K. We have that the payoff at T_a reads, as we have seen earlier, as

$$\begin{aligned}
\Pi_a :&= \Pi\text{CallCDS}_{a,b}(T_a) = \{\text{CDS}(T_a,T_a,T_b,R_{a,b}(T_a),\text{LGD}) - \text{CDS}(T_a,T_a,T_b,K,\text{LGD})\}^+ \\
&= [-\text{CDS}(T_a,T_a,T_b,K,\text{LGD})]^+ \\
&= \mathbf{1}_{\{\tau>T_a\}}\left(\mathbb{E}\left\{ - D(T_a,\tau)[\tau - T_{\beta(\tau)-1}]K\mathbf{1}_{\{\tau<T_b\}}\right.\right. \\
&\quad \left.\left. - \sum_{i=a+1}^{b} D(T_a,T_i)\alpha_i K\mathbf{1}_{\{\tau>T_i\}} + \mathbf{1}_{\{\tau<T_b\}}D(T_a,\tau)\text{LGD}|\mathcal{G}_{T_a}\right\}\right)^+ \\
&= \mathbf{1}_{\{\tau>T_a\}}\left(-K\int_{T_a}^{T_b}\mathbb{E}\left\{\exp\left[-\int_{T_a}^{u}(r_s+\lambda_s)\,\mathrm{d}s\right]\lambda_u|\mathcal{F}_{T_a}\right\}[u - T_{\beta(u)-1}]\,\mathrm{d}u\right. \\
&\quad - K\sum_{i=a+1}^{b}\alpha_i\mathbb{E}\left\{\exp\left[-\int_{T_a}^{T_i}(r_s+\lambda_s)\mathrm{d}s\right]|\mathcal{F}_{T_a}\right\} \\
&\quad \left. + \text{LGD}\int_{T_a}^{T_b}\mathbb{E}\left\{\exp\left[-\int_{T_a}^{u}(r_s+\lambda_s)\,\mathrm{d}s\right]\lambda_u|\mathcal{F}_{T_a}\right\}\mathrm{d}u\right)^+
\end{aligned}$$

If we take deterministic interest rates r this reads

$$\begin{aligned}\Pi_a = 1_{\{\tau>T_a\}}\Bigg\{&-K\int_{T_a}^{T_b}\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s\,\mathrm{d}s\right)\lambda_u|\mathcal{F}_{T_a}\right]P(T_a,u)[u-T_{\beta(u)-1}]\,\mathrm{d}u\\&-K\sum_{i=a+1}^{b}\alpha_i P(T_a,T_i)\mathbb{E}\left[\exp\left(-\int_{T_a}^{T_i}\lambda_s\,\mathrm{d}s\right)|\mathcal{F}_{T_a}\right]\\&+\mathrm{LGD}\int_{T_a}^{T_b}P(T_a,u)\mathbb{E}\left[\exp\left(-\int_{T_a}^{u}\lambda_s\,\mathrm{d}s\right)\lambda_u|\mathcal{F}_{T_a}\right]\mathrm{d}u\Bigg\}^+\end{aligned}$$

Define

$$H\left(t,T;y_t^\beta\right):=\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s\,\mathrm{d}s\right)|\mathcal{F}_t\right]$$

and notice that

$$\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s\,\mathrm{d}s\right)\lambda_T|\mathcal{F}_t\right]=-\frac{\mathrm{d}}{\mathrm{d}T}\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s\,\mathrm{d}s\right)|\mathcal{F}_t\right]=-\frac{\mathrm{d}}{\mathrm{d}T}H(t,T)$$

Write then

$$\begin{aligned}\Pi_a = 1_{\{\tau>T_a\}}\Bigg\{&K\int_{T_a}^{T_b}P(T_a,u)[u-T_{\beta(u)-1}]\frac{\mathrm{d}}{\mathrm{d}u}H(T_a,u)\,\mathrm{d}u\\&-K\sum_{i=a+1}^{b}\alpha_i P(T_a,T_i)H(T_a,T_i)-\mathrm{LGD}\int_{T_a}^{T_b}P(T_a,u)\frac{\mathrm{d}}{\mathrm{d}u}H(T_a,u)\,\mathrm{d}u\Bigg\}^+\end{aligned}$$

Note that the first two summations add up to a positive quantity, since they are expectations of positive terms. By integrating by parts in the first and third integral, we obtain, by defining $q(u):=-\mathrm{d}P(T_a,u)/\mathrm{d}u$,

$$\begin{aligned}\Pi_a = 1_{\{\tau>T_a\}}\Bigg(&\mathrm{LGD}-\int_{T_a}^{T_b}\{\mathrm{LGD}q(u)+KP\left(T_a,T_{\beta(u)}\right)\delta_{T_{\beta(u)}}(u)-K[u-T_{\beta(u)-1}]q(u)\\&-KP\left(T_a,T_{\beta(u)}\right)\delta_{T_{\beta(u)}}(u)+\mathrm{LGD}\delta_{T_b}(u)P(T_a,u)+K\,P(T_a,u)\}H(T_a,u)\,\mathrm{d}u\Bigg)^+\end{aligned}$$

where δ_x denotes the Dirac delta function centered at x. Define

$$h(u):=\mathrm{LGD}q(u)-K[u-T_{\beta(u)-1}]q(u)+\mathrm{LGD}\delta_{T_b}(u)P(T_a,u)+KP(T_a,u)$$

so that

$$\Pi_a = 1_{\{\tau>T_a\}}\left[\mathrm{LGD}-\int_{T_a}^{T_b}h(u)H\left(T_a,u;y_{T_a}^\beta\right)\mathrm{d}u\right]^+ \tag{21.39}$$

It is easy to check, by remembering the signs of the terms of which the above coefficients are expectations, that

$$h(u) > 0 \quad \text{for all } u$$

Now we look for a term $y^\star$ such that

$$\int_{T_a}^{T_b} h(u)H(T_a, u; y^\star)\,\mathrm{d}u = \mathrm{LGD} \tag{21.40}$$

It is easy to see that in general $H(t, T; y)$ is decreasing in y for all t, T. This equation can be solved, since $h(u)$ is known and deterministic and since H is given in terms of the CIR bond price formula. Furthermore, either a solution exists or the option valuation is not necessary. Indeed, consider first the limit of the left-hand side for $y^\star \to \infty$. We have

$$\lim_{y^\star \to \infty} \int_{T_a}^{T_b} h(u)H(T_a, u; y^\star)\,\mathrm{d}u = 0 < \mathrm{LGD}$$

which shows that for $y^\star$ large enough we always go below the value LGD. Then consider the limit of the left-hand side for $y^\star \to 0$:

$$\lim_{y^\star \to 0+} \int_{T_a}^{T_b} h(u)H(T_a, u; y^\star)\,\mathrm{d}u = \mathrm{LGD} + \int_{T_a}^{T_b} \Big(\mathrm{LGD}P(T_a, u)\frac{\partial H(T_a, u; 0)}{\partial u} + \{K[u - T_{\beta(u)-1}]q(u) + KP(T_a, u)\}H(T_a, u; 0)\Big)\mathrm{d}u$$

Now if the integral in the last expression is positive then we have that the limit is larger than LGD and by continuity and monotonicity there is always a solution $y^\star$ giving LGD. If instead the integral in the last expression is negative, then the limit is smaller than LGD and we have that Equation 21.40 admits no solution, in that its left-hand side is always smaller than the right-hand side. However, this implies in turn that the expression inside square brackets in the payoff Equation 21.39 is always positive and thus the contract loses its optionality and can be valued by taking the expectation without positive part, giving as option price simply $-\mathrm{CDS}(0, T_a, T_b, K, \mathrm{LGD}) > 0$, the opposite of a forward start CDS. In case $y^\star$ exists, instead, we may rewrite our discounted payoff as

$$\Pi_a = 1_{\{\tau > T_a\}} \left\{ \int_{T_a}^{T_b} h(u) \left[H(T_a, u; y^\star) - H\left(T_a, u; y^\beta_{T_a}\right) \right] \mathrm{d}u \right\}^+$$

Since $H(t, T; y)$ is decreasing in y for all t, T, all terms $\left[H(T_a, u; y^\star) - H\left(T_a, u; y^\beta_{T_a}\right)\right]$ have the same sign, which will be positive if $y^\beta_{T_a} > y^\star$ or negative otherwise. Since all such terms have the same sign, we may write

$$\Pi_a := 1_{\{\tau > T_a\}} Q_a = 1_{\{\tau > T_a\}} \left\{ \int_{T_a}^{T_b} h(u) \left[H(T_a, u; y^\star) - H\left(T_a, u; y^\beta_{T_a}\right) \right]^+ \mathrm{d}u \right\}$$

TABLE 21.6 Implied Volatility Patterns in the CIR++ Model

Parameter	$\kappa \uparrow$	$\mu \uparrow$	$\nu \uparrow$	$y_0 \uparrow$	$\rho \uparrow$
$\sigma_{a,b}^{\text{imp}}$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$

Now compute the price as

$$\mathbb{E}[D(0,T_a)\Pi_a] = P(0,T_a)\mathbb{E}(1_{\{\tau>T_a\}}Q_a) = P(0,T_a)\mathbb{E}\left[\exp\left(-\int_0^{T_a}\lambda_s\,\mathrm{d}s\right)Q_a\right]$$
$$= \int_{T_a}^{T_b} h(u)\mathbb{E}\left\{\exp\left(-\int_0^{T_a}\lambda_s\,\mathrm{d}s\right)\left[H\left(T_a,u;y^\star\right) - H\left(T_a,u;y^\beta_{T_a}\right)\right]^+\right\}\mathrm{d}u$$

From a structural point of view, $H(T_a,u;y^\beta_{T_a})$ are like zero-coupon bond prices in a CIR++ model with short-term interest-rate λ, for maturity T_a on bonds maturing at u. Thus, each term in the summation is $h(u)$ times a zero-coupon bond like call option with strike $K^\star_u = H(T_a,u;y^\star)$. A formula for such options is given for example in Equation 3.78 p. 94 of Brigo and Mercurio (2001).

If one maintains stochastic interest rates with possibly non-null ρ, then a possibility is to use the Gaussian mapped processes x^V and y^V introduced in Brigo and Alfonsi (2003) and to reason as for pricing swaptions with the G2++ model through Jamshidian's decomposition and one-dimensional Gaussian numerical integration, along the lines of the procedures in Brigo and Mercurio (2006). Clearly the resulting formula has to be tested against Monte Carlo simulation.

Finally, in the general SSRD model, one may compute the CDS option price by means of Monte Carlo simulations, equate this Monte Carlo price to Equation 21.28 applied to the same CDS option at $t=0$, and solve in $\sigma_{a,b}$. This $\sigma_{a,b}$ is then the implied volatility corresponding to the SSRD pricing model. The first numerical results we found in a number of cases point out the patterns of $\sigma_{a,b}$ in terms of SSRD model parameters that are listed in Table 21.6. For more details see Brigo and Cousot (2006). The patterns are reasonable. When κ increases (all other things being equal) the time-homogeneous core of the stochastic intensity has a higher speed of mean reversion and then randomness reduces more quickly in time, so that the implied volatility reduces. When μ increases, the asymptotic mean of the homogeneous part of the intensity increases, so that we have higher intensity and thus, since instantaneous volatility is proportional to the increased $\sqrt{y}$, more randomness. When ν increases, clearly randomness of λ increases so that it is natural for the implied volatility to increase. We also find that increasing y_0 (the initial point of the time-homogeneous part of the intensity) increases the implied volatility, while increasing the correlation ρ decreases the implied volatility.

21.8 CONCLUSIONS AND FURTHER RESEARCH

We considered several CDS payoff formulations. For some of them, we established equivalence with approximated defaultable floaters. We explained the CDS market quoting mechanisms and considered CDS pricing in an intensity framework. We derived a market

model for CDS options, and thus for callable defaultable floaters, given the above equivalence. We hinted at possible CDS option smile models and at a comparison of the market models with classic stochastic intensity models, such as for example the CDS-calibrated CIR++ model in Brigo and Alfonsi (2003), a comparison that is under more detailed investigation in Brigo and Cousot (2006). We also gave a CDS option formula under CIR++ stochastic intensity based on Jamshidian's decomposition. Moreover, further investigation on the possibility to link different CDS forward rate models, based on a fundamental set of candidate liquid CDS rates, is to be investigated, starting from the observed parallels with the LIBOR versus swap market models in the default-free interest rates derivatives setting. As a beginning, we provide in Brigo (2006) a first study of constant maturity CDS pricing under said parallels, where we also provide the explicit vector dynamics for all CDS rates under a single pricing measure, a vector dynamics that we have briefly recalled also here.

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CHAPTER 22

Arbitrage Pricing of Credit Derivatives*

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* Part of the results on single-name credit derivatives had appeared in a working paper under the title of "To recover or not to recover: that is not the question," by Wu (2005).

22.1 INTRODUCTION

In recent years, investors of credit markets have witnessed a rapid growth of liquidity in credit default swaps (CDS), options on the CDS (hereafter swaptions), tranches of collateralized debt obligations (CDO), and, more recently, exotic portfolio credit derivatives like bespoke single-tranche CDOs and CDOs of CDOs (so-called CDO^2). Evolving with the markets are the various pricing theories. For the pricing of single-name swaptions, market practitioners and a series of researchers, including Schönbucher (2000), Arvanitis and Gregory (2001), Jamshidian (2002), and Hull and White (2003), among others, have led to the adoption of the Black's formula as the market standard. For the pricing of CDO tranches, where the modeling of dependent defaults is essential, there are survival-time copula models represented by Li (2000), Gregory and Laurent (2003), Andersen, Sidenius, and Basu (2003), and Giesecke (2003); and structural models represented by Zhou (2001) and Hull et al. (2005). The Gaussian copula model formulated by Gregory and Laurent (2003) and later improved by Andersen et al. (2003) is quite well received and has also become the market standard. Despite the successes, there are still major problems remaining. A fundamental problem is, from modeling point of view, the detachment of the single-name credit derivatives market from the portfolio credit derivatives market, as there is no unified pricing framework for both markets yet. As a result, the notions of consistent pricing and hedging across the two markets are difficult to define. Although the Gaussian copula model can take the default probabilities estimated from the CDS markets as inputs, it has no capacity to utilize either spread dynamics or spread correlations observed in the CDS markets for CDO pricing.

There are also limitations with the popular portfolio credit derivatives models mentioned above. One of the major drawbacks of the survival-time copula approach for CDOs is that it does not constitute a proper dynamic model. The implication is that although this kind of models can be used to price CDO tranches, options on spread, of either single name or portfolio, cannot be priced under such models (e.g., exponential-copula model of Giesecke [2003]). With the Gaussian copula model, the market standard, default-time correlations are injected after the default times are mapped into normal random variables. There is, however, no economic intuition on how to input the correlations for these normal random variables. This problem is mitigated in the structural model of Hull et al. (2005), where a default is triggered when the value of a firm breaches a barrier (Black and Cox, 1976). However, structural models are difficult to use, as a number of model parameters are needed to be specified, including drifts, volatility, and correlations of the firm values, as well as default barriers. Given that firm values are not observable, a proper specification of the model is a daunting task by itself. In addition, firm-value based structural models do not naturally take the price information like credit spreads and implied spread volatilities from the single-name credit derivatives markets as inputs, hence the structural models do not ensure price consistency across the single-name and portfolio credit markets.

In this chapter, we intend to deliver a unified framework for the pricing of CDS options and CDOs. Motivated by the use of the Black's formula for the single-name swaptions, we developed a market model with forward credit spreads. Unlike usual structural models

with firm values, the state variables of our market model are observable quantities. For the pricing of single-name credit derivatives, the new model bears high analogy to the LIBOR market model. In a very natural way, such a model can be applied to pricing CDO tranches, using Monte Carlo simulations. Dynamically evolving CDS rates, CDS-rate correlations, and implied swaption volatilities, which are all observable in credit markets, are all utilized for pricing the codependence of defaults of credit portfolios. In addition, the new model also has the capacity to price default-time correlations, through using the technique of survival-time copulas.

Our dynamic model is developed under the notion of arbitrage pricing. We define risky zero-coupon bonds for a credit name as cash flows backed by the coupons of corporate bonds. With such a definition, risky zero-coupon bonds, the basic building blocks of our model, become tradable. Consequently, forward spreads can be replicated with risky bonds, and the hedging of CDS and CDS options can then be readily defined. If liquidity and other market insufficiency are ignored, we can calibrate the market model to risky bonds and spread derivatives (CDS and CDS options), and thus excludes arbitrage opportunities across these two markets. Survival-time copula, in principle, is the only input (for the model) that is unobservable in the credit market. When the volatilities of forward spreads of all names are set to zero, our market model reduces to the popular Gaussian copula model (Li, 2000; Gregory and Laurent, 2003). Hence, our model can be regarded as a dynamic extension of the standard model in current markets.

This chapter is organized as follows. First, Sections 22.2 through 22.7 concern with the single-name credit derivatives. In Section 22.2, we present the definition of risky zero-coupon bonds and explain the intuition behind this definition. In Section 22.3 we define risky forward rates and forward spreads. In Section 22.4 we deliver swap-rate formulae for two types of credit swaps. Section 22.5 is devoted to the calculation of par CDS rates of both floating-rate and fixed-rate bonds. In Section 22.6 we demonstrate the estimation of implied survival rates and implied recovery rates. In Section 22.7 we present the market model for single-name swaptions. In Sections 22.2 through 22.7, we have laid down the foundation for the pricing of portfolio credit derivatives. In Section 22.8, we introduce CDOs, describe the Monte Carlo method for CDO pricing under the market model, and demonstrate the capacity of the model in accommodating spread correlations and default-time correlations. Pricing examples with standardized CDOs are presented. Finally in Section 22.9 we conclude. Most technical details are put in the Appendix at the end of the chapter.

22.2 PRICING OF RISKY BONDS: A NEW PERSPECTIVE

Our model is set in the filtered probability space, $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where the filtration satisfies the usual conditions of right-continuity and completeness, and $\mathbb{Q}$ is the risk-neutral measure. All stochastic processes are adapted to $(\mathcal{F}_t)_{t\geq 0}$. Without loss of generality, we model the default time as the first jump time of a Cox process with an intensity (or hazard rate) process, $\lambda(t)$. For technical convenience and clarity of presentation, we assume independence between credit spreads and U.S. Treasury yields, independence between hazard rate and the recovery rate of the credit (with the understanding that this may be

against empirical findings [Duffee, 1998]). We also assume a single seniority across all bonds under the same credit name.

We begin with bond pricing. A defaultable coupon bond pays regular coupons until a default occurs or the maturity is reached. In case of a default, the market convention is that a creditor will receive a final payment that consists of a fraction of both principal and accrued interest. The schedule of the final payment varies. Without loss of generality, we assume that (1) the final payment is made at the next coupon date following default and (2) the last coupon accrues until the final payment date.* Let c be the coupon rate of a risky bond with tenor† $[T_m, T_n]$, τ be the default time, and R_τ be the recovery rate at the default time. Then, the cash flow at T_{j+1}, $m \le j \le n-1$ can be expressed as

$$\Delta T c \mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau (1 + \Delta T c) \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}$$

where $\Delta T = 1/(\text{coupon frequency})$ and $\mathbf{1}_{\tau \in \Omega}$ is the indicator function that equals to 1 if $\tau \in \Omega$, or 0 if otherwise. According to the arbitrage pricing theory (APT) (Harrison and Pliska, 1981), the bond is then priced as the risk-neutral expectation of discounted cash flows:

$$\begin{aligned} B^c(t) &= \sum_{j=m}^{n-1} E_t^Q \left\{ \frac{B(t)}{B(T_{j+1})} [\Delta T_j c \mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau (1 + \Delta T_j c) \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}] \right\} \\ &\quad + E_t^Q \left[\frac{B(t)}{B(T_n)} \mathbf{1}_{\{\tau > T_n\}} \right] \\ &= \sum_{j=m}^{n-1} P_{j+1}(t) \Delta T_j c E_t^{Q_{j+1}} \left(\mathbf{1}_{\{\tau > T_{j+1}\}} + R_\tau \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}} \right) \\ &\quad + \sum_{j=m}^{n-1} P_{j+1}(t) E_t^{Q_{j+1}} \left(R_\tau \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}} \right) + P_n(t) E_t^{Q_n} \left(\mathbf{1}_{\{\tau > T_n\}} \right) \end{aligned} \tag{22.1}$$

where, in addition,

$B(t)$ is the money market account of value $\exp\left(\int_0^t r_s \, ds\right)$, where r_t is the risk-free spot rate
$P_j(t)$ is the time-t price of the risk-free zero-coupon bond of \$1 notional value maturing at T_j, equals to $E_t^Q[B(t)/B(T_j)]$
ΔT_j is $T_{j+1} - T_j$, the length of the coupon interval $(T_j, T_{j+1}]$
$\mathbb{Q}_j$ is the T_j forward measure
$E_t^Q[\,\cdot\,]$ is the expectation under $\mathbb{Q}$ conditional on $\mathcal{F}_t$

The expression for payment upon a default, $R_\tau(1 + \Delta T_j c)$, conforms well with real-world practice: the compensation to the creditor is determined by the outstanding principal and

* The protection payment made only at a coupon date is a harmless idealization. There are other default payment schedules as well. For example, the last payment may occur at τ+90 days. Most payment schedules can be accommodated by an adjustment of the recovery rate.

† The coupon dates are $\{T_j\}_{j=m+1}^n$. The bond is said to be forward starting if $m > 0$.

the accrued interest of the defaulted bond, which are treated as in the same asset class, and future coupons are not taken into consideration (Schönbucher, 2004). The second line of Equation 22.1 results from the changes of measures, followed by a regrouping of the PVs of the coupons and the principal. The payout at time T_{j+1} is priced by using the T_{j+1}-forward measure.

Parallel to the U.S. Treasury market, we introduce here the C-strip and the P-strip of risky zero-coupon bonds that are backed separately by coupons and principals. It is not hard to see that the C-strip zero-coupon bonds should be defined as

$$\begin{aligned}\bar{P}_j(t) &= E_t^Q\left[\frac{B(t)}{B(T_j)}\left(\mathbf{1}_{\{\tau>T_j\}} + R_\tau \mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}\right)\right] \\ &= P_j(t)E_t^{Q_j}\left(\mathbf{1}_{\{\tau>T_j\}} + R_\tau \mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}\right) \triangleq P_j(t)D_j(t) \end{aligned} \tag{22.2}$$

while the P-strip zero-coupon bonds are defined as in the last line of Equation 22.1. Here in Equation 22.2, the $\triangleq$ means that a new variable $D_j(t)$ is defined through the equation. Unlike their Treasury counterparts, the risky zero-coupon bonds of the two strips have apparently different cash-flow structures. Given only the prices of risky coupon bonds, we cannot identify the PVs of zero-coupon bonds in either strips unless additional information is provided or additional assumptions are made.

In principle, zero-coupon bonds of the two strips can be generated through marketing the cash flows of the coupons and the principals separately. In reality, however, risky zero-coupon bonds are not traded and, as a result, direct price information is not available (with perhaps occasional exceptions in Japanese market). Nonetheless, the prices of risky zero-coupon bonds of both strips can be backed out from associated coupon bond prices and additional information like the CDS rates. The notion of C-strip zero-coupon bonds, in particular, is important for our model construction.

22.3 FORWARD SPREADS

To understand the product nature of CDS, we need to clarify risky forward rates introduced by Schönbucher (2000) and Brigo (2005). A risky forward rate should be defined as the fair rate on a defaultable loan for a future period of time, say, $(T_j, T_{j+1}]$, that is collateralized by the coupon flows of defaultable bonds of an entity. If a default of the bond occurs before T_j, the contract ceases to exist. If a default occurs between T_j and T_{j+1}, then a recovery value proportional to the recovery rate of the bonds applies. Assume the notional of the loan to be \$1. According to APT, the risky forward rate, denoted as $\hat{f}_j(t)$, must nullify the PV of the cash flows of the risky loan:

$$\begin{aligned} 0 &= E_t^Q\left[\frac{B(t)}{B(T_j)}\mathbf{1}_{\{\tau>T_j\}}\right] - E_t^Q\left\{\frac{B(t)}{B(T_{j+1})}[1+\Delta T_j \hat{f}_j(t)]\left(\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau \mathbf{1}_{\{T_j<\tau<T_{j+1}\}}\right)\right\} \\ &= P_j(t)E_t^{Q_j}(\mathbf{1}_{\{\tau>T_j\}}) - P_{j+1}(t)[1+\Delta T_j \hat{f}_j(t)]\, D_{j+1}(t) \\ &\triangleq P_j(t)\Lambda_j(t) - P_{j+1}(t)[1+\Delta T_j \hat{f}_j(t)]\, D_{j+1}(t) \end{aligned} \tag{22.3}$$

where $\Lambda_j(t)$ is the $\mathbb{Q}_j$ probability of survival until T_j and it is equal to the $\mathbb{Q}$ probability of survival until T_j due to the independence between U.S. Treasury yields and the default probability. Equation 22.3 gives rise to

$$\begin{aligned}\hat{f}_j(t) &= \frac{1}{\Delta T_j}\left[\frac{P_j(t)}{P_{j+1}(t)}\frac{\Lambda_j(t)}{D_{j+1}(t)} - 1\right] \\ &= \frac{1}{\Delta T_j}\left[\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)}\frac{\Lambda_j(t)}{D_j(t)} - 1\right]\end{aligned} \tag{22.4}$$

The two lines in the equation above lead to two alternative expressions of $\hat{f}_j(t)$. The first expression is in terms of the LIBOR rates,

$$\hat{f}_j(t) = f_j(t) + \frac{1+\Delta T_j f_j(t)}{\Delta T_j}\left\{\frac{E_t^{Q_{j+1}}[(1-R_\tau)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right\} \tag{22.5}$$

where $f_j(t)$ is the default-free forward rate for the period $(T_j, T_{j+1}]$ seen at time t, defined by

$$f_j(t) = \frac{1}{\Delta T_j}\left[\frac{P_j(t)}{P_{j+1}(t)} - 1\right], \quad t \le T_j$$

The second expression is in terms of defaultable effective forward rate (Schönbucher, 2000),

$$\hat{f}_j(t) = \bar{f}_j(t) - \frac{1+\Delta T_j \bar{f}_j(t)}{\Delta T_j}\left[\frac{E_t^{Q_j}\left(R_\tau \mathbf{1}_{\{T_{j-1}<\tau\le T_j\}}\right)}{D_j(t)}\right] \tag{22.6}$$

where $\bar{f}_j(t)$ is defined by

$$\bar{f}_j(t) = \frac{1}{\Delta T_j}\left[\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)} - 1\right], \quad t \le T_j$$

Note that $\bar{f}_j(t)$ should be understood as the effective rate of return over $(T_j, T_{j+1}]$ provided that no default occurs until T_{j+1}. It is pointed out in Brigo (2004) that $\bar{f}_j(t)$, in general, does not link directly to a financial contract. Putting Equations 22.5 and 22.6 together, we have the order

$$f_j(t) \le \hat{f}_j(t) \le \bar{f}_j(t) \tag{22.7}$$

Note that $\hat{f}_j(t)$ achieves the upper bound and the lower bound when $R_\tau = 0$ and $R_\tau = 1$, respectively. The bounds on $\hat{f}_j(t)$ should be regarded as no-arbitrage constraints.

Here, we reiterate the insight of nonseparability pointed out by Duffie and Singleton (1999), which means that the hazard rate and the loss rate cannot be determined from bond prices alone. In view of Equation 22.4, we can say that complete term structures of the survival probability and the recovery rate, $\Lambda_j(t)$ and $E_t^Q[R_\tau | T_{j-1} < \tau \le T_j]$, $j = 1, 2, \ldots$, can be uniquely determined from the term structures of $\hat{f}_j(t)$ and $\bar{f}_j(t)$, which in fact, as we

shall see, are informatively equivalent to the term structures of risky-bond yields and CDS rates.

Intuitively, a forward spread is defined as the difference between a risky forward rate and its corresponding risk-free forward rate:

$$S_j(t) = \hat{f}_j(t) - f_j(t), \quad j = 1, 2, \ldots$$

From Equation 22.5, we obtain

$$S_j(t) = [1 + \Delta T_j f_j(t)] \frac{E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}]}{\Delta T_j D_{j+1}(t)} \triangleq [1 + \Delta T_j f_j(t)] H_j(t) \tag{22.8}$$

According to its definition, $H_j(t)$ can be interpreted as the expected loss per risky dollar over $(T_j, T_{j+1}]$. Equation 22.8 can be rewritten as

$$1 + \Delta T_j \hat{f}_j(t) = [1 + \Delta T_j f_j(t)]\,[1 + \Delta T_j H_j(t)], \quad j = 1, 2, \ldots$$

By comparing definitions, we can say that $H_j(t)$ is the discrete-tenor version of the mean loss rate introduced in Duffie and Singleton (1999).

22.4 TWO KINDS OF DEFAULT PROTECTION SWAPS

A default protection swap consists of a fee leg (or premium leg) and a protection leg. Before the default of the reference entity, the protection buyer pays the protection seller a string of fees at regular time intervals. Upon default, the protection buyer either delivers the bond to the protection seller in exchange for par (so-called physical delivery), or receives from the protection seller a payment that is equal to the loss incurred (cash settlement). In this section, we consider two kinds of swaps: swaps for fixed-rate bonds and swaps for floating-rate bonds. Without loss of generality, we assume the notional value of the bonds to be \$1, and the coupon rate to be c, and LIBOR, $f_j(T_j)$, respectively. For the swap on the fixed-rate bond, the protection payment is $(1 - R_\tau)(1 + \Delta T c)$, while for the swap on the floating-rate bond, the protection payment is $(1 - R_\tau)[1 + \Delta T_j f_j(T_j)]$. The payments simply reflect the loss to the bond holders in case of a default.* Note that the swaps of the second kind depend only on the default status of the reference entity.

We proceed to the computation of fair default swap rates. Let us denote a swap rate by $\bar{s}$. In the CDS markets, the contractual cash flows of the fee leg (for the protection of \$1 notional) are typically

$$\bar{s}\Delta T_j \left[\mathbf{1}_{\{\tau > T_{j+1}\}} + \frac{(\tau - T_j)}{\Delta T_j}\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}\right], \quad j = 1, 2, \ldots$$

i.e., in case of a default between T_j and T_{j+1}, the protection buyer makes the final payment that is proportional to the time elapsed between the last fee payment and the default.

* The quantity $[1 + \Delta T_j f_j\,(T_j)]$ can be regarded as a constant dollar seen at time T_j.

From the financial engineering point of view, such a fee specification is troubling because the cash flow cannot be synthesized at the initiation of the swap contract. To make the swap contract replicable, we propose a slight modification to the fee leg: we let the cash flows be generated from the payouts of risky zero-coupon bonds:

$$\bar{s}\Delta T_j\left(\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau \mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}\right)$$

This new definition only changes the last piece of fee payment after default. The value of the fee leg is now

$$\begin{aligned} PV_{\text{fee}} &= \bar{s}\sum_{j=m}^{n-1}\Delta T_j P_{j+1}(t)E_t^{Q_{j+1}}\left(\mathbf{1}_{\{\tau>T_{j+1}\}} + R_\tau \mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}\right) \\ &= \bar{s}\sum_{j=m}^{n-1}\Delta T_j \bar{P}_{j+1}(t) \end{aligned} \tag{22.9}$$

which is now a tradable annuity, analogous to the fixed leg of default-free swaps.

For a swap of the first kind, the cash flow of the protection seller can be written as

$$V_{\text{prot}} = (1+\Delta Tc)\sum_{j=m}^{n-1}(1-R_\tau)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}$$

The PV of the protection payment is then

$$PV_{\text{prot}} = (1+\Delta Tc)\sum_{j=m}^{n-1}P_{j+1}(t)E_t^{Q_{j+1}}[(1-R_\tau)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}] \tag{22.10}$$

By equating the fee leg to the protection leg and making use of Equation 22.8, we obtain

$$\begin{aligned} \bar{s}_1 &= (1+\Delta Tc)\frac{\sum_{j=m}^{n-1}P_{j+1}(t)E_t^{Q_{j+1}}[(1-R_\tau)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{\sum_{j=m}^{n-1}\Delta T_j \bar{P}_{j+1}(t)} \\ &= (1+\Delta Tc)\sum_{j=m}^{n-1}\bar{\alpha}_j H_j(t) \end{aligned} \tag{22.11}$$

where

$$\bar{\alpha}_j(t) = \frac{\Delta T_j \bar{P}_{j+1}(t)}{\sum_{j=m}^{n-1}\Delta T_j \bar{P}_{j+1}(t)}$$

Note that the case $c = 0$ corresponds to a prototypical default swap, which only depends on the default status of the reference entity and dominates the liquidity of single-name credit derivatives markets.

For swaps of the second kind, the fair swap rate can be derived analogously as

$$\bar{s}_2 = \frac{\sum_{j=m}^{n-1} P_{j+1}(t)[1 + \Delta T_j f_j(t)] E_t^{Q_{j+1}}[(1 - R_\tau)\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}]}{\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)} = \sum_{j=m}^{n-1} \bar{\alpha}_j S_j(t) \tag{22.12}$$

Here, we have made use of the independence between credit spreads and the U.S. Treasury yields, as well as the martingale property of the forward rate: $E_t^{Q_{j+1}}[f_j(T_j)] = f_j(t)$. Note that a one-period CDS rate reduces to a risky forward rate.

Compared with the existing CDS rate formula (Schönbucher, 2004; Brigo, 2005), our definition simply states that a CDS rate equal to the weighted average of credit spreads, which does not require assumption on or explicit input of the recovery rate, and is analogous to the swap-rate formula in LIBOR markets.

22.5 PAR CREDIT DEFAULT SWAP RATES

A par CDS rate is a spread, when added to a corresponding par rate of default-free bond, yields the coupon rate of a risky par bond. We derive the par CDS rates for both floating-rate bonds and fixed-rate bonds.

Typically, a defaultable floating-rate bond (which is also called a defaultable floater) pays LIBOR plus a credit spread, denoted by s_F, until a default occurs, when a holder may obtain some recovered value of both principal and coupon. Hence, the cash flow of a defaultable floater at T_{j+1} can be written as

$$CF_{j+1} = \{1 + \Delta T_j [f_i(T_j) + s_F]\}(\mathbf{1}_{\{\tau > T_{j+1}\}} + R\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}) - \mathbf{1}_{\{\tau > T_{j+1}, j+1 < n\}}$$

The question here is: if the floater is to be priced at par, what should be the fair spread rate s_F? To answer this question, we imagine that the holder of the floater is also "long" a protection swap of the second kind. Then, his or her cash flow at time T_{j+1} is

$$\begin{aligned} CF_{j+1} = {} & \Delta T_j f_i(T_j)\mathbf{1}_{\{\tau > T_j\}} + \mathbf{1}_{\{T_j < \tau \le T_{j+1}\}} + \mathbf{1}_{\{\tau > T_n, j+1 = n\}} \\ & + (s_F - \bar{s}_2)(\mathbf{1}_{\{\tau > T_j\}} + R\mathbf{1}_{\{T_{j-1} < \tau \le T_j\}}) \end{aligned} \tag{22.13}$$

The first line in Equation 22.13 gives the cash flow of a rolling-forward CD that lasts until $T_{j_\tau} \wedge T_n$, where T_{j_τ} is the first fixing date after default, and such cash flows represent those of a par bond (that matures at $T_{j_\tau} \wedge T_n$). It then becomes clear that, for the defaultable floater to be priced at par, there must be

$$s_F = \bar{s}_2$$

i.e., the par default swap rate equals nothing else but the credit spread.

Given the clear relationship between a defaultable floater and a CDS, we come up with the following hedging strategy for swaps of the second kind: once such a swap is written, the hedger goes "long" a default-free floater and goes "short" a defaultable floater, both at par. Then, the net cash flow at any fixing date is zero.

Next, we derive the par CDS rate for a corresponding fixed-rate bond with tenor $[T_m,T_n]$. We let s_X denote the par CDS rate for the fixed-rate bond. Then, s_X can be determined by equating the PVs of fixed-rate and floating-rate par bonds:

$$\begin{aligned}[R_{m,n}(t)+s_X]\sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t) &= \sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t)[f_j(t)+\bar{s}_2]\\ &= \sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t)\hat{f}_j(t)\end{aligned}$$

where the PVs of the principals have been canceled, and $R_{m,n}(t)$ represents the corresponding swap rate (i.e., par rate) in LIBOR markets, defined by

$$R_{m,n}(t)=\sum_{j=m}^{n-1}\alpha_j f_j(t),\quad \alpha_j=\frac{\Delta T_j P_{j+1}(t)}{\sum_{k=m}^{n-1}\Delta T_k P_{k+1}(t)}$$

It follows that

$$\begin{aligned}s_X &= \sum_{j=m}^{n-1}\bar{\alpha}_j\hat{f}_j(t)-R_{m,n}(t)\\ &= \bar{s}_2+\sum_{j=m}^{n-1}(\bar{\alpha}_j-\alpha_j)f_j(t)\end{aligned}$$

Hence, the so-called credit spread is different for floating-rate bonds and fixed-rate bonds and a par CDS rate for fixed-rate bonds is close to $\bar{s}_2$ instead of $\bar{s}_1$ (when $c=0$).

It can be verified that the short position of the default swap on the risky coupon bond with coupon rate c can be hedged by the following:

1. Being "short" the risky coupon bond
2. Being "long" a risk-free floater at par
3. Being "long" $[c-\bar{R}_{m,n}(t)]$ units of the risky annuity, $\sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t)$

Here, $\bar{R}_{m,n}(t)=R_{m,n}(t)+s_X$ is the risky par yield.

22.6 IMPLIED SURVIVAL CURVE AND RECOVERY-RATE CURVE

In reality, neither $\{\hat{f}_j(t)\}$ or $\{\bar{f}_j(t)\}$ is directly observable. For CDS pricing and other applications, it is more convenient to make use of the term structures of forward hazard rate and forward recovery rate. We define the forward hazard rate for $(T_j, T_{j+1}]$ seen at time t by

$$\lambda_j(t) = \frac{1}{\Delta T_j}\left[\frac{\Lambda_j(t)}{\Lambda_{j+1}(t)} - 1\right] \quad \text{for } t \le T_j \tag{22.14}$$

and the forward recovery rate for the same period by

$$R_j(t) = E_t^Q[R_\tau | T_j < \tau < T_{j+1}] \quad \text{for } t \le T_j \text{ and } j \ge \eta(t) \tag{22.15}$$

where $\eta(t)$ is smallest integer such that $T_{\eta(t)} \ge t$. The survival probability, $\Lambda_j(t)$, relates to the hazard rates by

$$\Lambda_j(t) = [1 + (T_{\eta(t)} - t)\lambda_{\eta(t)-1}(T_{\eta(t)-1})]^{-1} \prod_{k=\eta(t)}^{j-1} (1 + \Delta T_k \lambda_k)^{-1} \tag{22.16}$$

The standard market practice is to back out the survival probabilities from CDS of various maturities, assuming a constant recovery rate (of 40%). Instead of doing the same thing with our model, we consider backing out simultaneously the implied hazard rates and recovery rates from CDSs and, in addition, corporate bond prices. We choose Citigroup as the credit name for a demonstration, as there is relatively richer credit information on this company. A snapshot of market quotations is provided in Tables 22.1 and 22.2, where the currency is U.S. dollars, the CDS rates are for $c = 0$, and the bond prices are "clean". To build the risk-free discount curve in U.S. dollars, we have used LIBOR rates up to 2 years and swap rates from 2 to 20 years. The interest-rate information is provided in Table 22.3. Figure 22.1 presents the forward-rate curve constructed using the yield data in Table 22.3.

We determine $\{\lambda_j, R_j\}$ through reproducing the CDS rates and the bond prices of Citigroup by the swap-rate formula (Equation 22.11) and the bond formula (Equation 22.1), respectively. Because the problem is under determined, we have adopted cubic-spline and linear interpolation for the hazard rates and the recovery rates, respectively, and imposed additional smoothness regularization. In our search algorithm, we have taken various initial guesses with $\lambda_j = 0$ and $0.0 \le R_j \le 0.6$, $\forall j$. Often, but not always, the search ends up in one of the two solutions, depicted in Figures 22.2 through 22.4, depending on the closeness of the initial recovery rate to either $R_0 = 0.0$ or $R_0 = 0.4$. Existence of more than one solution reflects the ill-posed nature of calibration problem, particularly with regard to the determination of the recovery rate.

We remark here the ill-posed nature of calibration problem is largely intrinsic. This is due to the insensitivity of the bond and CDS prices with respect to the change of the

TABLE 22.1 Citigroup CDS Rates (07/28/05, Bloomberg)

Maturity	**1Y**	**3Y**	**5Y**	**10Y**
Rates (%)	0.07	0.13	0.19	0.33

TABLE 22.2 Prices of Benchmark Citigroup Bonds (07/28/05, Bloomberg)

Maturity	Frequency	Coupon	Price
22/2/2010	Semi-annual	4.125%	98.123
1/10/2010	Semi-annual	7.25%	114.563
7/5/2015	Semi-annual	4.875%	97.563
18/5/2010	Quarterly	U.S. LIB+15bps	99.00
16/3/2012	Quarterly	U.S. LIB+12.5bps	99.80
5/11/2014	Quarterly	U.S. LIB+28bps	100.501

TABLE 22.3 U.S. Dollars Yield Data (07/28/05, Bloomberg)

Type of Rate	Term	Rate
LIBOR	3 mth	3.6931%
	6 mth	3.8435%
	12 mth	4.1731%
Swap	2 Y	4.3200%
	3 Y	4.3840%
	4 Y	4.4330%
	5 Y	4.4690%
	7 Y	4.5365%
	10 Y	4.6290%
	12 Y	4.6905%
	15 Y	4.7630%
	20 Y	4.8320%

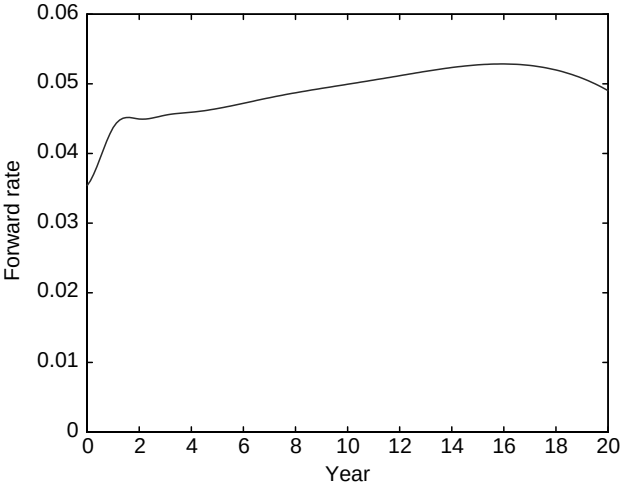


FIGURE 22.1 U.S. dollars forward rates (07/28/05).

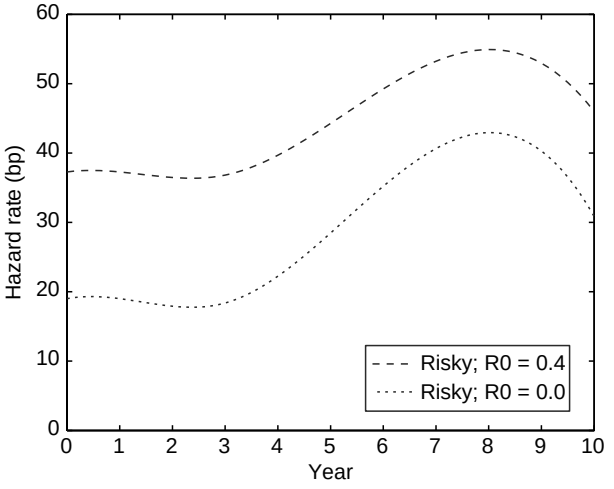


FIGURE 22.2 Implied hazard rates.

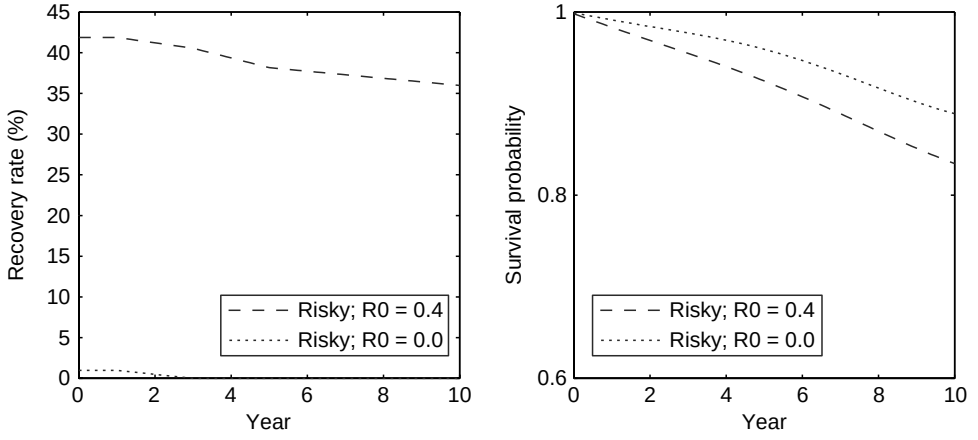


FIGURE 22.3 Implied default rates and recovery rates.

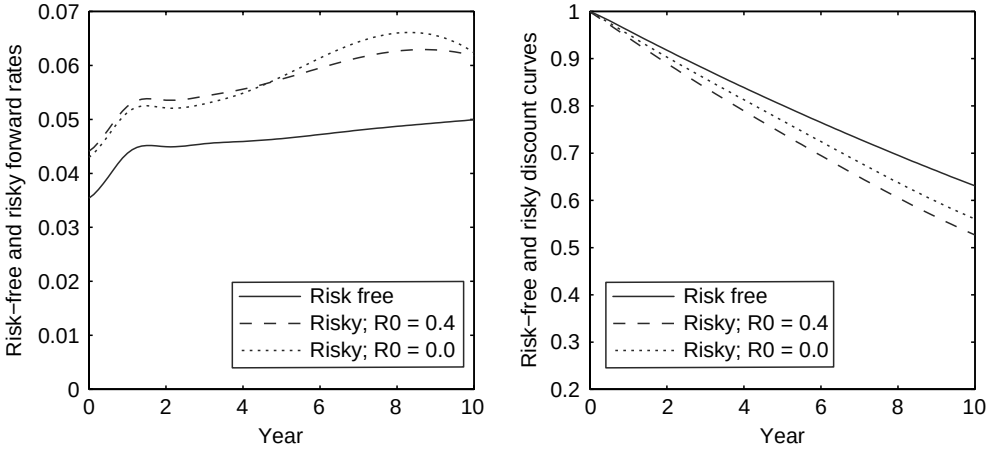


FIGURE 22.4 Forward rates and discount curves.

recovery rates,* particularly for a good credit name. As a consequence of the ill-posedness, implied recovery rates and default rates may depend on initial guesses, as are shown in Figures 22.2 and 22.3. To settle down to a single solution, additional regularizations (based on financial or mathematical considerations) are needed. On the other hand, the ill-posedness may not be a big concern. As is shown in Figure 22.4, risky forward rates and risky discount curve demonstrate noticeable stability. This is due to the complimentary effect between the hazard rates and the recovery rates in calibration.

By comparing the implied hazard rates with the CDS rates, we can say that the market quotations of CDS rates pretty much represent the risk-neutral hazard rates for zero recovery upon default. The quoted CDS spreads increase with time to maturity, a risk-neutral hazard rate will also increase with its maturity.

22.7 CREDIT DEFAULT SWAPTIONS AND AN EXTENDED MARKET MODEL

For either protection buyer or seller, an open position can be closed by either going "short" or going "long" of the same swap. Because the protection legs are exactly offset, the profit or loss for the pair of transactions comes from the difference of the fee legs, which is

$$\mathbf{1}_{\{\tau>t\}}[\bar{s}_{m,n}(t)-\bar{s}_{m,n}(0)]\sum_{j=m}^{n-1}\Delta T_j\bar{P}_j(t) \tag{22.17}$$

where $\bar{s}_{m,n}(t)$ is the prevailing CDS rate, either $\bar{s}_1$ or $\bar{s}_2$, seen at time t.

A credit swaption, meanwhile, is a contract that gives its holder the right, not the obligation, to enter into a forward-starting swap at time $T \le T_m$ with a predetermined swap rate, $\bar{s}^*$. Apparently, the profit/loss at maturity T of the swaption is

$$\mathbf{1}_{\{\tau>T\}}[\bar{s}_{m,n}(T)-\bar{s}^*]^+\sum_{j=m}^{n-1}\Delta T_j\bar{P}_j(T) \tag{22.18}$$

The standard practice in markets is to price the swaption by using Black's formula. Parallel to Schönbucher (2004), we define the default swap measure,† $\mathbb{Q}^S$, according to

$$\left.\frac{d\mathbb{Q}^S}{d\mathbb{Q}}\right|_{\mathcal{F}_t}=\frac{\bar{A}_{m,n}(t)}{\bar{A}_{m,n}(0)}\Big/\frac{B(t)}{B(0)} \tag{22.19}$$

where

$$\bar{A}_{m,n}(t)=\mathbf{1}_{\{\tau>t\}}\sum_{j=m}^{n-1}\Delta T_j\bar{P}_{j+1}(t)$$

* The insensitivity is also demonstrated in Brigo (2005).

† Schönbucher (2004) calls it a survival measure.

In term of the new measure, we can express the price of the default swaption as

$$\begin{aligned} C &= E_t^Q\left\{\frac{B(t)}{B(T)}\bar{A}_{m,n}(T)[\bar{s}_{m,n}(T)-\bar{s}^*]^+\right\} \\ &= \bar{A}_{m,n}(t)E_t^Q\left\{\left[\frac{\bar{A}_{m,n}(T)}{\bar{A}_{m,n}(t)}\Big/\frac{B(T)}{B(t)}\right](\bar{s}_{m,n}(T)-\bar{s}^*)^+\right\} \\ &= \bar{A}_{m,n}(t)E_t^{Q^S}\{[\bar{s}_{m,n}(T)-\bar{s}^*]^+\} \end{aligned} \tag{22.20}$$

A few comments are in order. First, $\bar{A}_{m,n}(t)$ is replicable until $t = T_m$ (beyond T_m, $\bar{A}_{m,n}(t)$ remains replicable if the recovery rate is zero). The implication is that the price given by Equation 22.20 is an arbitrage price. The default swap measure defined in Equation 22.19 generalizes the definition of Schönbucher (2004), whose annuity numeraire is nontradable unless the recovery rate is zero. Second, $\bar{A}_{m,n}(t) = 0$ in case of a default at $\tau < t$, which implies that $\mathbb{Q}^S$ is absolutely continuous with respect to, but not equivalent to, $\mathbb{Q}$. For swaption pricing, the lack of equivalence here is just harmless. Note that Brigo (2005) provides an alternative derivation of the Black's formula that preserves the measure equivalence, based on a theory of restricted measure (Jeanblanc and Rutkowski, 2000; Bielecki and Rutkowski, 2001). According to the interpretation of Schönbucher's survival measure by Bielecki and Rutkowski (2001) (Section 15.2.2), these two approaches should be equivalent.

Parallel to Schönbucher (2004), we can establish the following results. The proof is straightforward and thus is omitted for brevity.

Proposition 22.1 *Under the new measure $\mathbb{Q}^S$, there are*

(1) *$\mathbb{Q}^S$-default probability is zero*:

$$E^{Q^S}[\mathbf{1}_{\{\tau<T\}}] = 0$$

(2) *Swap rate $\bar{s}_{m,n}(t)$ of either kind of swap is a martingale:*

$$E_t^{Q^S}[\bar{s}_{m,n}(T)] = \bar{s}_{m,n}(t) \quad \forall T \in (t, T_m)$$

On the basis of the results of Proposition 22.1, we assume $\bar{s}_{m,n}(t)$ is lognormal martingale under $\mathbb{Q}^S$:

$$d\bar{s}_{m,n}(t) = \bar{s}_{m,n}(t)\bar{\gamma}_{m,n}dW_t^S \tag{22.21}$$

where

W_t^S is a one-dimensional Brownian motion under $\mathbb{Q}^S$

$\bar{\gamma}_{m,n}$ is the swap-rate volatility

The lognormality assumption (Equation 22.27) leads readily to Black's formula for credit swaptions:

$$C = \bar{A}_{m,n}(t)[\bar{s}_{m,n}(t)N(d_1) - \bar{s}^*N(d_2)] \tag{22.22}$$

with

$$d_{1,2} = \frac{\ln[\bar{s}_{m,n}(t)/\bar{s}^*] \pm \frac{1}{2}\bar{\gamma}_{m,n}^2(T-t)}{\bar{\gamma}_{m,n}\sqrt{T-t}}$$

A hedging strategy follows from Black's formula (Equation 22.22): at time $t < T$, the hedger maintains $N(d_1)$ units of the credit default swap; and maintains $\bar{s}_{m,n}(t)N(d_1) - \bar{s}^*N(d_2)$ units of the annuity $\sum_{j=m}^{n-1} \Delta T_j \bar{P}_{j+1}(t)$.

Next we show that the above swaption pricing approach can be justified under the market models with either mean loss rates:

$$\frac{dH_j(t)}{H_j(t)} = \mu_j^H(t)\,dt + \gamma_j^H(t) \cdot d\mathbf{W}_t \tag{22.23}$$

or forward spreads:

$$\frac{dS_j(t)}{S_j(t)} = \mu_j^S(t)\,dt + \gamma_j^S(t) \cdot d\mathbf{W}_t \tag{22.24}$$

where $\mathbf{W}_t$ is a multidimensional Brownian motion under $\mathbb{Q}$. Assume that the recovery rate is time stationary, $E^Q[R_\tau | T_{j-1} < \tau < T_j] = \bar{R} = \text{constant}$, we can derive that

$$\mu_j^H(t) = \gamma_j^H(t) \sum_{k=1}^{j} \frac{\Delta T_k H_k(t)}{1 + \Delta T_k H_k(t)} \left[\gamma_k^H - \left(1 - \frac{\bar{H}_k}{H_k}\right)\gamma_{k-1}^H\right] \tag{22.25}$$

where

$$1 - \frac{\bar{H}_k(t)}{H_k(t)} = -\frac{1 + \Delta T_k H_k(t)}{\Delta T_k H_k(t)} \frac{\bar{R}\Delta T_{k-1} H_{k-1}(t)}{1 - \bar{R} - \bar{R}\Delta T_{k-1} H_{k-1}(t)}$$

The drift term of $S_j(t)$ takes a more complex form:

$$\mu_j^S(t) = \frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)}\mu_j + \left[\frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)}\gamma_j - \sigma_{j+1}^D(t)\right]\left[\gamma_j^S(t) - \frac{\Delta T_j f_j(t)}{1 + \Delta T_j f_j(t)}\gamma_j(t)\right]$$

$$\sigma_{j+1}^D(t) = -\sum_{k=1}^{j} \frac{\Delta T_k S_k(t)}{1 + \Delta T_k \hat{f}_k(t)}\left[\gamma_j^S(t) - \frac{\Delta T_j}{1 + \Delta T_j f_j(t)}\gamma_j(t)\right] \tag{22.26}$$

Here, $f_j(t)$ follows a lognormal process under the LIBOR market model (Brace et al., 1997; Jamshidian, 1997; Miltersen et al., 1997), with drift μ_j and volatility γ_j, such that

$$\mu_j(t) = \gamma_j(t) \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)}{1 + \Delta T_k f_k(t)}\gamma_k(t) \tag{22.27}$$

The derivations of Equations 22.25 and 22.26 are left in Appendix.

Next, we proceed to the pricing of swaptions under the extended market model. We take the first kind of swaps as an example. The swap rate is given by Equation 22.11. By Ito's lemma, we can derive an approximate swap-rate process as follows:

$$\begin{aligned} d\,\bar{s}_{m,n}(t) &= \sum_{j=m}^{n-1} \frac{\partial\,\bar{s}_{m,n}(t)}{\partial H_j} H_j(t)\gamma_j^H \cdot d\mathbf{W}_t^S \\ &= \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \frac{\partial\,\bar{s}_{m,n}(t)}{\partial H_j} \frac{H_j(t)}{\bar{s}_{m,n}(t)} \gamma_j^H \cdot d\mathbf{W}_t^S \\ &\approx \bar{s}_{m,n}(t) \sum_{j=m}^{n-1} \frac{\partial\,\bar{s}_{m,n}(0)}{\partial H_j} \frac{H_j(0)}{\bar{s}_{m,n}(0)} \gamma_j^H \cdot d\mathbf{W}_t^S \\ &= \bar{s}_{m,n}(t)\bar{\gamma}_{m,n} \cdot d\mathbf{W}_t^S \end{aligned} \tag{22.28}$$

where

$$\bar{\gamma}_{m,n} \triangleq \sum_{j=m}^{n-1} \bar{\omega}_j \gamma_j^H, \quad \bar{\omega}_j \triangleq \frac{\partial\,\bar{s}_{m,n}(0)}{\partial H_j} \frac{H_j(0)}{\bar{s}_{m,n}(0)} \approx \bar{\alpha}_j \frac{H_j(0)}{\bar{s}_{m,n}(0)} \tag{22.29}$$

and $\mathbf{W}_t^S$ is a multidimensional Brownian motion under $\mathbb{Q}^S$, defined by

$$d\mathbf{W}_t^S = d\mathbf{W}_t - \sum_{j=m}^{n-1} \bar{\alpha}_j \bar{\sigma}_{j+1}(t)\,dt$$

and $\bar{\sigma}_{j+1}$ is the volatility of $\bar{P}_{j+1}(t)$. The lognormal process for swap rates (Equation 22.28) justifies the use of Black's formula (Equation 22.22) for swaptions. Note that, in the context of LIBOR market model, the approximations made in Equations 22.28 and 22.29 are known to be accurate enough for applications and have been justified with rigor (Brigo et al., 2004). The expressions in Equation 22.28 describe the relation between forward spread volatilities and swap-rate volatilities, which can be used in practice to gauge the relative price richness/cheapness of a swaption. The model for spreads, either Equation 22.23 or 22.24, can be calibrated to the implied volatilities of the default swaptions using the quadratic programming technology developed by Wu (2003) for market model calibrations.

Models more comprehensive than Equation 22.23 or 22.24 can be developed by including other risk dynamics like jumps, stochastic volatilities, and even correlations among multiple credit names (Eberlein et al., 2005). Such developments are largely parallel to existing extensions to the standard market model. Brigo (2005) and especially Schönbucher (2004) have made several extensions using swap rates as state variables.

22.8 PRICING OF CDO TRANCHES UNDER THE MARKET MODEL

In this section, we explain how to apply the market model for pricing CDOs. A CDO is a way to restructure the cash flows a portfolio of bonds (with various credit ratings). Tranches of ascending seniority are defined in terms of the percentages of the notional

principal value of the portfolio, and losses are always allocated to the most junior tranche that is still alive. The spread of each tranche is just the premium of protection payments, which, unlike the protection payments in a single-name CDS, are made periodically until all notional value is lost. Tranches are divided by attachment points. Take CDX IG, a standardized CDO, for example, the attachment points are 0%, 3%, 7%, 10%, 15%, 30%, and, of course, 100%. The equity tranche, which is the most junior tranche with attachment points 0% and 3%, will absorb the losses to the portfolio up to the first 3% and then ceases to exist. Subsequent losses will then be bearded by the next tranche with attachment points of 3% and 7%, which is called the mezzanine tranche. Losses to other senior tranches are determined similarly. For each tranche, the premium of protection is calculated based on the outstanding notional principal value. To express the remaining outstanding notional principals of the tranches, we introduce the following notations:

$[P_D^i, P_U^i]$ — the attachment points for the ith tranche, in percentage;
$D^{(k)}(T_{j+1})$ — the forward price of the outstanding notional value of the kth name at T_{j+1}, equal to $\mathbf{1}_{\{\tau > T_{j+1}\}} + \bar{R}\mathbf{1}_{\{T_j < \tau < T_{j+1}\}}$;

Then, the total outstanding notional at T_{j+1} for the portfolio in percentage is

$$D^P(T_{j+1}) = \frac{1}{K}\sum_{k=1}^{K} D^{(k)}(T_{j+1}) \tag{22.30}$$

where K be the number of credit names in the portfolio. The outstanding notional at T_{j+1} for the ith tranche can be expressed as

$$\begin{aligned} O_i(T_{j+1}) = \frac{1}{P_U^i - P_D^i}\Big\{ &\left[D^P(T_{j+1}) - 1 + P_U^i\right]^+ \\ &- \left[D^P(T_{j+1}) - 1 + P_D^i\right]^+ \Big\} \end{aligned} \tag{22.31}$$

The loss to the ith tranche over (T_j, T_{j+1}) is thus

$$O_i(T_j) - O_i(T_{j+1})$$

The expression in Equation 22.31 reiterates the fact that the outstanding notional for any tranche at any cash flow day can be regarded as a spread option on the total outstanding notional value of the portfolio backing the CDO.

Next, we consider the pricing of the premium rate on a tranche. Let s_i be the premium rate on the ith tranch, then the value of the fee leg is

$$PV_{\text{fee}} = s_i \sum_{j=m}^{n-1} \Delta T_j P_{j+1}(t)\, E_t^{Q_{j+1}}[O_i(T_{j+1})]$$

while the value of the protection leg is

$$PV_{\text{prot}} = \sum_{j=m}^{n-1} P_{j+1}(t)\, E_t^{Q_{j+1}}[O_i(T_j) - O_i(T_{j+1})]$$

The formula for the premium rate on the ith tranche is then

$$s_i = \frac{\sum_{j=m}^{n-1} P_{j+1}(t)\, E_t^{Q_{j+1}}[O_i(T_j) - O_i(T_{j+1})]}{\sum_{j=m}^{n-1} \Delta T_j P_{j+1}(t)\, E_t^{Q_{j+1}}[O_i(T_{j+1})]}$$

In view of Equation 22.31, we understand that the key to CDS rate calculation lies in the valuation of a sequence of call options of the form

$$E_t^{Q_{j+1}}\left\{\left[D^P(T_{j+1}) - X\right]^+\right\}, \quad j = m, \ldots, n-1 \tag{22.32}$$

We now consider the valuation of the above options by Monte Carlo simulation method. In view of the definition of $D^P(T_{j+1})$ in Equation 22.30, we need to simulate $D^{(k)}(T_{j+1})$, $k = 1, \ldots, K$. For simplicity, we assume constant recovery rate, $E^Q[R_\tau | T_{j-1} < \tau < T_j] = \bar{R} =$ constant. We then can express $D^{(k)}(T_{j+1})$ as

$$D^{(k)}(T_{j+1}) = \mathbf{1}_{\{\tau > T_j\}}\left(\mathbf{1}_{\{u > \lambda_j^{(k)}(T_j)\Delta T_j\}} + \bar{R}\mathbf{1}_{\{u < \lambda_j^{(k)}(T_j)\Delta T_j\}}\right) \tag{22.33}$$

where u obeys the uniform distribution in (0,1), denoted by $U(0,1; \Sigma_u)$, where Σ_u stands for the correlation matrix of u's. If taking a Gaussian copula for u's, we can proceed as follows:

1. Do a Choleski decomposition of the input correlation matrix

$$\Sigma_g = AA^T \tag{22.34}$$

2. Simulate $K(T_j)$ (which is the number of surviving firms at time T_j) independent standard normal random variables, $\{\tilde{\varepsilon}_i\}_{i=1}^{K(T_j)}$
3. Transform $\{\tilde{\varepsilon}_i\}_{i=1}^{K(T_j)}$ to correlated normal random variables

$$\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{K(T_j)} \end{pmatrix} = A \begin{pmatrix} \tilde{\varepsilon}_1 \\ \vdots \\ \tilde{\varepsilon}_{K(T_j)} \end{pmatrix}$$

4. Transform the normal random variables to uniform random variables

$$(u_1, \ldots, u_{K(T_j)}) = [N(\varepsilon_1), \ldots, N(\varepsilon_{K(T_j)})]$$

If, furthermore, we abide by the industrial convention to assume a uniform pairwise correlation such that $corr(\varepsilon_k, \varepsilon_j) = \rho > 0$ for any k and l, then steps 1–3 above are simplified into the calculations of

$$\varepsilon_k = \sqrt{1-\rho}\tilde{\varepsilon}_k + \sqrt{\rho}\tilde{\varepsilon}_c \tag{22.35}$$

where $\tilde{\varepsilon}_c$ and $\tilde{\varepsilon}_k$, $k = 1, \ldots, K(T_j)$ are independent standard normal random variables.

The evolution of $\lambda_j^{(k)}(t)$ follows from that of $H_j^{(k)}(t)$. Under discrete approximation, we have

$$H_j^{(k)}(T_j) = H^{(k)}(T_{j-1}) \exp\left\{ \left[\mu_j^H(T_{j-1}) - \frac{1}{2} \left\| \gamma_j^H \right\|^2 \right] \Delta T_j + \gamma_j^H \Delta W^{(k)} \right\} \tag{22.36}$$

Here, we have locally freeze $\mu_j^H(T_{j-1})$, defined in Equation 22.25. The hazard rate at T_j is then

$$\lambda_j^{(k)}(T_j) = \frac{H_j^{(k)}(T_j)}{1 - [1 + \Delta T_j H_j^{(k)}(T_j)]\bar{R}} \tag{22.37}$$

In the evolution of $H_j^{(k)}(t)$, we can incorporate the correlations of the credit spreads observed in the single-name CDS market.

We are now ready to describe the algorithm for CDO pricing. The simulation of correlated defaults is the focus of the algorithm. Note that we need to input two set of correlations. The first set is for the correlations, Σ_H, of CDS rates, which is the state variables for the market model. This set of correlations can be observed from the market. The second set is for the correlations of default times, Σ_u, which is not quite observable and is dealt with the technique of Gaussian copula. Let T be the maturity of the CDO, ΔT the time interval for premium payments, $J = T/\Delta T$ the maximal number of the premium payments, and M be the number of Monte Carlo simulation paths. We develop the following algorithm for pricing the options in Equation 22.32.

/* Algorithm for pricing options on $D_j^P(T_j)$, $j = 1, \ldots, J$ */
  For $j = 1 : J$
    $V_j = 0$
  end
  For $m = 1 : M$
    For $k = 1 : K$
      $D^{(k)}(T_0) = 1$
    end
    $K(T_0) = K$
    For $j = 1 : J$
      Generate $\{\Delta W^{(k)}\}_{k=1}^{K(T_{j-1})} \sim N(0, \Delta T \Sigma_H)$
      Generate $\{u_k\}_{k=1}^{K(T_{j-1})} \sim U(0, 1; \Sigma_u)$
      Put $D^P(T_j) = 0$
      $l = 0$
      For $k = 1 : K$ repeat
        If $D^{(k)}(T_{j-1}) = 1$, then
          $D^{(k)}(T_j) = 1$

/* Calculate the hazard rate $\lambda^{(k)}(T_{j-1})$ */

$$\lambda^{(k)}(T_{j-1}) = H_{j-1}^{(k)}(T_{j-1})/(1-(1+\Delta T_j H_{j-1}^{(k)}(T_{j-1}))\bar{R})$$

/* Simulate default over (T_{j-1}, T_j) for the k^{th} name */

$$l = l+1$$

If $u_l < \lambda^{(k)}(T_{j-1})\Delta T_j$

$$D^{(k)}(T_j) = \bar{R}$$

$$K(T_j) = K(T_j) - 1$$

end if

$$D^P(T_j) \leftarrow D^P(T_j) + D^{(k)}(T_j)$$

/* Simulate the hazard rate $H^{(k)}(T_j)$ according to the market model */

$$H^{(k)}(T_j) = H^{(k)}(T_{j-1}) \exp\left((\mu_j^H(T_{j-1}) - \tfrac{1}{2} \| \gamma_j^H \|^2)\Delta T_j + \gamma_j^H \Delta W^{(k_s)}\right)$$

end if

end if

/* Calculate the payoff of the option */

$$V_j \leftarrow V_j + (D^P(T_j)/K - X)^+$$

end

end

/* Average payoff */

For $j = 1 : J$

$$V_j \leftarrow V_j/M$$

end

/* The end of the algorithm */

One can see that the entire algorithm is rather easy to implement and the computation time is about J times more than that of the Gaussian copula method of Li (2000).

Next, let us examine the ability of the model to back out correlations implied by various tranches of two standardized CDOs, namely, CDX IG and iTraxx IG. The quotes on August 24, 2004 are listed in Table 22.4.* The LIBOR and swap rates for the same day are listed in Table 22.5.

Without loss of generality, we make a few reasonable simplifications in the handling of data. We assume that the curve of forward spreads is flat and equal to the index of the respective maturity, which implies that the CDS rate of any maturity equals to the value of forward spreads. The CDS rate volatility is set at the constant level of either 50% or 100%, which represents the usual range of implied swaption volatilities (Schönbucher, 2004; Brigo, 2005). The recovery rate is taken to be $\bar{R} = 40\%$, abiding to the industrial convention. Owing to the lack of correlation data for CDS rates, we let the CDS rates and the Gaussian copula for default times share the same pairwise correlation. We take the number of paths to be $M = 10{,}000$ and the number of time stepping to be $\Delta t = 0.25$. The implied

* The data are taken from Hull et al. (2005).

TABLE 22.4 Quotes on 08/24/04

	CDX IG Tranches					
	0%–3%	**3%–7%**	**7%–10%**	**10%–15%**	**15%–30%**	**Index**
5 year quotes	40.02	295.71	120.50	43.00	12.43	59.73
10 year quotes	58.17	632.00	301.00	154.00	49.50	81.00
	iTraxx IG Tranches					
	0%–3%	**3%–6%**	**6%–9%**	**9%–12%**	**12%–22%**	**Index**
5 year quotes	24.10	127.50	54.00	32.50	18.00	37.79
10 year quotes	43.80	350.17	167.17	97.67	54.33	51.25

TABLE 22.5 U.S. Dollars Yield Data (08/24/04, Bloomberg)

Type of Rate	**Term**	**Rate**
LIBOR	3 mth	1.760%
	6 mth	1.980%
	12 mth	2.311%
Swap	2 Y	2.840%
	3 Y	3.265%
	4 Y	3.592%
	5 Y	3.890%
	6 Y	4.100%
	7 Y	4.295%
	8 Y	4.455%
	9 Y	4.595%
	10 Y	4.710%

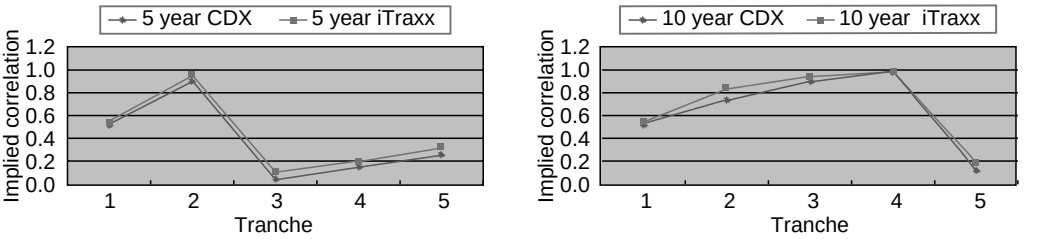


FIGURE 22.5 Implied correlations for $\|\gamma^H\| = 0.5$.

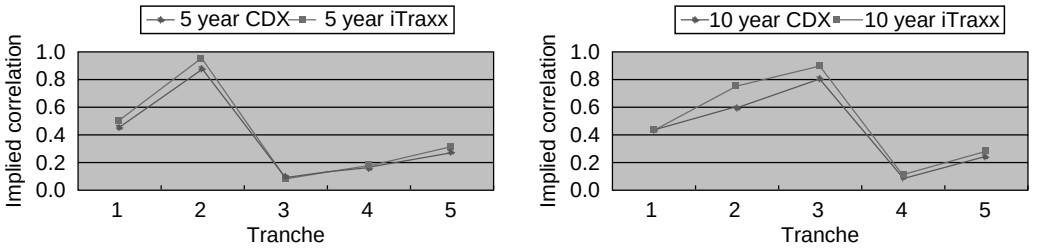


FIGURE 22.6 Implied correlations for $\|\gamma^H\| = 1$.

correlations of various tranches,* obtained through trial and error, are listed in Figures 22.5 and 22.6. The average accumulated default numbers for both CDOs, for a pairwise correlation of 20%, are shown in Figures 22.7 and 22.8.

Let us comment on the results. Under the market model, the implied correlation curves do not quite look like a smile. We have let $\|\gamma^H\|$, the CDS rates volatility, vary from 50% to 100%, and witnessed a gradual deformation of the smile curve. Remarkably, the smile curves of CDX IG and iTraxx IG stay close to each other, which is interesting but an explanation is not yet available. The histograms of average accumulated number of defaults look reasonable. The bigger numbers of defaults for CDX IG are consistent with the bigger numbers of spreads across all tranches. The algorithm is not optimized, but it has been very robust. The pricing of an entire CDO takes about 20 s in a PC with Intel Pentium 4 CPU (3.06 GHz, 504 MB RAM).

We tend to believe that the higher implied correlation for the mezzanine tranche is caused by the assumption of flat CDS rates across all maturities, which is against the fact that a CDS rate increases with maturity, as is seen in Table 22.1 for the CDS rates of Citigroup. If we bootstrap the forward spreads against the CDS rates, we should see that the forward spread curve has a bigger slope of increase than that of the CDS rate curve. The assumption of flat forward spread curve should have produced more defaults in short horizon yet less

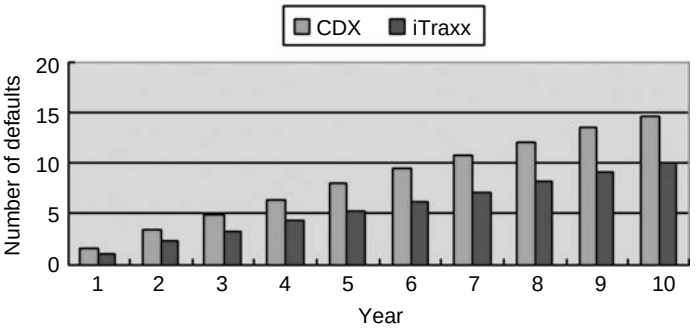


FIGURE 22.7 Accumulated number of defaults for $\|\gamma^H\| = 0.5$.

* Note that the implied correlation for a tranche may not necessarily be unique.

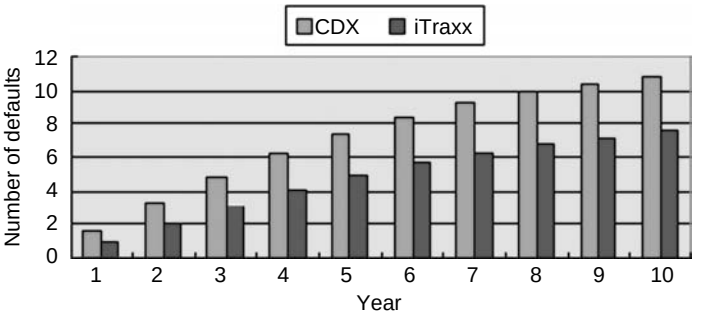


FIGURE 22.8 Accumulated number of defaults for $\|\gamma^H\| = 1$.

defaults for long horizon, which in turn can cause higher price fore junior tranches yet lower prices for senior tranches, as is reflected by the level of implied correlations.

We want to point out here that we have tried to reproduce the CDO spreads without default correlation, by letting the random variables $\{u_k\}$ be independent. The results are not good, which thus support a widely held opinion among market participants that the spread correlations are insufficient for pricing CDOs, for that we also need default correlations.

22.9 CONCLUSIONS

In this chapter, we develop a market model with the newly defined forward credit spreads. We justify the use of Black's formula for CDS options and present a natural algorithm for pricing CDO tranches. Model calibration and the calculations of implied CDO correlations are demonstrated with market data and the results show that the current model has the capacity to calibrate to both CDS options and CDOs. There are a number of outstanding theoretical, computational, and empirical issues left for further studies, which include allowing stochastic recovery rate, developing analytical solutions to CDO spreads, and studying the hedging performance of the model.

ACKNOWLEDGMENT

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APPENDIX: DETAILS OF SOME DERIVATIONS

Derivation of $\hat{f}_j(t)$:

The first line of Equation 22.4 leads to

$$\hat{f}_j(t) = \frac{1}{\Delta T}\left[\frac{P_j(t)}{P_{j+1}(t)} - 1\right] + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left[\frac{\Lambda_j(t)}{D_{j+1}(t)} - 1\right]$$

$$= f_j(t) + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left[\frac{E_t^{Q_j}\left(\mathbf{1}_{\{\tau > T_j\}}\right) - E_t^{Q_{j+1}}\left(\mathbf{1}_{\{\tau > T_{j+1}\}} + R\mathbf{1}_{\{T_j < \tau \le T_{j+1}\}}\right)}{D_{j+1}(t)}\right]$$

$$= f_j(t) + \frac{1}{\Delta T}\frac{P_j(t)}{P_{j+1}(t)}\left\{\frac{E_t^{Q_{j+1}}[(1-R)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right\}$$

$$= f_j(t) + \frac{1+\Delta T_j f_j(t)}{\Delta T}\left\{\frac{E_t^{Q_{j+1}}[(1-R)\mathbf{1}_{\{T_j<\tau\le T_{j+1}\}}]}{D_{j+1}(t)}\right\} \tag{22.A1}$$

In the third row of the equation, we have again used the independence between the hazard rate and Treasury yields. The second line of Equation 22.4 leads to

$$\hat{f}_j(t) = \frac{1}{\Delta T}\left[\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)} - 1\right] - \frac{1}{\Delta T}\frac{\bar{P}_j(t)}{\bar{P}_{j+1}(t)}\left[1 - \frac{\Lambda_j(t)}{D_j(t)}\right]$$

$$= \bar{f}_j(t) - \frac{1+\Delta T_j \bar{f}_j(t)}{\Delta T_j}\left[\frac{E_t^{Q_j}(R\mathbf{1}_{\{T_{j-1}<\tau\le T_j\}})}{D_j(t)}\right] \tag{22.A2}$$

Derivation of the drift term of $H_j(t)$:

According to its definition, we have

$$H_j(t) = \frac{1}{\Delta T_j}\left[\frac{1+\Delta T_j \hat{f}_j(t)}{1+\Delta T_j f_j(t)} - 1\right]$$

$$= \frac{1}{\Delta T_j}\left[\frac{\frac{\bar{P}_j}{\bar{P}_{j+1}}\cdot\frac{\Lambda_j(t)}{D_j}}{\frac{P_j}{P_{j+1}}} - 1\right]$$

$$= \frac{1}{\Delta T_j}\left[\frac{D_j}{D_{j+1}}\cdot\frac{\Lambda_j(t)}{D_j} - 1\right] \tag{22.A3}$$

For simplicity, we assume that the recovery rate is time stationary, meaning that $E_t^{Q_{T_j}}[R_\tau] = \bar{R} = \text{constant}$. Our first step is to express Λ_j/D_j in terms of H_j. In fact,

$$\frac{\Lambda_j}{D_j} = \frac{\Lambda_j + \bar{R}(\Lambda_{j-1} - \Lambda_j) - \bar{R}(\Lambda_{j-1} - \Lambda_j)}{D_j}$$

$$= 1 - \frac{\bar{R}\Delta T_{j-1}}{1-\bar{R}}\frac{(1-\bar{R})(\Lambda_{j-1} - \Lambda_j)}{\Delta T_{j-1} D_j}$$

$$= 1 - \frac{\bar{R}}{1-\bar{R}}\Delta T_{j-1} H_{j-1} \tag{22.A4}$$

According to the LIBOR model, the risk-neutral process of $H_j(t)$ is

$$dH_j(t) = H_j\left(\gamma_j^H d\mathbf{W}_t + \mu_j^H dt\right) \quad \text{for all } j$$

We thus have

$$\begin{aligned} d\left(\frac{\Lambda_j(t)}{D_j(t)}\right) &= -\frac{\bar{R}}{1-\bar{R}}\Delta T_{j-1} dH_{j-1}(t) \\ &= -\frac{\bar{R}}{1-\bar{R}}\Delta T_{j-1} H_{j-1}(t)\left[\gamma_{j-1}^H(t) d\mathbf{W}_t + \mu_{j-1}^H dt\right] \\ &= \left(\frac{\Lambda_j}{D_j} - 1\right)\left[\gamma_{j-1}^H(t) d\mathbf{W}_t + \mu_{j-1}^H dt\right] \end{aligned} \tag{22.A5}$$

It then follows that

$$\begin{aligned} dH_j(t) &= \frac{1}{\Delta T_j}\left[d\left(\frac{D_j}{D_{j+1}}\right)\left(\frac{\Lambda_j}{D_j}\right) + \left(\frac{D_j}{D_{j+1}}\right) d\left(\frac{\Lambda_j}{D_j}\right) + d\left(\frac{D_j}{D_{j+1}}\right) d\left(\frac{\Lambda_j}{D_j}\right)\right] \\ &= \frac{1}{\Delta T_j}\left[\frac{\Lambda_j}{\Delta_j}\frac{D_j}{D_{j+1}}\left(\sigma_j^D - \sigma_{j+1}^D\right)\cdot d\mathbf{W}_t + \frac{D_j}{D_{j+1}}\left(\frac{\Lambda_j}{D_j} - 1\right)\gamma_{j-1}^H(t) d\mathbf{W}_t\right] \\ &\quad + \text{drift terms} \\ &= \frac{1}{\Delta T_j}\left\{(1+\Delta T_j H_j)\left(\sigma_j^D - \sigma_{j+1}^D\right) d\mathbf{W}_t + \left[(1+\Delta T_j H_j) - \frac{D_j}{D_{j+1}}\right]\gamma_{j-1}^H d\mathbf{W}_t\right\} \\ &\quad + \text{drift terms} \\ &= \left\{\frac{1+\Delta T_j H_j}{\Delta T_j}\left(\sigma_j^D - \sigma_{j+1}^D\right) + \left[H_j - \frac{1}{\Delta T_j}\left(\frac{D_j}{D_{j+1}} - 1\right)\right]\gamma_{j-1}^H\right\} d\mathbf{W}_t \\ &\quad + \text{drift terms} \\ &\triangleq \left[\frac{1+\Delta T_j H_j}{\Delta T_j}\left(\sigma_j^D - \sigma_{j+1}^D\right) + \left(H_j - \bar{H}_j\right)\gamma_{j-1}^H\right] d\mathbf{W}_t + \text{drift terms} \end{aligned} \tag{22.A6}$$

where σ_j^D is the percentage volatility of $D_j(t)$. By equating the volatility term above to $\gamma_j^H H_j$, we obtain

$$\frac{1+\Delta T_j H_j}{\Delta T_j}\left(\sigma_j^D - \sigma_{j+1}^D\right) = H_j\gamma_j^H - (H_j - \bar{H}_j)\gamma_{j-1}^H \tag{22.A7}$$

We then have the recursive formula for the volatility σ_j^D as

$$\begin{aligned} \sigma_{j+1}^D &= \sigma_j^D - \frac{\Delta T_j H_j}{1+\Delta T_j H_j}\left[\gamma_j^H - \left(1 - \frac{\bar{H}_j}{H_j}\right)\gamma_{j-1}^H\right] \\ &= \sigma_1^D - \sum_{k=1}^{j}\frac{\Delta T_k H_k}{1+\Delta T_k H_k}\left[\gamma_k^H - \left(1 - \frac{\bar{H}_k}{H_k}\right)\gamma_{k-1}^H\right] \end{aligned} \tag{22.A8}$$

Here,

$$1 - \frac{\bar{H}_j(t)}{H_j(t)} = -\frac{1+\Delta T_j H_j(t)}{\Delta T_j H_j(t)}\frac{\bar{R}\Delta T_{j-1} H_{j-1}(t)}{1-\bar{R}-\bar{R}\Delta T_{j-1} H_{j-1}(t)}$$

In addition,

$$\sigma_1^D = \bar{\sigma}_1 - \sigma_1$$

where $\bar{\sigma}_1$ and σ_1 are the volatilities of $\bar{P}_1$ and P_1, respectively. Unless a default is imminent, we may comfortably put $\sigma_1^D = 0$ and hence obtain an expression of σ_{j+1}^D in terms of $\gamma_k^H, k < j$.

The drift term of H_j is given by

$$\begin{aligned}
\mu_j^H H_j &= \frac{1}{\Delta T_j}\left[-\frac{\Lambda_j}{D_j}\frac{D_j}{D_{j+1}}\left(\sigma_j^D - \sigma_{j+1}^D\right)\sigma_{j+1}^D + \frac{D_j}{D_{j+1}}\left(\frac{\Lambda_j}{D_j} - 1\right)\mu_{j-1}^H\right] \\
&\quad + \frac{D_j}{D_{j+1}}\left(\sigma_j^D - \sigma_{j+1}^D\right)\left(\frac{\Lambda_j}{D_j} - 1\right)\gamma_{j-1}^H \\
&= \frac{1}{\Delta T_j}\left\{-\Delta T_j\left[\gamma_j^H H_j - \gamma_{j-1}^H(H_j - \bar{H}_j)\right]\sigma_{j+1}^D + \Delta T_j \mu_{j-1}^H (H_j - \bar{H}_j)\right. \\
&\quad \left. + \Delta T_j (H_j - \bar{H}_j)\left(\sigma_j^D - \sigma_{j+1}^D\right)\gamma_{j-1}^H\right\} \\
&= -H_j \gamma_j^H \sigma_{j+1}^D + (H_j - \bar{H}_j)\left(\mu_{j-1}^H + \gamma_{j-1}^H \sigma_j^D\right) \\
&= H_j\left[-\gamma_j^H \sigma_{j+1}^D + \left(1 - \frac{\bar{H}_j}{H_j}\right)\left(\mu_{j-1}^H + \gamma_{j-1}^H \sigma_j^D\right)\right]
\end{aligned} \tag{22.A9}$$

Hence,

$$\mu_j^H = -\gamma_j^H \sigma_{j+1}^D + \left(1 - \frac{\bar{H}_j}{H_j}\right)\left[\mu_{j-1}^H(t) + \gamma_{j-1}^H \sigma_j^D\right] \tag{22.A10}$$

or

$$\mu_j^H + \gamma_j^H \sigma_{j+1}^D = \left(1 - \frac{\bar{H}_j}{H_j}\right)\left[\mu_{j-1}^H(t) + \gamma_{j-1}^H \sigma_j^D\right] \tag{22.A11}$$

If we make an insignificant assumption that the chance of imminent default between t and T_0 is zero, then we have $\mu_0^H(t) + \gamma_0^H \sigma_1^D = 0$, and, consequently,

$$\mu_j^H = -\gamma_j^H \sigma_{j+1}^D$$

Derivation of the drift term of $S_j(t)$:

The drift and volatility terms of S_j can now be derived with ease. Since

$$S_j = (1 + \Delta T_j f_j) H_j$$

we have

$$\begin{aligned} dS_j &= \Delta T_j H_j df_j + (1 + \Delta T_j f_j)\, dH_j + \Delta T_j df_j\, dH_j \\ &= \Delta T_j H_j f_j (\mu_j dt + \gamma_j \cdot d\mathbf{W}_t) + (1 + \Delta T_j f_j)\, H_j \left(\mu_j^H dt + \gamma_j^H \cdot d\mathbf{W}_t\right) + \Delta T_j f_j H_j \gamma_j \gamma_j^H dt \\ &= S_j \left\{ \left[\frac{\Delta T_j f_j}{1 + \Delta T_j f_j} \left(\mu_j + \gamma_j \gamma_j^H\right) + \mu_j^H \right] dt + \left(\frac{\Delta T_j f_j}{1 + \Delta T_j f_j} \gamma_j + \gamma_j^H \right) \cdot d\mathbf{W}_t \right\} \\ &\triangleq S_j \left(\mu_j^S dt + \mu_j^S \cdot d\mathbf{W}_t\right) \end{aligned} \tag{22.A12}$$

where μ_j^S and γ_j^S relate to μ_j^H and γ_j^H through

$$\begin{aligned} \mu_j^S &= \mu_j^H + \frac{\Delta T_j f_j}{1 + \Delta T_j f_j} \left(\mu_j + \gamma_j \gamma_j^H\right) \\ \gamma_j^S &= \gamma_j^H + \frac{\Delta T_j f_j}{1 + \Delta T_j f_j} \gamma_j \end{aligned} \tag{22.A13}$$

By substituting the above expressions for μ_j^H and γ_j^H in Equations 22.A8 and 22.A10, we obtain Equation 22.26.

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CHAPTER 23

An Empirical Analysis of CDO Data

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23.1 INTRODUCTION

Over the last decade, the growth in the credit derivatives market has been enormous. First, the market for credit default swaps (CDSs) on single names has become very liquid. This was followed by the rapid development of the synthetic collateralized debt obligation (CDO) market over the last couple of years. Following this large growth in liquidity there has been a need for models which can be calibrated to market quotes. Amongst practitioners, the one factor Gaussian copula has become standard. In order to allow for calibration an ad hoc method has become popular, known as the base correlation approach, presented in McGinty et al. (2004). This method cannot be regarded as a model explaining the observed market quotes, but rather as a fix of the standard one factor Gaussian copula model, much like implied volatilities in the equity derivatives markets. When considering nonstandard tranches, i.e., tranches with nonstandard attachment and detachment levels, tranches with nonstandard maturities or tranches on nonstandard indices, some mapping choices have to be made. There are multiple ways to map the observed base correlation skew to different maturities or different baskets and three of them are considered in this chapter.

In this chapter we perform an empirical analysis of market prices for CDO tranches, using a data set ranging from December 2004 up to November 2006. The data set consists of three different sets of quotes. Quotes of standard interest rate products are available and these are used to determine risk-free discount factors. For each reference entity entire CDS term structures are available, which are used to determine marginal distributions of the default times. Finally quotes for synthetic CDO tranches are available and these are used to determine the market-implied default dependency structure. Using only the data on synthetic CDO tranche quotes we first investigate whether correlation effects can be discovered from these quotes directly. Therefore the influence of the credit risk of the underlying index is filtered out by means of regression analysis. The resulting residuals should then capture correlation effects.

The second part of the analysis uses the base correlation method to imply the default dependency structure from the market. First, the base correlation method is used to investigate the behavior of market-implied credit correlation over time. Secondly, three different mapping methods are considered and their performance is investigated by means of out-of-sample tests. An out-of-sample test is considered where tranche quotes with five year maturity on a certain index are mapped to tranches with either seven or ten year maturities on the same index. In addition out-of-sample tests are performed where

mapping is done using quotes for five year tranches on one index and mapping the resulting base correlations to five, seven, and ten year tranches on the other index.

The outline of this chapter is as follows. In the next section the synthetic CDO market is discussed. The mechanics of the products are discussed and the quoting convention is explained. In addition it is shown that pricing of these products can be reduced to determining the expected loss on equity tranches. The third section focuses on the modeling of default dependency by means of copulas and the one factor Gaussian copula in particular. In addition, the base correlation method is described, which has become the industry standard approach for visualizing default correlation implied in the market. Section four describes the data which have been used in the empirical analysis and some summary statistics are presented. In addition, the increase in liquidity in the synthetic CDO market is briefly touched upon. Section five presents the results of the empirical analysis. First the synthetic CDO tranche quotes are considered directly without using any default correlation model. The second part of this section applies the base correlation method to the available data. First, the behavior of the different base correlations with a five year maturity is considered. Second, the base correlation method is used in combination with different mapping methods in order to investigate the performance of three different mapping methods by means of out-of-sample tests. Finally, Section 23.6 concludes this chapter.

23.2 SYNTHETIC CDO TRANCHES

In a collateralized debt obligation, CDO, a pool of assets functions as collateral against which notes are issued. These notes represent tranches of the structure and strictly regulate the seniority of the notes. In the following figure (Figure 23.1), the structure of a CDO contract is graphically represented. Here on the left-hand side one can see the pool of collateral, while on the right-hand side the CDO structure is represented where four different tranches are issued, the equity, the junior mezzanine, the senior mezzanine, and the super senior tranche. In the case of a default event in the pool of collateral the first losses are absorbed by the equity tranche investor. Once the cumulative losses increase beyond the size of the equity tranche, the junior mezzanine tranche is affected and so forth. Clearly the equity tranche investor is much more exposed to losses in the pool of collateral and therefore this investor is compensated by receiving a larger coupon. Rating agencies can be involved to assign credit ratings to the different tranches. Usually the equity tranche is not rated, due to the large risk involved. As the name suggests these tranches are similar to investing in equity, while the more senior notes can be compared to investing in bonds.

Traditionally these CDO tranches are backed by a pool of cash assets, such as bonds, student loans, or credit card receivables. Due to issues such as different maturities and prepayment of assets such structures usually require a manager to manage the cash flows and invest in new assets. It was realized that one could also create the pool of collateral synthetically by means of CDSs. The clear advantage here is that one can select CDSs with equal maturity leading to easy to predict cash flows which are only affected by a default event of one of the underlying reference entities. More information about different types of CDOs can be found in Anson et al. (2004) or Tavakoli (2003).

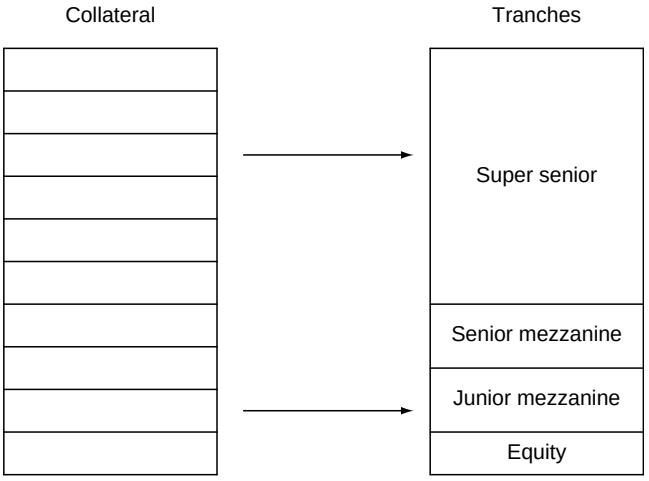


FIGURE 23.1 CDO structure with the pool of collateral assets on the left-hand side and four different tranches on the right-hand side. The risky equity tranche is shown at the bottom of the CDO structure and is followed by a junior mezzanine tranche, a senior mezzanine tranche, and a super senior tranche.

23.2.1 Standard Tranches and Quoting Conventions

In recent years the market for synthetic CDO tranches has become increasingly liquid. Two reference indices have been defined, known as CDX NA IG and iTraxx Europe, which contain 125 North American reference entities and 125 European reference entities, respectively. All these names are high grade issuers and on March 20th and September 20th of each year the indices are rolled, leading to new series. At a roll date names for which credit worthiness has deteriorated are possibly replaced by names with better credit quality. In addition to a possible change of index composition, at a roll date the maturity of quoted synthetic CDO tranches also change with six months. Standard maturities are at 20th June and 20th December of each year. For instance, after an index roll at 20th March, the new on the run tranches reference the new index and have maturities at 20th June. Thus, initially the five year quoted CDO tranches have a tenor of five years and three months. Just before the next roll date the tenor has reduced to four years and nine months.

Five tranches on the iTraxx index are quoted in the market with attachment and detachment levels: 0%–3%, 3%–6%, 6%–9%, 9%–12%, and 12%–22%. For the tranches on the CDX index detachment levels are slightly different at: 0%–3%, 3%–7%, 7%–10%, 10%–15%, and 15%–30%. All tranches have quarterly payments and the Actual/360 day-count convention is used to determine premium payments. For the equity tranche the quote is given as a percentage of the notional to be paid up front by the protection buyer, in addition to premium payments of 500 bp on an annual basis, paid quarterly. Typically the five tranches are quoted in combination with a notional delta exchange. This means that a protection buyer can buy protection for a certain tranche in return for the quoted premium

TABLE 23.1 Market Quotes for the Five Standard Synthetic CDO Tranches on iTraxx, with a Tenor of Five Years

iTraxx	Tranche	Bid	Offer	Delta
Ref. 24 bp	0–3	11.625	11.875	25.5
	3–6	54.5	54.75	5
	6–9	13.25	14.25	1.5
	9–12	5	6	0.5
	12–22	2	3	0.3

For each tranche the bid and offer quotes are shown as well as the notional delta exchange. In addition a reference spread is shown. Quotes are from the 1st of November 2006 and thus maturity for the quoted tranches is December 20th, 2011.

and in addition will sell protection on the index CDS with notional equal to the tranche notional times the notional delta exchange. The reason for this delta exchange is that both parties are likely to hedge their credit delta risk. Trading the delta hedge with each other, in combination with the tranche, is thus simple and avoids extra costs due to the bid/offer spread. The quotes also consist of the index spread against which the notional delta exchange part of the trade is executed and this is known as the reference spread. Both reference spread and delta exchanges are agreed periodically by a dealer poll, where the reference spread follows the index spread closely.

Table 23.1 gives an example of a set of synthetic CDO quotes on the iTraxx index with maturity of five years.

Thus, for instance an investor can sell credit protection on the 3%–6% tranche for 1 million notional, receiving 54.5 bp. Simultaneously, the investor buys protection on the index CDS for 5 million notional against a spread of 24 bp.

The synthetic CDO market on these indices started with quoting for a five year tenor only. In recent years the range of maturities has expanded to include seven and ten year maturities as well. However, the major part of the analysis presented in this chapter focuses on tranches with a five year maturity as these are most liquid.

23.2.2 Pricing of Synthetic CDO Tranches

In order to price a synthetic CDO tranche the focus will be on the distribution of the cumulative loss of a certain basket at future points in time. Let $L^{\mathcal{B}}(t)$ denote the percentage loss on basket $\mathcal{B}$ at time t. The percentage loss on an equity tranche, i.e., a tranche with zero attachment, and detachment level u is given by

$$L_u^{\mathcal{B}}(t) = \frac{1}{u} \min\left(L^{\mathcal{B}}(t),\, u\right) \tag{23.1}$$

Next a general tranche is considered, having both an attachment as well as a detachment level. The percentage loss of a tranche with attachment l and detachment u is given by

$$\begin{aligned} L^{\mathcal{B}}_{l,u}(t) &= \frac{\min\left(\max\left(L^{\mathcal{B}}(t)-l,0\right),\, u-l\right)}{u-l} \\ &= \frac{\min\left(L^{\mathcal{B}}(t),u\right)-\min\left(L^{\mathcal{B}}(t),\, l\right)}{u-l} \\ &= \frac{u\cdot L^{\mathcal{B}}_u(t)-l\cdot L^{\mathcal{B}}_l(t)}{u-l} \end{aligned} \tag{23.2}$$

Thus, the loss on any tranche can always be written in terms of the loss on equity tranches. Modeling of tranches will focus on the modeling of the cumulative loss of the basket over time, $L^{\mathcal{B}}(t)$. Once a certain model is chosen, the focus will be on the expected loss on a certain equity tranche at future times, i.e., $\mathbb{E}[L^{\mathcal{B}}_u(t)]$. This expected loss is calculated under the risk neutral measure as the model for the cumulative loss will be consistent with the market-observed values for CDSs. The expected loss needs to be evaluated for different times in the future due to discounting and due to premium payments which are made over outstanding tranche notional, which is expected to decrease over time.

First the value of the protection leg of an equity tranche is considered using a discrete approximation over time. Let $T_0,\, T_1,\ldots,\, T_{M-1},\, T_M = T$ denote a partition of the time interval, where T_0 denotes the valuation date and T_M the maturity. The value, $P^{\mathcal{B},T}_u$, of the protection leg of an equity tranche with detachment u on basket $\mathcal{B}$ with maturity T can then be approximated using the following expression:

$$\begin{aligned} P^{\mathcal{B},T}_u &\approx \mathbb{E}\left[\sum_{i=1}^{M} D\left(\frac{1}{2}(T_{i-1}+T_i)\right)\,\left(L^{\mathcal{B}}_u(T_i)-L^{\mathcal{B}}_u(T_{i-1})\right)\right] \\ &= \sum_{i=1}^{M} D\left(\frac{1}{2}(T_{i-1}+T_i)\right)\left(\mathbb{E}\left[L^{\mathcal{B}}_u(T_i)\right]-\mathbb{E}\left[L^{\mathcal{B}}_u(T_{i-1})\right]\right) \end{aligned} \tag{23.3}$$

Here $D(t)$ denotes the value at the valuation date of a risk-free payment of one unit made at time t. In the expression the midpoint assumption is used as it is assumed that all protection payments during the period are made at the middle of the period under consideration. Similarly one can derive the value, $A^{\mathcal{B},T}_u$, of the premium leg in the case of 1 bp premium. Here we consider the case of premium payments made over outstanding notional, which is the most commonly traded variation of synthetic CDO tranche swaps. The premium payment dates, $T_k,\, k = 1,\ldots,N$, can be used for discretization. Furthermore, we let T_0 denote the valuation date.

$$\begin{aligned} A^{\mathcal{B},T}_u &\approx \mathbb{E}\left[\sum_{k=1}^{N} \delta_k\, D\left(T_k\right)\frac{1}{2}\left(1-L^{\mathcal{B}}_u(T_k)+1-L^{\mathcal{B}}_u(T_{k-1})\right)\right] \\ &= \sum_{k=1}^{N} \delta_k\, D\left(T_k\right)\left(1-\frac{1}{2}\left(\mathbb{E}\left[L^{\mathcal{B}}_u(T_k)\right]+\mathbb{E}\left[L^{\mathcal{B}}_u(T_{k-1})\right]\right)\right) \end{aligned} \tag{23.4}$$

Here the term δ_k denotes the day count fraction which applies to the premium payment to be made at time T_k. There are some important differences when comparing Equations 23.3

and 23.4. First, the discretization for the protection leg is general, while for the premium leg the payment dates have been used. Second, discounting for the protection leg is done in the middle of a period, while for the premium leg the end of the period is used, which corresponds to the payment date.

Due to the simple linear relationships it is straightforward to derive the following relationships for general tranches:

$$\begin{aligned} P_{l,u}^{\mathcal{B},T} &= \frac{u \cdot P_u^{\mathcal{B},T} - l \cdot P_l^{\mathcal{B},T}}{u - l} \\ A_{l,u}^{\mathcal{B},T} &= \frac{u \cdot A_u^{\mathcal{B},T} - l \cdot A_l^{\mathcal{B},T}}{u - l} \end{aligned} \tag{23.5}$$

As the value of the protection leg is defined for one unit of tranche notional, one has to weigh using the attachment and detachment levels of the contract. This relation shows that one can write both protection and premium leg values of any tranche in terms of two equity tranches. This relationship is the main observation used in the base correlation model discussed in Section 23.3.2. Given the values of the protection leg and premium leg for 1 bp premium, one could easily determine the value, $V_{l,u}^{\mathcal{B},T}$, of a CDO with contracted premium π^c, or one could determine the fair premium π^f of that tranche:

$$\begin{aligned} V_{l,u}^{\mathcal{B},T} &= P_{l,u}^{\mathcal{B},T} - \pi^c \cdot A_{l,u}^{\mathcal{B},T} \\ \pi^f &= \frac{P_{l,u}^{\mathcal{B},T}}{A_{l,u}^{\mathcal{B},T}} \end{aligned} \tag{23.6}$$

Here the value is given from the point of view of the protection buyer. In case an up-front payment is specified, one could easily determine the value by subtracting this up-front payment. Note that this is only relevant at initiation of a contract.

23.3 MODELING DEFAULT DEPENDENCY

Default dependency is usually modeled using copula methods, see Nelsen (1999) for more details on copulas. The main advantage of copulas is that they allow one to split modeling of marginal distributions from the modeling of the dependency structure. The one factor Gaussian copula is a special case of the more general Gaussian copula, presented in Li (2000), where some additional structure is enforced which simplifies the model.

23.3.1 One Factor Gaussian Copula

The market standard model has become the one factor Gaussian copula. Default events are modeled by means of latent variables and default dependency is modeled by correlating the latent variables in some specific way. For the one factor Gaussian copula model, the latent variables are modeled as follows:

$$\begin{aligned} X_i &\equiv \rho_i \cdot Y + \sqrt{1 - \rho_i^2} \cdot \xi_i \\ \tau_i \leq t &\Leftrightarrow X_{\mathrm{i}} \leq \chi_i(t) \end{aligned} \tag{23.7}$$

Here τ_i denotes the default time for reference entity *i*. X_{i} denotes the latent variable for name *i* and a default event has occurred before time *t*, when the latent variable is smaller than some time dependent threshold function $\chi_i(t)$. The latent variables of the different names are correlated through the common factor, *Y*, which is assumed to have a standard normal distribution. The second term depends on the idiosyncratic random variable ξ_i, which also has a standard normal distribution. All the random normals, i.e., *Y* and ξ_i are independent. It is easy to see that the latent variables, X_i, have a standard normal distribution as well. In order to make sure that the distribution of default times are calibrated to the market, one must use the distribution of these latent variables to determine the thresholds $\chi_i(t)$:

$$\chi_i(t) = \Phi^{-1}(p_i(t)) \tag{23.8}$$

Here $p_i(t)$ is the marginal default probability for name *i* for the period up to time *t*, which is obtained by bootstrapping to CDS quotes from the market. Φ denotes the cumulative distribution function for the standard normal distribution.

One can observe that the latent variables X_i and X_j, for $i \neq j$, as defined in Equation 23.7 have correlation $\rho_i \cdot \rho_j$. In practice, due to the large number of names to model, the parameters ρ_i are often chosen to be constant over all names, i.e., $\rho_i = \rho$, $\forall i$. In this case the default dependency structure is driven by a single correlation parameter.

As discussed in Section 23.2.2 the valuation of synthetic CDO tranches comes down to modeling $\mathbb{E}[L_u^{\mathcal{B}}(t)]$. Under the class of one factor copula models this can be done relatively fast using the observation that, conditional on the realization of the common factor, all remaining sources of risk are independent.

$$\mathbb{E}\left[L_u^{\mathcal{B}}(t)\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[L_u^{\mathcal{B}}(t)|Y = y\right]\phi(y)dy \tag{23.9}$$

Here ϕ denotes the probability density function for the standard normal distribution. The problem has thus been reduced to finding the expected loss on an equity tranche conditional on the realization of the common factor.

One can use a simplified version of the methods described in Andersen et al. (2003), Hull and White (2004), and Laurent and Gregory (2003) and we will illustrate this method using an example where the probability of exactly *n* defaults at some future time *t* will be determined. The approach can be seen as a recombining binomial tree which has *N* steps, each step corresponding to one name in the basket. At horizonal step *i* the probability of an up-move will be the default probability of name *i* conditional on the realization of the common factor $Y = y$: $p_i(t|y)$. Thus, the probability of a down-move will be $1 - p_i(t|y)$. Every up-move corresponds to a default event, while a down-move corresponds to survival. Working from the root of the tree back to the end values, one obtains the probability of exactly *n* defaults, for $n = 0, \ldots, N$. To illustrate this, let $\delta_{i,j}$ denote the path probability at step *i* and height *j*. We assume that *i* runs from $0, \ldots, N$, and $j = 0, \ldots, i$.

Thus we start the algorithm by setting $\delta_{0,0} = 1$, after which the algorithm continues as follows:

$$\delta_{i,j} = \begin{cases} (1 - p_i) \cdot \delta_{i-1,j} & \text{if } j = 0 \\ p_i \cdot \delta_{i-1,j-1} + (1 - p_i) \cdot \delta_{i-1,j} & \text{if } 0 < j < i \\ p_i \cdot \delta_{i-1,j-1} & \text{if } j = i \end{cases} \tag{23.10}$$

It is important to note that this algorithm can only be applied with independent default probabilities. For brevity we have used $p_i = p_i(t|y)$. If one is done working through the tree, the values $\delta_{N,j} = \delta_{N,j}(t|y)$ give the conditional probability of exactly j defaults in the basket of N names at time t. Integrating these terms out over the distribution of the common factor will result in the probability of exactly j defaults, $\pi(j, t)$ for $j = 0, \ldots, N$. Thus, we get

$$\pi(j, t) = \int_{-\infty}^{\infty} \delta_{N,j}(t|y)\phi(y)dy \tag{23.11}$$

To calculate this integral one has to resort to numerical integration techniques, for instance Gauss-Legendre quadratures as described in Abramowitz and Stegun (1972). During numerical integration one has to build the entire tree as shown in Equation 23.10 for each chosen value for the common factor realization, y.

From this one can easily determine the expected loss on a certain tranche. Alternatively one could determine the expected loss on a tranche conditional on the realization of the common factor and integrate out subsequently. This will lead to the same result as first integrating over probabilities. As can be seen from the pricing formulas 23.3 and 23.4 this procedure has to be repeated to determine the expected loss on a tranche for different future times.

23.3.2 Base Correlation

In recent years the market for tranche CDSs has become very liquid leading to a need for a marking-to-market approach. Similar to implied volatilities, the notion of implied correlation can be introduced for these products using the standard one factor Gaussian copula. Assuming that each pair of reference entities has a constant correlation parameter, one can derive a correlation such that the model price exactly matches the price observed in the market. This results in a tranche specific correlation parameter known as the implied, or compound correlation. One drawback of such an approach is that one is left with correlations which are a function of both attachment as well as detachment level. The base correlation method, due to McGinty et al. (2004), overcomes this issue by switching to equity tranches or base tranches. In Section 23.2.2 it was shown that the expected loss of a certain tranche can always be written as the difference of expected losses for two equity tranches, see Equation 23.5. The idea of the base correlation method is that all tranches are reduced to base tranches and correlation depends on the detachment level of the tranche. One can iteratively solve for the correlations belonging to each detachment level for which

a quote is available. The base correlation method has become the industry standard for valuing CDO tranches, see Reyfman (2004) or Friend and Rogge (2005) for instance.

Let $Q_{l,u}$ denote the quote of a tranche with attachment percentage l and detachment percentage u. Further, let $U_{l,u}$ denote the up-front value in percentages of tranche notional. A set of base correlations is calculated per basket and per maturity for which quotes are available. In the derivations below the basket and maturity are fixed and for brevity their dependency is not shown in the protection and premium values. We want to determine ρ_u recursively such that the model-implied fair spread equals the spread quotes in the market.

$$Q_{l,u} = \frac{u \cdot P_u(\rho_u) - l \cdot P_l(\rho_l) - (u-l) \cdot U_{l,u}}{u \cdot A_u(\rho_u) - l \cdot A_l(\rho_l)} \tag{23.12}$$

For the quoted equity tranche the attachment percentage is 0 and thus both P_l and A_l are zero. One can use a numerical root-finding routine, see for instance Press et al. (2002), to obtain the value of ρ_u such that the quote is exactly matched by the model. Once this has been determined, one can move to the next tranche and its attachment level, l, equals the detachment level of the previous tranche under consideration. Thus ρ_l is already known, as this is ρ_u solved in the previous step. This can be used to determine the next base correlation:

$$\begin{aligned} Q_{l,u}(u \cdot A_u(\rho_u) - l \cdot A_l(\rho_l)) &= u \cdot P_u(\rho_u) - l \cdot P_l(\rho_l) - (u-l) \cdot U_{l,u} \\ Q_{l,u} &= \frac{u \cdot P_u(\rho_u) - (u-l) \cdot \tilde{U}_{l,u}}{A_u(\rho_u)} \end{aligned} \tag{23.13}$$

In the last line the terms already known, i.e., those depending on the attachment level l have been grouped in the adjusted up-front value:

$$\tilde{U}_{l,u} \equiv U_{l,u} + \frac{l \cdot P_l(\rho_l) - l \cdot A_l(\rho_l) \cdot Q_{l,u}}{u-l} \tag{23.14}$$

From the final line of Equation 23.13 one can see that the problem of finding ρ_u is reduced to finding the implied correlation of an equity tranche with detachment level u. Moreover as the value of an equity tranche is monotonically decreasing in default correlation, this relationship shows that the base correlation is unique, something which is not necessarily the case for implied correlations for general tranches. Moreover the monotonicity makes numerical root-finding a straightforward exercise.

Note that this algorithm can be applied only when quotes are available for successive tranches, where the detachment level of one tranche equals the attachment of another. Furthermore, a quote for the equity tranche is required as well.

The following figure (Figure 23.2) shows both compound correlations as well as base correlations using market data of the 1st of November 2006, for tranches on the CDX index with maturity of five years. Both curves are plotted as a function of the detachment level. The compound correlation for the equity tranche is around 14%, after which it decreases to a level close to 6.5% for the junior mezzanine tranche. For more senior

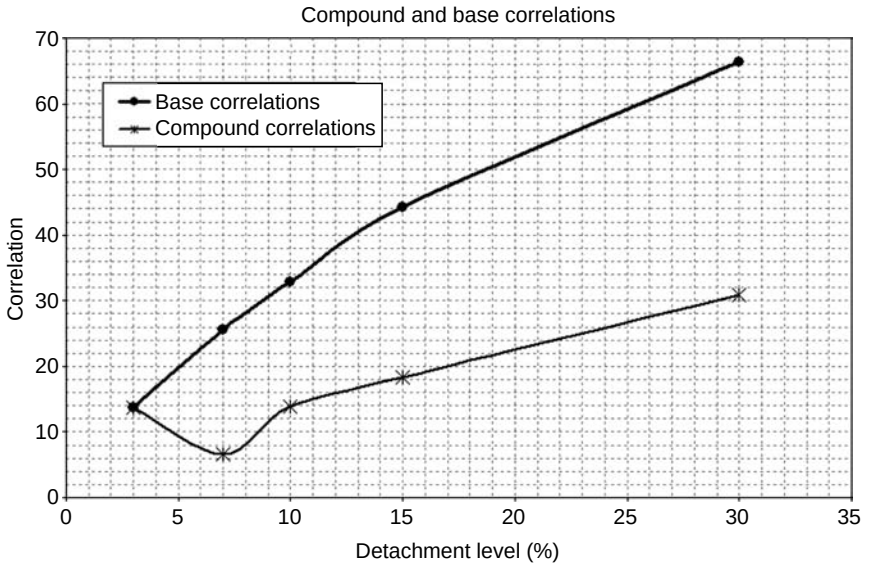


FIGURE 23.2 Compound and base correlations plotted against the detachment level of the tranches. Quotes on standard tranches on the DJ CDX basket are used from the 1st of November 2006.

tranches it increases again from roughly 14% to 18% to reach a level of 31% for the most senior tranche. Further, one can observe a close to linear relationship between base correlations and detachment level. By construction the base correlation for the equity tranche equals the compound correlation. The base correlations increase to roughly 66% for the most senior tranche. The fact that the lines are not horizontal clearly illustrates the model's inability to fit the market with a single correlation parameter.

In order to apply the base correlation method to nonstandard tranches, for example tranches with nonstandard attachment and detachment levels, tranches with different maturities, or tranches on bespoke baskets, some mapping approach is required. Ideally such an approach would take the credit risk in the underlying basket over the relevant period into account. For example a 0%–3% tranche on a basket which has fair spread of 30 bp is very different from a 0%–3% tranche on a basket having a fair spread at 100 bp. The following three different mapping methods have been described in literature and will be considered in this study:

- Direct *d*etachment–*d*etachment mapping, *dd*
- *D*etachment as a *p*ercentage of expected basket loss, *dp*
- Expected *l*oss *p*ercentage mapping, *lp*

The first method states that the base correlation for any basket just depends on the detachment level only. It does not take into consideration differences in credit risk of different portfolios. The second method is described in McGinty et al. (2004) where it is

proposed to divide the detachment level by the expected loss on the entire basket over the same time-span, resulting in a scaled detachment level. From this one can define a relationship between scaled detachment and base correlation, which defines the map. In order to determine the correlation for a tranche with different maturity or on a different basket one first divides the detachment level by the expected loss of the underlying basket up to the relevant maturity and then determines the number using the relationship defined earlier. The third and final mapping method is the most advanced and is described in Reyfman et al. (2004). For each combination of detachment level and corresponding base correlation one first determines the expected loss on the tranche, expressed as a percentage of expected loss on the total basket, leading to five combinations of expected loss percentage and base correlation. When mapping to a different maturity or different basket one first implies detachment levels on this basket such that the same combinations of expected loss percentage and base correlations are found. This requires a numerical root-finding technique. Finally the resulting detachment levels and corresponding base correlations can be used in combination with interpolation and extrapolation rules to determine the correlation to use for a certain tranche.

23.3.3 Other Default Dependency Models

The base correlation approach cannot be seen as a proper model of defaults, but merely as a method to visualize market quotes in terms of correlations. It does not allow one to determine prices on bespoke baskets, without the use of mapping methods. Difficulties will increase even further when prices for more complex credit derivatives are required, such as a CDO of CDOs, or CDO^2. For these reasons many alternative models have been proposed in the literature, usually extending the one factor Gaussian copula in one way or the other. For instance, Hull and White (2004) propose to model both the common factor and the idiosyncratic risk terms using a Student t distribution. Other extensions allow for randomness of the correlation parameter. A mixture copula for instance assumes that the correlation parameter can take on a limited number of values each with some probability. Another random correlation model proposed by Andersen and Sidenius (2004) allows the correlation to be dependent on the common factor. In this case one can model behavior where a bad state of the economy, reflected through the realization of the common factor, goes accompanied by large default correlation. In van der Voort (2007) additional idiosyncratic default risk is introduced by an external source such as fraud. Although these models do a reasonable job fitting to market quotes and can be used to model more complex derivatives, the base correlation method is still the industry standard approach for pricing tranches and to communicate about default correlation. A comparative analysis of a number of different one factor copula models is given in Burtschell et al. (2005).

23.4 DATA

In order to apply the base correlation method a large amount of input is required. For this study market data have been used which have been made available to us by ABN Amro. For the analysis we have used data from December 2004 up to November 2006. From the

modeling description given in Section 23.3 it is clear that market data are needed for three parts of the model: risk-free discounting, marginal default distributions, and the default dependency. Each of these aspects is related to specific types of market instruments which are discussed below.

23.4.1 Default Free Discounting

In order to approximate risk-free discounting, we have used quotes for money market accounts and plain vanilla interest rate swaps. From these quotes a risk-free discounting curve is constructed using standard methods, see for instance Hull (2005). For each day of the data set a risk-free discounting curve is constructed for both the U.S. dollar as well as for the Euro.

23.4.2 CDS Term Structures

As our data set ranges from December 2004 up to November 2006, different series for both indices are involved due to the roll of the indices as discussed in Section 23.2.1. For the iTraxx Europe index this means that series 2 up to series 6 are included. For CDX the data also involve five different compositions, namely series 3 up to series 7.* It is worth noting that when rolling from series 4 to series 5 at September 20th 2005, Ford and General Motors, were removed from the basket, amongst other reference entities. Due to the rolls the total number of different reference entities is thus larger than 250. For each of these reference entities a CDS term structure is available at each date in the period. A piecewise constant default intensity curve is used, which allows the model to be calibrated to the CDS quotes using a bootstrap method, see Schönbucher (2003) or Brigo and Mercurio (2006) for instance. Recovery rates are assumed to be 40% for all names. As discussed in Houweling and Vorst (2005) the choice of the recovery rate does not have a large influence on pricing, as long as one calibrates to observed CDS quotes. From the resulting marginal default distributions for all names, one could easily determine the fair spread for an index CDS on each of the two indices. This determines the total amount of default risk present in each of the baskets and is therefore a large determinant of the values of the different tranches as well. Figure 23.3 shows the evolution of the fair spread for an index CDS on iTraxx and CDX with five year maturity.

From the figure one can observe the decrease of the fair spread for a five year CDS on the CDX index around the roll date of September 20th, 2005, which is partly due to the removal of Ford and General Motors. Further, one can observe that both indices show a similar behavior over time.

23.4.3 Synthetic CDO Quotes

Once the marginal default distributions are calibrated to CDS quotes the remaining part is to include default dependency. Quotes for synthetic CDO tranches are available which, in combination with the base correlation method, determine the dependency structure.

* For a composition of the different iTraxx and CDX series the reader is referred to the Markit website: www.markit.com.

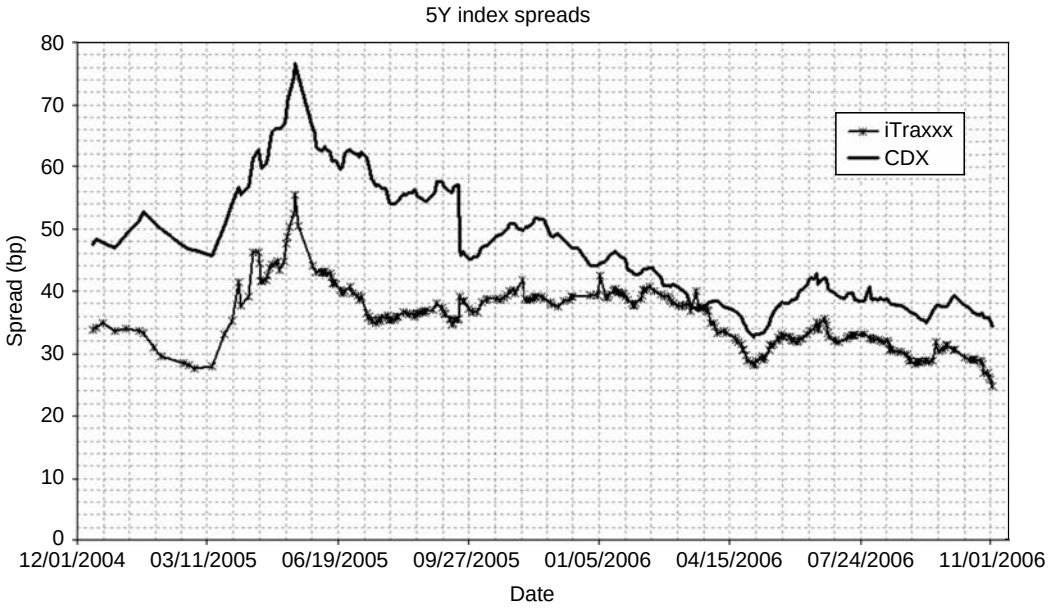


FIGURE 23.3 Fair spread for a five year index credit default swap on iTraxx and CDX from December 2004 up to November 2006.

Due to historical reasons quotes for tranches with a five year maturity are available from December 2004 up to November 2006, while quotes for tranches with seven and ten year maturity are available only over the period from July 2005 up to November 2006.

Table 23.2 presents a summary of statistics of both the quotes on iTraxx as well as those on CDX, all with a maturity of five years. In addition, summary statistics are presented for the two reference spreads. For each series of tranche quotes or reference spread the table presents the average, the standard deviation, the maximum, and minimum values. The last row presents the correlation of the quote under consideration with the reference spread.

One can observe that quotes have moved a lot over the past two years, as can be derived from the standard deviations as well as the large differences between minimum and maximum quotes of the different tranches. Further, one can observe that for each of the ten different time series of tranche quotes there exists a large degree of correlation with the reference spread. Especially the equity tranches and the most senior tranches seem to be largely correlated with the value of the reference spread. This relationship is investigated in more detail in Section 23.5.1.

23.4.4 Liquidity

As discussed the synthetic CDO market has grown rapidly over the last few years. In order to investigate whether there is evidence of this growth in the available data set, the bid/offer spread of the quotes can be considered, which are available for each tranche over the entire period. A smaller bid/offer spread would indicate a larger degree of liquidity for the

TABLE 23.2 Summary Statistics for Quotes with a Five Year Maturity on Both the European iTraxx Index as well as the North-American CDX Index

iTraxx	**0–3**	**3–6**	**6–9**	**9–12**	**12–22**	**Reference**
Avg.	24.7	86.3	26.4	13.5	7.4	35.15
Std. Dev.	6.3	31.6	10.6	6.6	4.9	5.9
Max.	49.6	191.0	63.0	35.0	25.0	57.0
Min.	10.4	43.8	13.0	5.0	2.1	23.0
Corr. Ref.	95%	82%	84%	88%	89%	100%
CDX	**0–3**	**3–7**	**7–10**	**10–15**	**15–30**	**Reference**
Avg.	37.0	126.1	31.6	15.4	7.9	47.6
Std. Dev.	9.1	50.6	14.8	7.7	4.4	10.0
Max.	62.0	260.0	74.5	34.0	20.0	76.0
Min.	22.9	65.0	12.0	6.5	3.1	34.0
Corr. Ref.	98%	86%	75%	91%	94%	100%

Note: For each tranche the average (Avg.), the standard deviation (Std. Dev.), the maximum (Max.), the minimum (Min.), and the correlation with the reference quote (Corr. Ref.) are shown.

synthetic CDO tranches. In Table 23.3 the average bid/offer spread of the first ten days is compared to the average bid/offer spread of the last ten days for each synthetic CDO tranche with a five year maturity.

From the table one can observe that bid/offer spreads have decreased dramatically, roughly with a factor three. Although the spread of both underlying indices has decreased slightly as well this large decrease in bid/offer spread clearly indicates a large increase in synthetic CDO tranche liquidity over the last two years. A similar effect can be observed for the tranches on iTraxx and CDX with either seven year or ten year maturity. Relative decreases in bid/offer spreads were similar although one has to note that the period for which seven and ten year data are available only ranges from July 2005 up to November 2006.

TABLE 23.3 Bid/Offer Spread of the Different Tranches on the Two Indices

iTraxx	**December 2004**	**November 2006**	**CDX**	**December 2004**	**November 2006**
0–3	0.67	0.24	0–3	1.33	0.24
3–6	3.30	0.60	3–7	4.35	1.70
6–9	3.30	0.95	7–10	5.40	1.40
9–12	3.75	0.95	10–15	5.75	1.23
12–22	2.00	0.76	15–30	2.58	0.95

A ten day average at the beginning of the data set is compared to a ten day average at the end of the data set.

23.5 EMPIRICAL ANALYSIS

23.5.1 Correlation Moves in Quotes

In the previous section it was shown that there exists a large correlation between quotes of tranches and the reference spread of the quote. Here we investigate this relationship in more detail using iTraxx and CDX tranche quotes with a five year maturity. Clearly, in a CDO structure, such as shown in Figure 23.1 any change in the risk of the pool of collateral assets must be translated to the different tranches of the CDO structure. When taking a closer look at modeling one finds that, apart from interest rates, the fair spread is a function of both the credit risk of the underlying index as well as default correlation. The effect of interest rate moves is typically small as it affects both legs in a similar manner and will therefore be ignored here. As before let Q denote the fair spread of a synthetic CDO tranche and S the fair spread of a CDS on the underlying index with the same maturity as the synthetic CDO tranche. Further let ρ denote a parameter for default correlation. Under these settings one has $Q = Q(S, \rho)$ and thus:

$$dQ = \frac{\partial Q}{\partial S} dS + \frac{\partial Q}{\partial \rho} d\rho \tag{23.15}$$

This equation shows that after correcting for index spread movements, changes in default dependency remain the only driver of changes in quotes. We are interested in levels rather than changes in the quotes and therefore it is assumed that there exists a simple linear relationship between the tranche quote and the corresponding basket spread. In order to investigate correlation effects, one should thus adjust for the risk in the portfolio, which is reflected in the index spread, i.e., the spread of an index CDS on the index with same maturity as the CDO tranche. As a proxy for this index spread the reference spread is used, which is part of the quote. This can be done by means of an Ordinary Least Squares, OLS, regression where one regresses the observed quote on the corresponding reference spread. Let Q_i^k denote the mid-quote for tranche k, for observation i. Assume that the iTraxx tranches are indexed $1, \ldots, 5$ and CDX tranches $6, \ldots, 10$ in order of seniority. Further let the corresponding reference spread be denoted by R_i^k, which is equal for all iTraxx quotes and is also equal for all CDX quotes. The following simple linear relationship is considered:

$$Q_i^k = \alpha^k + \beta^k \cdot R_i^k + \varepsilon_i^k \tag{23.16}$$

Here the disturbances ε_i^k are assumed to be i.i.d. normally distributed. This regression analysis has been performed for each of the available time series of quotes and regression results are shown in Appendix A. Let a^k and b^k denote the estimate values for α^k and β^k, respectively. One can observe from the regression results that for each time series the estimated coefficient b^k is highly significant and that the coefficients of determination are reasonably large. Thus, the reference spread explains a large part of the variation in the observed market quotes of synthetic CDO tranches. For each of the time series a second regression analysis has been performed where a quadratic form is considered by including the squared reference spread. However, for none of the series this improved explanatory power of the model significantly.

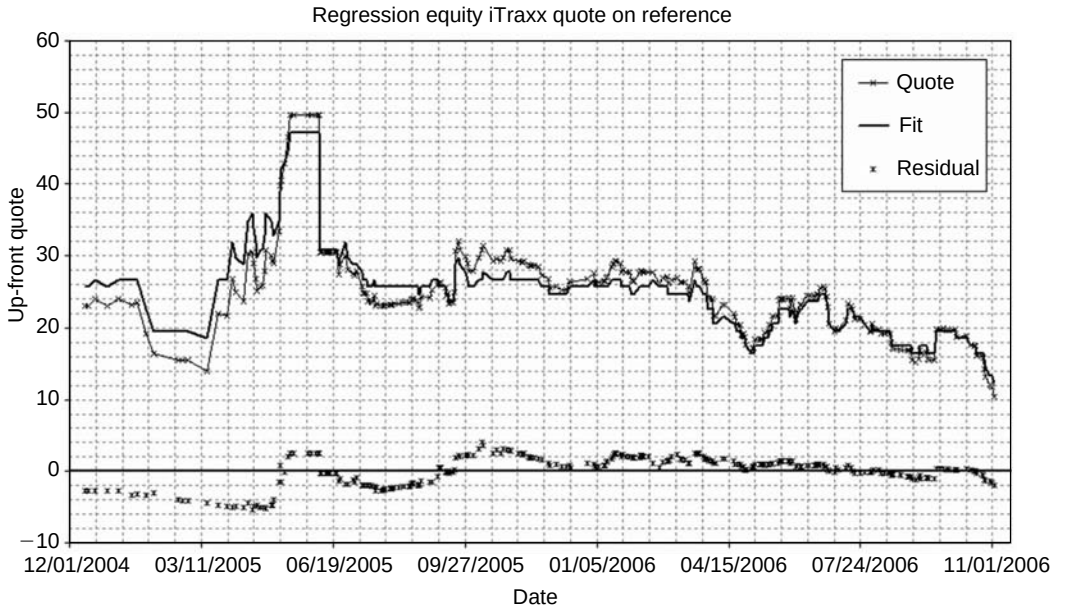


FIGURE 23.4 Regression of iTraxx equity quote on the reference spread. Both the original data as well as the fit and the residuals are presented.

Using the estimated parameters as shown in Appendix A, one can determine the estimated residuals as follows:

$$e_i^k \equiv Q_i^k - a^k - b^k \cdot R_i^k \quad \text{for } k = 1, \ldots, 10 \tag{23.17}$$

As an example the series of iTraxx equity quotes is considered. As can be seen from the appendix the estimated coefficients are $a^1 = -11.1$ and $b^1 = 1.02$ and the coefficient of determination is close to 90%. In Figure 23.4 we present the time series of quotes along with the estimated values using the reference quote. In addition the estimated residuals are plotted.

As can be seen from the figure the simple linear relationship provides a good fit. Further one can observe that the estimation residuals do not seem to be independent but clearly follow a certain pattern over time. Furthermore, one can clearly observe the increase in residuals at the beginning of May 2005. This increase means that, after correcting for moves in reference spread, the equity quote has moved upward. An increase in the value of protection for the equity tranche goes accompanied by a decrease in default correlation. This is exactly what was observed in the base correlations during the May 2005 events.

This simple OLS analysis can be applied to all different tranches which are quoted and one can observe clear patterns for the errors over time for each of these series. In Table 23.4 the correlation matrix of the ten different residuals series is presented.

From the table one can observe some interesting results. First one can observe that for both indices the residuals for the equity tranche are negatively correlated with those for the other tranches. Note that this makes intuitive sense as when prices for the equity

TABLE 23.4 Correlation of the Regression Residuals for the Different Series of Synthetic CDO Quotes

Correlation of Errors		iTraxx					CDX				
		0–3%	**3–6%**	**6–9%**	**9–12%**	**12–22%**	**0–3%**	**3–7%**	**7–10%**	**10–15%**	**15–30%**
iTraxx	0–3	100	−64	−76	−78	−75	80	−54	−73	−75	−49
	3–6	−64	100	87	74	48	−64	93	83	70	−1
	6–9	−76	87	100	83	60	−74	82	89	87	30
	9–12	−78	74	83	100	85	−80	73	80	80	44
	12–22	−75	48	60	85	100	−78	47	59	63	62
CDX	0–3	80	−64	−74	−80	−78	100	−67	−76	−74	−57
	3–7	−54	93	82	73	47	−67	100	84	73	1
	7–10	−73	83	89	80	59	−76	84	100	90	28
	10–15	−75	70	87	80	63	−74	73	90	100	50
	15–30	−49	−1	30	44	62	−57	1	28	50	100

Note: These residuals can be regarded as reference corrected quotes and are expected to capture correlation moves in the synthetic CDO market.

tranche are high relative to the reference spread, or average portfolio spread, it must be that the more senior tranches have lower prices relative to average spread. Another interesting observation is the large correlation of the residuals for the different underlying baskets. For instance residuals for quotes on equity tranches have a correlation of 80%, while for the junior mezzanine tranche this is even 93%. Thus, in Figure 23.3 it was shown that index CDS spreads for iTraxx and CDX moved closely and the table above suggests that the default dependency structure also moves closely between both indices.

23.5.2 Empirical Base Correlations

As discussed in Section 23.3.2 the base correlation method has become the industry standard approach which is used to visualize market-implied correlations. This method has been applied to the data and in Appendix B the 5 base correlation series for iTraxx are shown, as well as the ones for the CDX, all for a five year maturity. When considering the base correlation for the equity tranches for both iTraxx and CDX one can clearly observe the May 2005 event. For both indices base correlations for the equity tranche dropped from around 20% to just below 10%. For the base correlations with higher detachment levels corresponding to the iTraxx one can observe similar drops. However, for the CDX base correlations corresponding to higher detachment levels one can see an increase.

As can be seen from the figures in Appendix B the correlations move closely together. Table 23.5 presents the correlations of the different base correlations series. These numbers confirm that base correlations for the different detachment levels move closely together and in addition show that the base correlations corresponding to the two different indices move closely together. One cannot directly compare these numbers with the correlations of the regression residuals presented in Table 23.4. The reason is that the latter are based on the tranches seen in the market, which correspond to different parts of the CDO structure. When considering the base tranches, i.e., equity tranches with different detachment levels,

TABLE 23.5 Correlation of the Different Base Correlations

Correlation of Base Correlation Series		iTraxx					CDX				
		3%	6%	9%	12%	22%	3%	7%	10%	15%	30%
iTraxx	3	100	89	80	75	65	90	79	70	57	58
	6	89	100	98	96	89	74	92	89	81	81
	9	80	98	100	99	94	64	91	91	86	86
	12	75	96	99	100	96	59	89	91	87	88
	22	65	89	94	96	100	46	76	80	75	80
CDX	3	90	74	64	59	46	100	76	62	48	49
	7	79	92	91	89	76	76	100	98	93	90
	10	70	89	91	91	80	62	98	100	98	95
	15	57	81	86	87	75	48	93	98	100	96
	30	58	81	86	88	80	49	90	95	96	100

Note: For both iTraxx and CDX base correlations for five different detachment levels are determined for each date in the data set. This results in a time series for each base correlation and the table presents correlation for these series.

there is a large amount of overlap. One could however compare the base correlations corresponding to the equity tranche, i.e., with detachment level of 3%, to the residuals determined in Section 23.5.1. For iTraxx the correlation for these two time series equals −94%, while for CDX the correlation is −83%. As discussed earlier, large residuals from the OLS analysis for the equity tranches goes accompanied by low base correlations, explaining the negative sign of the two correlations between the time series of residuals and the time series of base correlations. These large correlations give further evidence that the quotes, when adjusted for portfolio credit risk, indeed give a good indication of the default dependency implied in the market.

23.5.3 Mapping Methodologies

We now turn to the different mapping methodologies as described in Section 23.3.2. As discussed the base correlation method is not a proper model which becomes clear when one wants to mark-to-market nonstandard tranches. For instance tranches with nonstandard attachment and detachment levels, tranches with nonstandard maturities, or tranches on bespoke baskets. As only five base correlations are available, one has to resort to interpolation/extrapolation techniques in combination with mapping methods. Here we consider the three different mapping methodologies described earlier on:

1. Direct *d*etachment–*d*etachment mapping, *dd*
2. *D*etachment as a *p*ercentage of expected basket loss, *dp*
3. Expected *l*oss *p*ercentage mapping, *lp*

In the test all different mapping methods are used in combination with natural splines to determine a base correlation for each detachment level. Linear extrapolation is used for

detachment levels smaller than 3%. Note that the choice of interpolation technique is a complex problem as well, as one might introduce arbitrage opportunities, especially for short-term CDO tranches. However, this topic is not considered in this chapter.

For testing purposes we shall use the quotes on standard tranches with a five year maturity for building the base correlation skew, as these are by far the most liquid correlation instruments. After building the skew we apply one of the three different mapping methods in order to determine the model value of tranches with different maturities or on different underlying baskets. Comparing these model-generated values with the actual values results in an error of the model compared to the market value.

Let S denote the set of tranches used for testing the mapping method. By definition when considering a tranche with up-front value and premium contracted as implied by the quotes, the correct value should be zero. Let V_i^j denote the model value for tranche i under mapping method j, when the up-front percentage and contracted premium correspond to the quoted numbers.* As the market-implied value of the tranche is zero, the error made by the mapping method is thus equal to V_i^j. These errors can be used to define the Root Mean Squared Error, for a mapping method j given a set of tranches S. We refer to this as the value-based Root Mean Squared Error, or RMSE^V:

$$\text{RMSE}^V(j,S) \equiv \sqrt{\frac{1}{N_S}\sum_{i\in S}\left(V_i^j\right)^2} \tag{23.18}$$

Here $N_S \equiv \sum_{i\in S} 1$ denotes the number of tranches in the testing set. Instead of using the value of the tranche, one could alternatively look at the fair up-front value or fair premium implied by the model, to determine the errors. The market quotes the up front or premium, while a mapping method results in a model-implied fair up-front value or fair premium, leading again to an error. Let Q_i denote the quote for tranche i and π_i^j the model-implied up-front value or fair spread for the same tranche under mapping method j. In addition to the value-based RMSE one could consider a quote-based Root Mean Squared Error, RMSE^Q:

$$\text{RMSE}^Q(j,S) \equiv \sqrt{\frac{1}{N_S}\sum_{i\in S}\omega_i\cdot\left(Q_i-\pi_i^j\right)^2} \tag{23.19}$$

Here the terms ω_i allow one to give a certain weight to the different tranches. One could for instance set $\omega_i = 1/Q_i^2$ and consider the relative errors, a choice which is made throughout the remainder of this section.

When looking at the value-based RMSE one can expect that a large part of the result is caused by equity and junior mezzanine tranches. When using the quote-based RMSE with weights as above, the difficulty arises that quotes for the most senior tranches can be very

* In order to allow for a fair comparison, the notional delta exchange is taken into account by determining its value and correcting the model-implied value for this.

low, making large relative errors likely and causing the $RMSE^Q$ to put a large weight on these senior tranches. Thus, the $RMSE^V$ is likely to put more weight on the equity and junior tranches, while the $RMSE^Q$ is expected to put more weight on the more senior tranches. For this reason, both of them are considered in the performance test for the different base correlation mapping methods.

23.5.3.1 Mapping in the Time Domain

First the performance of the different mapping methods is investigated when varying the maturity of the tranches. Using quotes on five year synthetic CDO tranches and one of the mapping methods, one can determine the model-implied quote for a standard tranche with either seven or ten years maturity. As quotes are available for these products one can easily determine the error made due to the use of a mapping method. From these errors one can determine the different RMSEs as described above, and compare the performance of the different mapping methods among each other. In Appendix C the value-based RMSE is presented for each of the five tranches on both iTraxx and CDX, with either seven year or ten year maturity. In addition the RMSE is aggregated over both indices and both maturities for each tranche separately. Further the RMSE is aggregated over all tranches for a certain combination of index and maturity. Finally the RMSE is determined over all tranches for each mapping method.

As can be seen from the results the mapping based on the expected loss percentage, *lp*, shows the best performance. With the exception of the equity tranche it shows the best performance for all tranches. A slightly worse performance is found for the scaled detachment method, *dp*, where the detachment is expressed as a fraction of expected loss of the total basket. The simple detachment-based mapping, *dd*, shows the worst performance, even though for equity tranches it appears to generate the closest fit. A reason for this might be the chosen extrapolation method for the other two mapping methodologies, which is not an issue for the *dd* method.

Table 23.6 shows the RMSE using all available tranches. Both the value-based RMSE is shown as well as the RMSE based on relative errors for fair premia.

From this table one can observe that the expected loss percentage mapping, *lp*, shows the best performance under both measures. However, the average relative error in terms of the quotes is still at a level of 18%.

TABLE 23.6 Total RMSE under Two Approaches for All Three Different Mapping Methods

Mapping	**$RMSE^V$**	**$RMSE^Q$**
Detachment–detachment, *dd*	3.78	0.46
Detachment percentage, *dp*	3.38	0.28
Expected loss percentage, *lp*	2.54	0.18

Note: $RMSE^V$ is based on the value of the tranche when contracted premium equals fair premium. The $RMSE^Q$ is based on the quoted numbers and relative errors are used.

TABLE 23.7 Total RMSE under Two Approaches for All Three Different Mapping Methods

Mapping	$RMSE^V$	$RMSE^Q$
Detachment–detachment, *dd*	3.21	0.42
Detachment percentage, *dp*	3.48	0.32
Expected loss percentage, *lp*	2.75	0.23

Note: $RMSE^V$ is based on the value of the tranche when contracted premium equals fair premium. The $RMSE^Q$ is based on the quoted numbers and relative errors are used. iTraxx quotes are used to map to CDX tranches and vice versa.

23.5.3.2 Mapping to Bespoke Baskets

A similar test can be performed in order to test the different mapping methods when applied to bespoke baskets. Quotes for bespoke baskets are not available, but one could use quotes on iTraxx tranches and subsequently value CDX tranches. The model-generated quotes can then be compared to the market quotes and the performance of the different mapping methods can be investigated. For this test we have used the five year quotes on the iTraxx index to build a base correlation skew. Using one of the three mapping methods the standard tranches on CDX with maturities of five, seven, and ten years, are priced and results are compared with the correct values. In Appendix D one can find the value-based RMSE split over the different mapping methods, the different tranches, and the different maturities. Again the results show that the expected loss percentage mapping, *lp*, performs best in most cases. Further one can observe that the simple detachment mapping, *dd*, again performs best for equity tranches. It is interesting to see that this is only the case when mapping from CDX to iTraxx tranches. When mapping from iTraxx to CDX tranches the *dd* mapping method shows the worst performance.

Table 23.7 shows the RMSE using all available tranches. Both the value-based RMSE is shown as well as the RMSE based on relative errors for fair premia. From this table one can observe that the expected loss percentage mapping, *lp*, shows again the best performance under both measures. However, the average relative error in terms of the quotes is still at a level of 23%. Further one can observe that the simple detachment mapping, *dd*, performs better than the scaled detachment level, *dp*, when considering the value-based RMSE. For the RMSE based on relative error in fair premium, the results are the other way around. This is caused by the large relative errors for the simple detachment mapping method for senior tranches.

23.6 CONCLUSION

Over the last years the market for synthetic CDO tranches has undergone a rapid development together with the research efforts in order to value these products consistently with the observed market quotes. A widely used model for imposing default correlation is the one factor Gaussian copula approach and calibration is done using the base correlation method. One of the main drawbacks is that the base correlation method is not a proper

model and mapping methods are required when pricing tranches with nonstandard attachment and detachment levels, nonstandard maturities, or tranches on bespoke baskets.

In this chapter we have focussed on default correlation implied by market quotes. Using a large data set with quotes for synthetic CDO tranches between December 2004 and November 2006 we have considered different aspects of market-implied default correlation. First the quotes have been used directly, using a simple linear regression to correct for the level of default risk of the reference index. The remaining residuals show a clear pattern over time, indicating moves in market-implied default dependency. One could clearly observe the May 2005 events in these residuals. Furthermore, it was shown that the correlation of these residuals amongst each other has been large and of expected sign, where residuals for equity quotes were negatively correlated with those for the other tranches. In addition it was shown that there exists a large correlation between the time series of the residuals for equity tranches and the time series of base correlations corresponding to the 3% detachment level.

In addition the data were used to investigate the performance of three different mapping methods. These methods determine the correlation for a nonstandard tranche using the base correlations implied from market quotes. Two out-of-sample tests have been performed. First quotes for tranches with a five year maturity on a certain index were mapped through the base correlation approach to tranches with seven and ten year maturity on the same index. This allowed us to investigate the performance of the different mapping methods in the maturity domain. Secondly, quotes for five year tranches on a certain index were mapped to tranches with five, seven and ten year maturities referencing the other index. This allowed us to investigate the performance of the mapping methods when mapping to different baskets. It was found that the method based on the expected loss percentage outperforms the use of the method of scaled detachment levels and simple detachment only mapping. However, in all cases the errors were still very large with relative errors in quotes around levels of 20%.

APPENDIX A REGRESSION RESULTS: QUOTES REGRESSED ON REFERENCE SPREAD

TABLE 23.A1 Results of Regression Analysis, where the Quote for Each Tranche is Regressed on the Reference Spread

iTraxx	**0–3**			**CDX**	**0–3**		
R^2	0.898			R^2	0.956		
Variable	**Coeff.**	**Std. Err.**	***t*-Ratio**	**Variable**	**Coeff.**	**Std. Err.**	***t*-Ratio**
Constant	−11.139	0.719	−15.502	Constant	−5.268	0.569	−9.259
Ref. Spread	1.022	0.020	50.617	Ref. Spread	0.888	0.012	75.886
iTraxx	**3–6**			**CDX**	**3–7**		
R^2	0.676			R^2	0.742		
Variable	**Coeff.**	**Std. Err.**	***t*-Ratio**	**Variable**	**Coeff.**	**Std. Err.**	***t*-Ratio**
Constant	−68.661	6.386	−10.752	Constant	−81.994	7.660	−10.704
Ref. Spread	4.417	0.180	24.603	Ref. Spread	4.370	0.157	27.754
iTraxx	**6–9**			**CDX**	**7–10**		
R^2	0.706			R^2	0.560		
Variable	**Coeff.**	**Std. Err.**	***t*-Ratio**	**Variable**	**Coeff.**	**Std. Err.**	***t*-Ratio**
Constant	−26.618	2.040	−13.051	Constant	−21.302	2.926	−7.280
Ref. Spread	1.512	0.057	26.363	Ref. Spread	1.111	0.060	18.475
iTraxx	**9–12**			**CDX**	**10–15**		
R^2	0.773			R^2	0.824		
Variable	**Coeff.**	**Std. Err.**	***t*-Ratio**	**Variable**	**Coeff.**	**Std. Err.**	***t*-Ratio**
Constant	−21.109	1.115	−18.928	Constant	−18.003	0.964	−18.671
Ref. Spread	0.986	0.031	31.445	Ref. Spread	0.702	0.020	35.435
iTraxx	**12–22**			**CDX**	**15–30**		
R^2	0.798			R^2	0.879		
Variable	**Coeff.**	**Std. Err.**	***t*-Ratio**	**Variable**	**Coeff.**	**Std. Err.**	***t*-Ratio**
Constant	−18.892	0.786	−24.023	Constant	−11.981	0.460	−26.043
Ref. Spread	0.749	0.022	33.876	Ref. Spread	0.417	0.009	44.140

On the left-hand side results are shown for iTraxx, while on the right-hand side results for CDX are shown. For each regression the coefficient of determination, R^2 is shown and for each parameter its estimate, standard error, and *t*-ratio.

APPENDIX B BASE CORRELATIONS

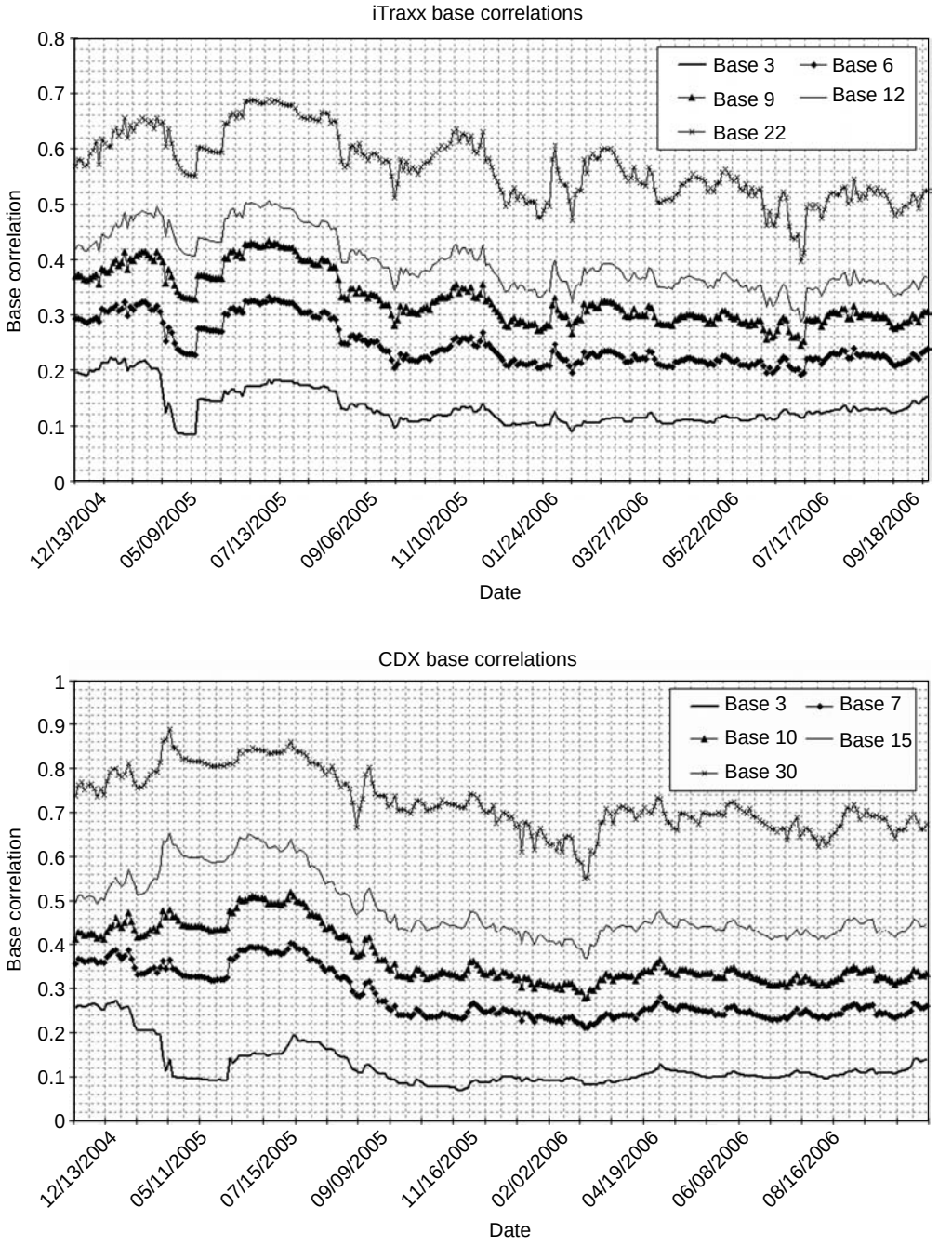


FIGURE 23.A1 iTraxx and CDX base correlations from December 2004 up to November 2006.

APPENDIX C PERFORMANCE OF MAPPING METHODS TO MATURITY

TABLE 23.A2 Root Mean Square Error, RMSE of Three Different Mapping Methods

			Tranche 1	Tranche 2	Tranche 3	Tranche 4	Tranche 5	Total
iTraxx	7Y	*dd*	2.83	1.32	1.17	0.88	0.24	1.55
		dp	2.94	3.72	0.62	0.28	0.08	2.14
		lp	2.29	2.51	0.35	0.21	0.13	1.53
	10Y	*dd*	2.58	10.11	3.33	2.90	1.18	5.10
		dp	6.20	6.16	5.31	1.32	0.31	4.61
		lp	5.89	4.28	2.92	0.57	0.41	3.52
CDX	7Y	*dd*	2.79	1.36	1.15	0.93	0.26	1.54
		dp	2.58	2.89	0.61	0.19	0.14	1.76
		lp	2.31	1.93	0.42	0.18	0.19	1.36
	10Y	*dd*	2.04	10.68	3.00	2.81	1.23	5.23
		dp	3.94	5.51	6.00	0.99	0.16	4.07
		lp	3.99	3.97	3.58	0.95	0.39	3.02
All		*dd*	2.59	7.35	2.38	2.11	0.86	3.78
		dp	4.20	4.76	3.98	0.85	0.19	3.38
		lp	3.94	3.32	2.29	0.56	0.30	2.54

Map "*dd*" simply maps based on the relevant detachment level. Map "*dp*" uses the method described in McGinty et al. (2004), where detachment is expressed as a percentage of the expected loss on the entire basket. Finally, map "*lp*" considers mapping based on the expected loss on a tranche as a percentage of total expected loss, as described in Reyfman et al. (2004). Quotes with 5Y maturities on iTraxx and CDX are used as input and errors in value are determined for tranches with 7Y and 10Y maturities, after mapping. The final row considers both indices and both maturities for the tranche. The final column combines the five different tranches.

APPENDIX D PERFORMANCE OF MAPPING TO BESPOKE

TABLE 23.A3 Root Mean Square Error, RMSE of Three Different Mapping Methods

			Tranche 1	Tranche 2	Tranche 3	Tranche 4	Tranche 5	Total
iTraxx	5Y	*dd*	3.70	1.42	0.66	0.38	0.32	1.81
		dp	4.71	1.74	0.64	0.32	0.35	2.27
		lp	3.71	0.80	0.29	0.18	0.17	1.71
	7Y	*dd*	1.88	1.99	0.83	0.69	0.31	1.32
		dp	4.99	3.73	0.63	0.23	0.16	2.80
		lp	4.50	3.15	0.49	0.26	0.10	2.47
	10Y	*dd*	1.95	8.81	2.73	2.84	1.22	4.44
		dp	8.13	6.67	5.20	1.23	0.19	5.28
		lp	7.95	5.08	3.49	0.74	0.42	4.51
CDX	5Y	*dd*	4.03	1.36	0.59	0.26	0.16	1.92
		dp	4.21	1.92	0.33	0.13	0.11	2.08
		lp	3.83	1.08	0.26	0.10	0.19	1.79
	7Y	*dd*	5.17	1.72	2.03	1.04	0.26	2.64
		dp	2.89	4.96	1.24	0.22	0.18	2.63
		lp	2.56	3.08	0.41	0.18	0.25	1.81
	10Y	*dd*	3.73	10.67	4.19	3.11	1.11	5.59
		dp	3.13	7.35	7.74	1.74	0.29	5.04
		lp	3.17	5.57	4.51	0.62	0.35	3.52
All		*dd*	3.62	5.50	2.14	1.71	0.68	3.21
		dp	4.96	4.68	3.64	0.84	0.23	3.48
		lp	4.58	3.44	2.22	0.40	0.26	2.75

Map "*dd*" simply maps based on the relevant detachment level. Map "*dp*" uses the method described in McGinty et al. (2004), where detachment is expressed as a percentage of the expected loss on the entire basket. Finally, map "*lp*" considers mapping based on the expected loss on a tranche as a percentage of total expected loss, as described in Reyfman et al. (2004). Quotes with 5Y maturities on iTraxx are used as input and errors in value are determined for tranches on CDX with 5Y, 7Y, and 10Y maturities, after mapping. This is repeated using CDX quotes as input and pricing iTraxx tranches. The final row considers both indices and both maturities for the tranche. The final column combines the five different tranches.

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CHAPTER 24

Pricing Tranched Credit Products with Generalized Multifactor Models

Manuel Moreno, Juan I. Peña, and Pedro Serrano

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24.1 INTRODUCTION

The market of credit tranched products is one of the fastest growing segments in the credit derivative industry. As an example, Tavakoli (2003) reports an increase in market size from almost $19 billions in 1996, to $200 billions in 2001. Recent reports estimate market size to be $20 trillions in 2006 (BBA Credit Derivatives Report, 2006). As a result, an increasing attention of the financial sector audience has focused on the pricing of these new products.

The development of pricing models for multiname derivatives is relatively recent. As was pointed out by Hull and White (2004), the standard approach on the credit risk literature tends to subdivide the pricing models for multiname derivatives in two groups: structural models, which are those inspired in Merton's (1974) model or Black and Cox (1976), or the intensity-based models, like Duffie and Gârlenau (2001). Roughly speaking, their differences remain on how the probability of default of a firm is obtained: using their fundamental variables—assets and liabilities—as in the case of the structural models, or using directly market spreads, as in the intensity models approach. Up to a point, the structural-based approach has been extensively implemented by the financial sector, maybe due to the extended use of industrial models like Vasicek (1991) or Creditmetrics model of Gupton et al. (1997).* However, recent academic literature analyze the prices of collateralized debt obligations (CDO) tranches using intensity models, as Longstaff and Rajan (2008). We refer to Bielecki and Rutkowski (2002) for a general presentation of structural- and intensity-based models.

This chapter presents an extension of the standard gaussian model of Vasicek (1991), in line with the structural models literature. Basically, Vasicek's (1991) model assumes that the value of a firm is explained by the weighted average of one common factor for every asset plus an independent idiosyncratic factor. By means of linking the realization of one systematic factor to every firm's values, Vasicek (1991) provides a simple way to reduce the complexity of dealing with dependence relationships between firms. Gibson (2004) and Gregory and Laurent (2004, 2005) provide additional insights about the risk features of this model.

Our approach relies on the connection between the changes of value of a firm and the sum of two factors: systematic and idiosyncratic. Our approach overcomes the limitations

* It is worth mentioning that the appearance of techniques within the structural framework that diminishes the traditional high computing cost of multiname credit derivatives (see Andersen et al. (2003) or Glasserman and Suchitabandid (2006), among others) has contributed to the wide usage of structural models.

of the standard Gaussian model: the different areas or regions of correlation that could compose a credit portfolio (Gregory and Laurent, 2004). This chapter presents a model that captures some of the facts found in real data. Motivated by this fact, this chapter proposes an extension to the two-Gaussian asset classes as in Schönbucher (2003). This chapter extends the existing literature in three ways. Firstly, the assumption of asset homogeneity is relaxed by introducing two asset classes. Secondly, we consider an additional source of systematic risk by including another common factor related with industry factors. Finally, the normality assumption on common factors is relaxed.

This chapter is divided as follows: Section 24.2 presents the model. Section 24.3 studies the sensitivity to correlation and to changes in credit spreads. Section 24.4 addresses the econometric modeling. Finally, some conclusions are presented in Section 24.5.

24.2 MODEL

To motivate our model, we discuss some empirical features found in CDO data that are not captured by the standard Gaussian models and propose an extension to the asset class models posited by Schönbucher (2003). Notation is taken from Mardia et al. (1979).

24.2.1 Standard Gaussian Model

The Gaussian model introduced by Vasicek (1991) has become a standard in the industry. Basically, it addresses in a simple and elegant way the key input in CDOs price: the correlation in default probabilities between firms affects the price of the CDO.

Usually, a CDO is based on a large portfolio of firms bonds or CDS.* Let $\mathbf{V}_{n\times 1}$ (subscript denotes matrix dimension) be a random vector with mean zero and covariance matrix $\boldsymbol{\Sigma}$. As standard notation in multivariate analysis, we will define the p-factor model as

$$\mathbf{V}_{n\times 1} = \boldsymbol{\Lambda}_{n\times p}\mathbf{F}_{p\times 1} + \mathbf{u}_{n\times 1} \tag{24.1}$$

where $\mathbf{F}_{p\times 1}$ and $\mathbf{u}_{n\times 1}$ are random variables with different distributions.

The interpretation of the model (Equation 24.1) is the following:

- Vector $\mathbf{V}_{n\times 1}$ represents the value of the assets for each of the individual n-obligors.
- Vector $\mathbf{F}_{p\times 1}$ captures the effect of systematic factors—business cycle, industry, etc.—that affect the whole economy.
- By contrast, $\mathbf{u}_{n\times 1}$ represents the idiosyncratic factors that affect each of the n-companies.
- Finally, $\boldsymbol{\Lambda}_{n\times p}$ is called the loading matrix, and determines the correlation between each of the n-firms.

By assumption, we have that

* CDX or iTraxx indexes are composed by 125 firms.

$$E(\mathbf{F}) = \mathbf{0}, \quad \mathrm{V_{ar}}(\mathbf{F}) = \mathbf{I} \tag{24.2}$$

$$E(\mathbf{u}) = \mathbf{0}, \quad \mathrm{Cov}(\mathbf{u_i}, \mathbf{u_j}) = 0, \ i \neq j \tag{24.3}$$

and

$$\mathrm{Cov}(\mathbf{F}, \mathbf{u}) = \mathbf{0} \tag{24.4}$$

where $\mathbf{I}$ is the identity matrix.

We will also assume that vector $\mathbf{u}_{n\times 1}$ is standardized to have zero mean and unit variance.

Using a simplified form of Equation 24.1, Vasicek (1991) assumes that firm's values in the asset pool backing the CDO are affected by the sum of two elements: on one hand, a common factor to every firm that represents the systematic component represented by the factor F; on the other hand, an idiosyncratic component modeled by a noise ε_i. Both are assumed to be standard N(0,1) random variables

$$V_i = \rho_{i1} F_1 + \sqrt{1 - \rho_{i1}{}^2}\varepsilon_i, \quad \text{with } i < n \tag{24.5}$$

Using Equation 24.5 it is possible to capture the correlation between different firms in a portfolio. As Equation 24.5 reveals, Vasicek (1991) assumes that correlation coefficient is homogeneous for each pair of firms.* Additionally, the simplicity of their assumptions permits a fast computation of CDO prices under this framework.

As an immediate consequence of Equation 24.5, a further step is given by generalizing the number of factors that affects firms values. Thus, Lucas et al. (2001) consider the following factor model:

$$V_i = \sum_{j=1}^{p} \rho_{ij} F_j + \sqrt{1 - \sum_{j=1}^{p} \rho_{ij}^2} u_i, \quad \text{with } i < n \tag{24.6}$$

where V_i represents the value of the i-company, F_j, $j = 1, \ldots, p$ capture the effect of p-systematic factors, and u_i is the idiosyncratic factor associated to the i-firm. Needless to say, assumptions on the distribution of common and idiosyncratic factors (F and u, respectively) could be imposed. Hull and White (2004) also include an extension to many factors, including the t-Student distributed case. Finally, Glasserman and Suchitabandid (2006), in a recent paper, implements numerical approximations to deal with these multi-factor structures within a Gaussian framework.

For getting intuition about the behavior of these different models, Figure 24.1 exhibits the loss distribution generated by three alternative models:

- Standard Gaussian model (Vasicek, 1991),

* For a detailed exposition of assumptions in structural models see Bielecki and Rutkowski (2002).

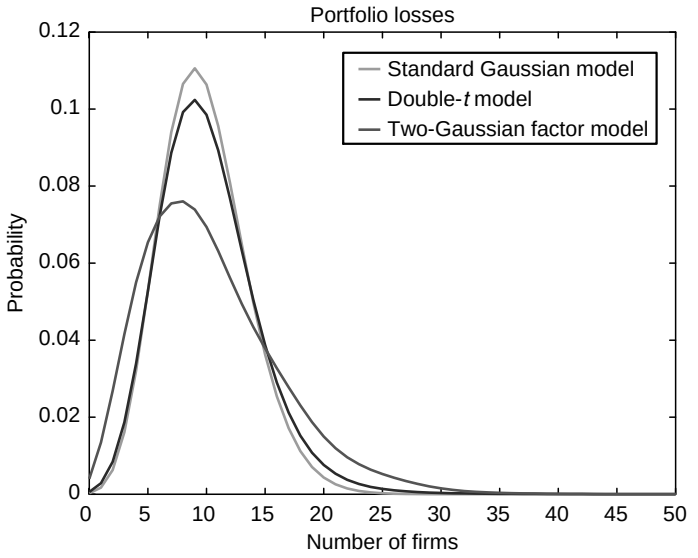


FIGURE 24.1 Total loss distribution of a 100-firms portfolio for different models. Individual default probabilities are fixed at 1%. Correlation for the standard Gaussian and *t*-Student distribution (with six degrees of freedom) is 0.3. Correlations for the double Gaussian model are 0.1, 0.3.

$$V_i^G = \rho_{i1}F_1 + \sqrt{1-\rho_{i1}^2}\varepsilon_i \quad \text{and} \quad F_1,\, \varepsilon_i \sim N(0,1)$$

- One-factor double-*t* model (Hull and White, 2004),

$$V_i^S = \rho_{i1}F_1 + \sqrt{1-\rho_{i1}^2}\varepsilon_i$$

  where F_1 and ε_i follows *t*-Student distributions, with six degrees of freedom.
- Two-factor Gaussian model (Hull and White, 2004, or Glasserman and Suchitabandid, 2006),

$$V_i^{DG} = \rho_{i1}F_1 + \rho_{i2}F_2 + \sqrt{1-\rho_{i1}^2-\rho_{i2}^2}\varepsilon_i$$

  with F_1, F_2, $\varepsilon_i \sim N(0,1)$ and $\rho_{i1}^2 + \rho_{i2}^2 < 1$.

As we will see later, the loss distribution plays a role crucial in the CDO pricing. Different distributions are computed for a portfolio of 100 firms, with constant default intensities of 1% per year. Correlation parameters are 0.3 for all models, except for two-Gaussian factor, with 0.1 and 0.3. The double-*t* model considers a common and idiosyncratic factors each distributed as *t*-Student with six degrees of freedom. The picture shows that, under the same correlation parameters, the standard Gaussian model assigns lower probability to high losses (see for instance the case of 20 firms) than the double-*t* model. By

TABLE 24.1 Spreads of Different Tranches for a 100-Names CDO under Different Correlation Parameters

Model	Standard Gaussian		Two-Gaussian Factors		*t*-Student	
Correlation	**0.1**	**0.3**	**0.1/0.1**	**0.1/0.3**	**0.1**	**0.3**
Attachment Points (%)	**Tranche Spread (Basis Points)**					
0–3	2117	1320	2186	1656	2485	2135
3–10	86	110	508	653	131	233
10–100	0.02	1	7	29	1	9

considering an additional factor, as in the two-Gaussian case, the probability of high losses is substantially higher than in previous cases.

Table 24.1 presents the spreads (in basis points) of a 5 years CDO portfolio composed by 100 firms. The individual default probabilities are constant and fixed at 1%. The recovery rate is 40%, a standard in the market. Finally, the different correlation parameters are displayed in the table. We remember that all spreads are obtained considering only one asset class. As Table 24.1 shows, results are consistent with the loss distribution lines presented in Figure 24.1, the double *t*-Student distribution gives prices systematically bigger than those obtained for the standard Gaussian model, keeping constant the correlation. In the two-Gaussian factor model prices, we observe a mixture of effects due to different combinations of correlations in the portfolio.

24.2.2 Two-by-Two Model

Generally, as considered by Schonbücher (2003) or Lando (2004), a credit portfolio is composed by different asset classes or buckets, attending to criteria of investment grade, noninvestment grade assets, or industry, among others. The exposure of a credit portfolio to a set of common risk factors could be significant between groups, but should be homogeneous within them. In line with this, the idea of two groups of assets treated in different ways is a more realistic assumption.

As was pointed out by Gregory and Laurent (2004), the one-factor model imposes a limited correlation structure on the credit portfolio, which is not realistic. Initially, one can argue that increasing the number of factors could be enough to capture a richer structure of correlation within the portfolio. However, this is not yet consistent with the idea of heterogeneity correlation among groups, due to the fact that every asset is exposed to the same degree of correlation. By contrast, a richer correlation structure could be imposed in the portfolio if these two groups are treated in different ways.

To illustrate these ideas, Figure 24.2 shows the yearly percentage default rates for investment and noninvestment grades. Rates for investment data have been multiplied by 10 with the intention of clarifying the exposition. Figure 24.2 reveals that correlation between these two asset groups varies through time: periods with high degree of default in noninvestment grade assets do not match with high default rates for investment grade. With reference to this idea, Figure 24.3 displays the correlation coefficients computed using a moving window of 5 years. This also provides some additional insights on the degree of correlation between asset classes: the picture shows how the correlation among different

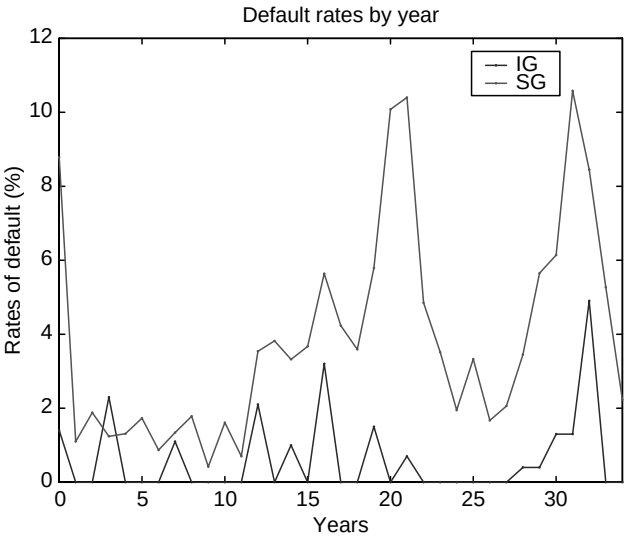


FIGURE 24.2 Default (yearly) rates for investment and noninvestment grades. Rates for investment data have been multiplied by 10 for comparing the data. (From Hamilton, D.T., Varma, P., Ou, S., and Cantor, R., Default and Recovery Rates of Corporate Bond Issuers, 1920–2004, Moody's Investor Service, January, 2005.)

groups changes during the sample period, from negative correlation (1975, 1977, or 1990) to zero (1993, 1996) or highly positive (1982, 1995, or 2000). These differences in default rates through time could reveal the existence of an idiosyncratic component between asset classes. This fact could support the idea of modeling the behavior of different assets in a different way.

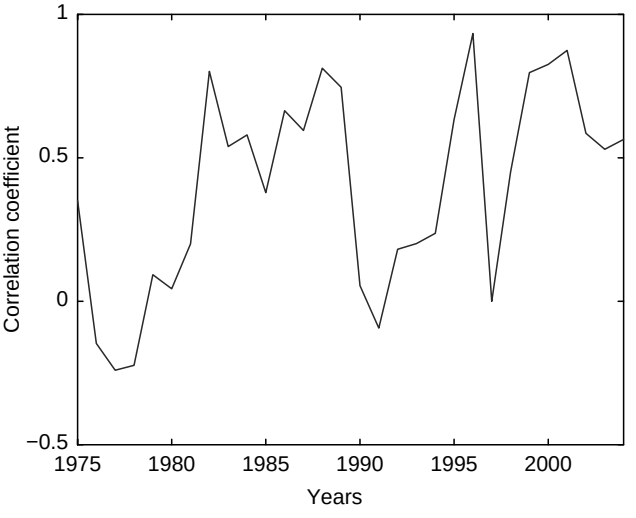


FIGURE 24.3 Correlation coefficients using a moving window of 5 years.

This chapter considers a family of models that take account the existence of these different asset classes or regions, in line with the suggestions of Gregory and Laurent (2004). We analyze a model in the line of the two assets–two Gaussian factor model of Schönbucher (2003), where no distinction is made between the obligors that belong to the same class. Our model generalizes the model posited by Schönbucher (2003) by considering a *t*-Student distribution, which assigns a higher probability to high default events. Empirical evidence seems to go in this direction (Mashal and Naldi, 2002). Our work contributes to the existing literature in the analysis of these asset class models. To the best of our knowledge, no similar studies have yet been reported in this direction.

The model we propose is a two-by-two factor model as follows:*

$$\begin{aligned} V_{i1} &= \alpha_{11}F_1 + \alpha_{12}F_2 + \beta_1 u_{i1} \\ V_{i2} &= \alpha_{21}F_1 + \alpha_{22}F_2 + \beta_2 u_{i2} \end{aligned} \tag{24.7}$$

where, V_{i1}, V_{i2} $(i < n)$ represents the value of the *i*-company which belongs to the different asset class, F_j, $j = 1, 2$ captures the effect of systematic factors—business cycle and industry—with independent *t*-distributions with n_j degrees of freedom, and u_{i1}, u_{i2} are idiosyncratic factors distributed also with n_{i1}, n_{i2} degrees of freedom, respectively.

Under the assumptions of factor models, F_j are scaled to have variance 1, then $\alpha_{11} = \rho_{11}\sqrt{\frac{n_1-2}{n_1}}$ and so on. Idiosyncratic errors are also scaled, for instance $\beta_1 = \sqrt{\frac{n_{i1}-2}{n_{i1}}}\sqrt{1-\alpha_{11}^2-\alpha_{12}^2}$. Finally, we assume the same default barriers K_{i1}, K_{i2} for the obligors of the same class.

Needless to say, the model can be easily generalized to the case of *m*-asset classes as follows:

$$V_{i,m} = \sum_{h=1}^{p} \rho_{mh}F_h + u_{i,m}\sqrt{1-\sum_{h=1}^{p}\rho_{mh}^2}$$

where, $V_{i,m}$ represents the value of the *i*-company which belong to the *m*-asset class, F_j, $j = 1, \ldots, p$ capture the effect of systematic factors, and $u_{i,m}$ is the idiosyncratic factor corresponded to *i*-firm of the *m*-asset class. Generally speaking, assumptions relying on distribution factors or more asset classes could also be proposed. However, a trade-off between accuracy, parsimony, and computing efficiency must be considered.

24.2.3 Conditional Default Probabilities

Without loss of generality, we omit the subscript that refers to the *i*-firm for the ease of exposition. We want to study the probability of default for the *i*-firm that belongs to an asset class *m*, with $m = 1, 2$,

* For the sake of brevity, we omit the graph of the loss distribution implied by this model.

$$P[V_m < K|\mathbf{F} = \mathbf{f}] = P\left[u_m \leq \frac{K - \alpha\mathbf{F}}{\beta_m}|\mathbf{F} = \mathbf{f}\right] = T_m\left(\frac{K - \alpha\mathbf{f}}{\beta_m}\right)$$

where, T_m denotes the distribution function of a t-student with n_m degrees of freedom for the i-firm, and $\mathbf{F}$, α are the common vector factors and their coefficients, respectively.

It is usual to calculate the probability of having k default events, conditional to realization of vector factor $\mathbf{F}$ as

$$P[X = k|\mathbf{F}] = \sum_{l=0}^{k} b(l, N_1, T_1)\, b(k - l, N_2, T_2)$$

where $b(l, N, T)$ denotes the binomial frequency function, which gives the probability of observing l successes with probability T, where N represents the number of firms that belong to each asset class. In the same manner, the unconditional probability of k default events is obtained considering all possible realizations of factors $\mathbf{F}$

$$P[X = k] = \sum_{l=0}^{k} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} b(l, N_1, T_1)\, b(k - l, N_2, T_2)\psi(\mathbf{f})\, \mathrm{d}\mathbf{f}$$

where $\psi(\mathbf{f})$ denotes the probability density function of $\mathbf{F}$.

Finally, the total failure distribution is just obtained as the sum of all the defaults up to level k,

$$\begin{aligned} P[X \leq k] &= \sum_{r=0}^{k} P[X = r] \\ &= \sum_{r=0}^{k} \sum_{l=0}^{r} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} b(l, N_1, T_1)\, b(r - l, N_2, T_2)\psi(\mathbf{f})\, \mathrm{d}\mathbf{f} \end{aligned} \tag{24.8}$$

Lando (2004) refers to this problem as a different buckets problem in the sense that by means of multinomial distributions, we compute the total loss distribution of the portfolio. Our approach to this point would be the computation of this set of bucket probabilities but we will adopt a different approach.

24.2.4 Loss Distribution

As pointed out in Lando (2004), calculation of multinomial expressions like Equation 24.8 is burdensome. Instead of computing Equation 24.8 by brute force, we use a simple idea, due to Andersen et al. (2003) that could improve the efficiency in terms of computing cost.*

Andersen et al. (2003) provide an efficient algorithm, which has become widely used by the industry. Roughly speaking, the main idea behind this is to observe what happens to the total loss distribution of a portfolio when we increase its size by one firm.

* Their contribution has been also explored and extended in Hull and White (2004).

Consider a portfolio that includes n credit references and let $p_n(i|\mathbf{f})$ be the probability of default of i firms in this portfolio conditional on factors $\mathbf{f}$. Let $q_{n+1}(\mathbf{f})$ be the default probability of an individual firm that is added to this portfolio. These two probabilities are conditional on the realization of the common factor vector $\mathbf{f}$.

Consider the total loss distribution (LD) in this portfolio. Intuition says that the probability of i-defaults in this portfolio—conditional on $\mathbf{f}$—can be written as

$$LD(i|\mathbf{f}) = p_n(i|\mathbf{f}) \times (1 - q_{n+1}(\mathbf{f})) + p_n(i-1|\mathbf{f}) \times q_{n+1}(\mathbf{f}), \quad 0 < i < n+1$$

where the first term reflects that default is due to i-firms included in the initial portfolio while the new reference (just included in the portfolio) survives. In a similar way, the second term reflects the new firm (just included in the portfolio) defaults while the other $i-1$ defaulted firms were previously included in the original portfolio.

Moreover, for the extreme cases of default firms, we have

$$\begin{aligned} p_{n+1}(0|\mathbf{f}) &= p_n(0|\mathbf{f}) \times [1 - q_{n+1}(\mathbf{f})] \\ p_{n+1}(n+1|\mathbf{f}) &= p_n(n|\mathbf{f}) \times q_{n+1}(\mathbf{f}) \end{aligned}$$

Then, taking into account the last firm just included in the portfolio, these equations reflect that no firm defaults or all of them default, respectively.

As an example, consider an initial portfolio including two firms. Adding a third firm, the total default probabilities are given by

$$\begin{aligned} p_3(0|\mathbf{f}) &= p_2(0|\mathbf{f})(1 - q_3(\mathbf{f})) \\ p_3(1|\mathbf{f}) &= p_2(1|\mathbf{f})(1 - q_3(\mathbf{f})) + p_2(0|\mathbf{f})q_3(\mathbf{f}) \\ p_3(2|\mathbf{f}) &= p_2(2|\mathbf{f})(1 - q_3(\mathbf{f})) + p_2(1|\mathbf{f})q_3(\mathbf{f}) \\ p_3(3|\mathbf{f}) &= p_2(2|\mathbf{f})q_3(\mathbf{f}) \end{aligned}$$

Using this iterative procedure, we can compute the unconditional total LD by considering all possible realizations of $\mathbf{f}$.*

24.3 RESULTS FOR TWO-BY-TWO MODEL

In this section, we present some results of the model given by Equation 24.7. Firstly, we discuss the spreads obtained using the two-by-two approach. Secondly, we give an approach useful for cases of high degrees of freedom based on Cornish–Fisher expansions, which is useful in terms of computing cost.

* See Andersen et al. (2003) or Gibson (2004) for more details.

24.3.1 Numerical Results

To give some results of the model, given by Equation 24.7, we analyze different cases for a two asset classes, 5 year CDO with 100 firms with quarterly payments. We also assume that the recovery rate is fixed and equal at 40%. Risk-neutral default individual default probabilities are fixed at 1% and 5% for assets belonging to class 1 and class 2, respectively. For ease of explanation, the size of each asset class in the portfolio is the same (50% for each one). More results concerning the size of the portfolio are provided in Section 24.4.

Table 24.2 displays the main results obtained. First row corresponds to the three simulated base cases: the standard Gaussian model* with one factor, two asset classes; the two assets–two Gaussian factor; finally, the two assets–two *t*-Student factors.† *t*-Student distributions have been fixed at five degrees of freedom for idiosyncratic and systematic factors. Second row displays the correlation parameters of each asset class with both factors. For example, 0.1/0.3 in the two-Gaussian case refers to a correlation of 0.1 (0.3) for elements of class 1 (2) with *the* two systematic factors. In line with Gregory and Laurent (2004), our idea is to check the behavior of the CDO to a portfolio exposed to two different degrees of correlation.

All the simulations have been carried out for different tranches values. Looking at the riskier tranche (equity tranche), we observe that for the same degree of correlation, its spread is systematically bigger for the two *t*-Student case than for the two-Gaussian factors. One conclusion that arises from Table 24.2 is that the two assets–one-Gaussian factor spreads for equity tranche are close to those values obtained for two assets–two *t*-Student factors.

As expected, the same does not apply for mezzanine tranches: the addition of more factors provides more weight to extreme default events, which results in an increase in spread of mezzanine tranches. The same conclusions apply to senior tranche.

24.3.2 Approximation for *n* Infinite

The asymptotic relationship between a *t*-distributed random variable and a normal random variable by means of the Cornish–Fisher expansion could be interesting for cases of big degrees of freedom. Shaw (2006) provides the Cornish–Fisher expansion for a *t*-distributed random variable, with zero mean and unit variance, in terms of a standard

TABLE 24.2 Spreads of Different Tranches for a 100-Names CDO under Different Correlation Parameters

Model	**Standard Gaussian**		**Two-Gaussian Factors**		***t*-Student**	
Correlation	**0.1/0.1**	**0.1/0.3**	**0.1/0.1**	**0.1/0.3**	**0.1/0.1**	**0.1/0.3**
Attachment Points (%)	**Tranche Spread (Basis Points)**					
0–3	7166	5016	5421	3154	6899	4977
3–10	1040	929	1581	1401	1254	1302
10–100	5	11	43	122	27	65

* Standard Gaussian model has been computed using a Gauss–Hermite quadrature with eight nodes.
† *t*-Student simulations have been computed using a Simpson's quadrature with 25 points.

TABLE 24.3 Spreads of Different Tranches for a 50-Names CDO under Different Correlation Parameters

Model	Two-Gaussian Factors	*t*-Student		CF *t*-Student	
Correlation	0.1/0.3	0.1/0.3	0.1/0.3	0.1/0.3	0.1/0.3
Degrees of Freedom ($n1/n2$)	∞/∞	10/10	15/15	10/10	15/15
Attachment Points (%)	Tranche Spread (Basis Points)				
0–3	3015	3924	3711	3270	3296
3–10	1380	1236	1198	1025	1083
10–100	289	160	160	166	160

normal random variable distribution. This reduces considerably the computational time as, in this case, it is possible to use a double Gauss–Hermite quadrature, instead of a Simpson quadrature. Basically, Shaw (2006) provides the relationship

$$s = z + \frac{1}{4n} z(z^2 - 3) + \frac{1}{96n^2} z(5z^4 - 8z^2 - 69) + \cdots \tag{24.9}$$

where s is the t-distributed random variable with n degrees of freedom, and z is a standard normal random variable. To check the accuracy of the approximation in Equation 24.9, Table 24.3 displays the spreads (in basis points) for different tranches in a two asset classes, 50-named CDO with quarterly payments under different correlation parameters for various degrees of freedom. As in the previous section, correlation parameters correspond to factors of each asset classes. As in previous examples, risk-neutral default probabilities have also been fixed at 0.1 and 0.3 for each asset class. The recovery rate is fixed at 40%.

Basically, two general models have been computed: two-Gaussian factors and a two t-Student factor. The last column refers to the two t-Student factor approximation using the Cornish–Fisher expansion. Correlation coefficients and degrees of freedom for each model are displayed in the table.* The Gaussian case is presented to get intuition of how far we are from the asymptotic result. As expected, the larger the degree of freedom, the higher the accuracy of the results. The differences between the equity tranche range from 20% for 10 degrees of freedom to 12% for 15 degrees of freedom case. Similarly, considering the mezzanine cases, differences go from 21% to 11%. There are no substantial changes in the case of senior tranche. It is worth remembering that computations under the exact t-Student distribution have been done using a numerical quadrature and, so, they are exposed to numerical errors.

24.4 SENSITIVITY ANALYSIS

Now, we are interested in the prices of the CDO under two different scenarios: changes in portfolio size and correlation. Firstly, we present some standard measures in risk management as the value at risk (VaR) and the conditional value-at-risk (VaR) measures. Secondly, we analyze the sensitivity of different tranches to changes in correlation.

* The two-Gaussian factor model is equivalent to a two t-Student factor model with infinite degrees of freedom.

24.4.1 Value at Risk and Conditional Value at Risk

Value at risk and conditional value at risk are usually taken as representative risk measures for a portfolio. VaR is defined as the percentile of the distribution of portfolio losses given a certain level of confidence (Duffie and Pan, 1997). Artzner et al. (1999) enumerates some limitations of the VaR measure and discuss some interesting properties of a proper measure of risk. According to this, we also include the CVaR measure,* defined as the expected loss in a portfolio conditional to a certain loss threshold u, that is,

$$CVaR_u = E[x|x > u]$$

where the sign of the inequality has been changed because we are working directly with the total loss distribution.

Table 24.4 includes the VaR and CVaR measures for a 100-named CDO composed by two different risky asset classes under different correlation parameters and different proportions in the portfolio. The (yearly) risk-neutral default probabilities for each obligor of asset class 1 have been fixed at 0.01, and 0.05 for obligors belonging to asses class 2 to asset class 2. For the sake of simplicity, all the factors—systematic and idiosyncratic—in the model (Equation 24.7) have been fixed at five degrees of freedom. The first column includes the correlation coefficients corresponding to both factors of each class (i.e., 0.1/0.3 means a correlation coefficient of 0.1 (0.3) for both factors in asset class 1 (2)). Correlation coefficients equal to zero refers to independence case between obligors. For analyzing the response of the loss distribution generated by the model (Equation 24.7) with respect to different sizes of the portfolio, the remaining columns show different percentage sizes of the portfolio. The first term refers to the portfolio percentage of class 1, and so on.

TABLE 24.4 VaR and CVaR Measures for a 100-Named CDO Composed by Two Different Risky Asset Classes under Different Correlation Parameters and Different Proportions in the Portfolio

	VaR 99%			CVaR 99%		
Correlation	**25/75**	**50/50**	**75/25**	**25/75**	**50/50**	**75/25**
0.0/0.0	10	9	7	18	9	9
0.1/0.1	25	19	13	35	27	19
0.1/0.3	57	40	23	67	48	30
0.3/0.3	61	47	33	72	61	50

Note: Degrees of freedom are fixed at 5 for all factors. The first (second) coefficient refers to the proportion (in percentage) of asset class 1 (2) in the portfolio. Yearly risk-neutral probabilities have been fixed to 0.01 for each obligor of asset class 1, and 0.05 for those of asset class 2.

* This measure was posited by Acerbi et al. (2001).

Needles to say that two main results arise form Table 24.4. Firstly, the higher the correlation the higher expected extreme loss as measured by VaR and CVaR, as expected. Secondly, an increase in the percentage of the risky asset (asset class 2) produces an increase in the losses of the portfolio.

24.4.2 Sensitivity to Correlation

To analyze the sensitivity to correlation of the model (Equation 24.7), we have created a CDO based on a portfolio of 50 names. Individual default probabilities have been fixed at 1% for asset class 1 and 5% for asset class 2, respectively. To reduce the computational cost of the implementation, we have set the distribution of the two systematic factors as *t*-Student ones with 15 degrees of freedom. We have used the results of Shaw (2006) developed in Section 24.2.5, without loss of generality. All simulations have been performed using a double Hermite quadrature with 64 nodes. Idiosyncratic factors have been fixed at 5 for each asset class. To search differences in the portfolio composition, we have applied our study to two different-sized portfolio: equally weighted portfolio (50% asset class 1, 50% asset class 2) and risky portfolio (25% asset class 1, 75% asset class 2).

Figures 24.4 and 24.5 display the spreads obtained under different sets of correlations for the equally weighted and risky portfolios. In general, the convexity pattern of the equity-senior curves remains constant in both cases, which is consistent with the preference (aversion) for risk on equity (senior) tranche investors, as expected.

Regarding changes in correlation, Figure 24.4 reveals that correlation with risky asset classes are, by large, responsible for changes in the value of equity spreads. When it comes

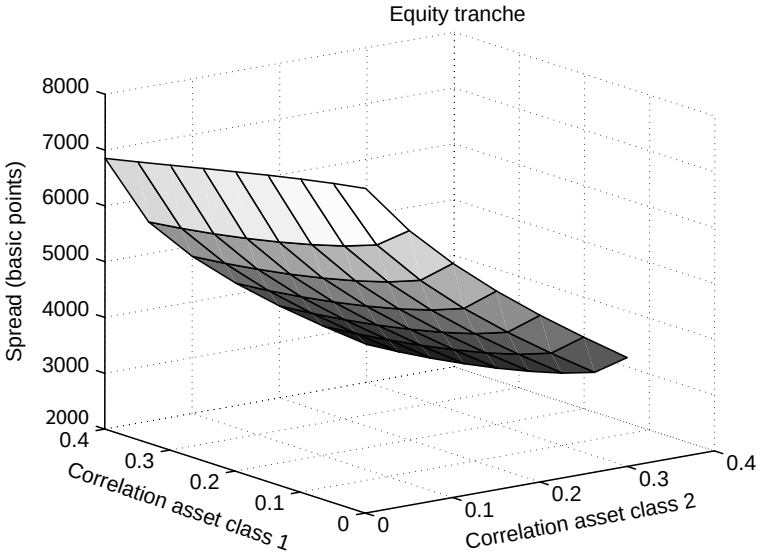


FIGURE 24.4 Spreads under different correlations for a equally weighted (50%–50%) portfolio.

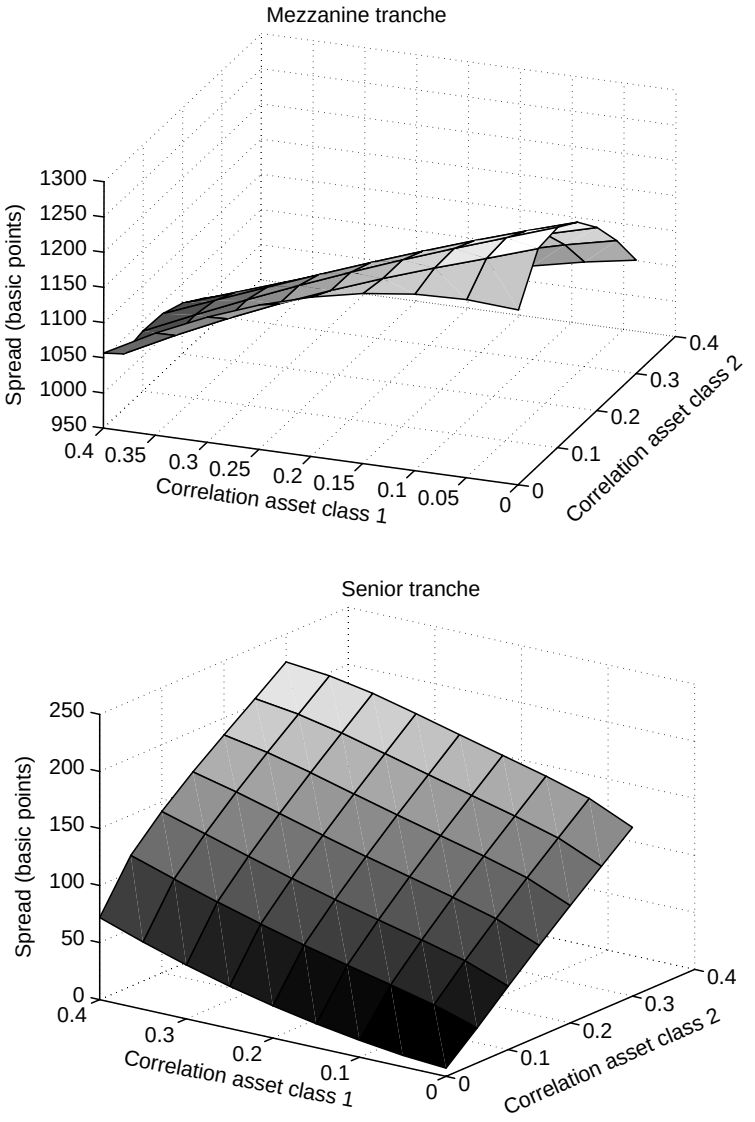


FIGURE 24.4 (continued)

to the risky portfolio (Figure 24.5), it is interesting to observe how changes produced by the correlation in asset class 1 or 2 (see Figure 24.5a and b) produce almost the same effects.

As it is also expected, an increase in global correlation raises the spreads of senior tranche. A higher correlation increases the probability of big losses, which is reflected in the senior spreads. This feature could be also mentioned (in a different scale) to the case of the risky portfolio in Figure 24.5.

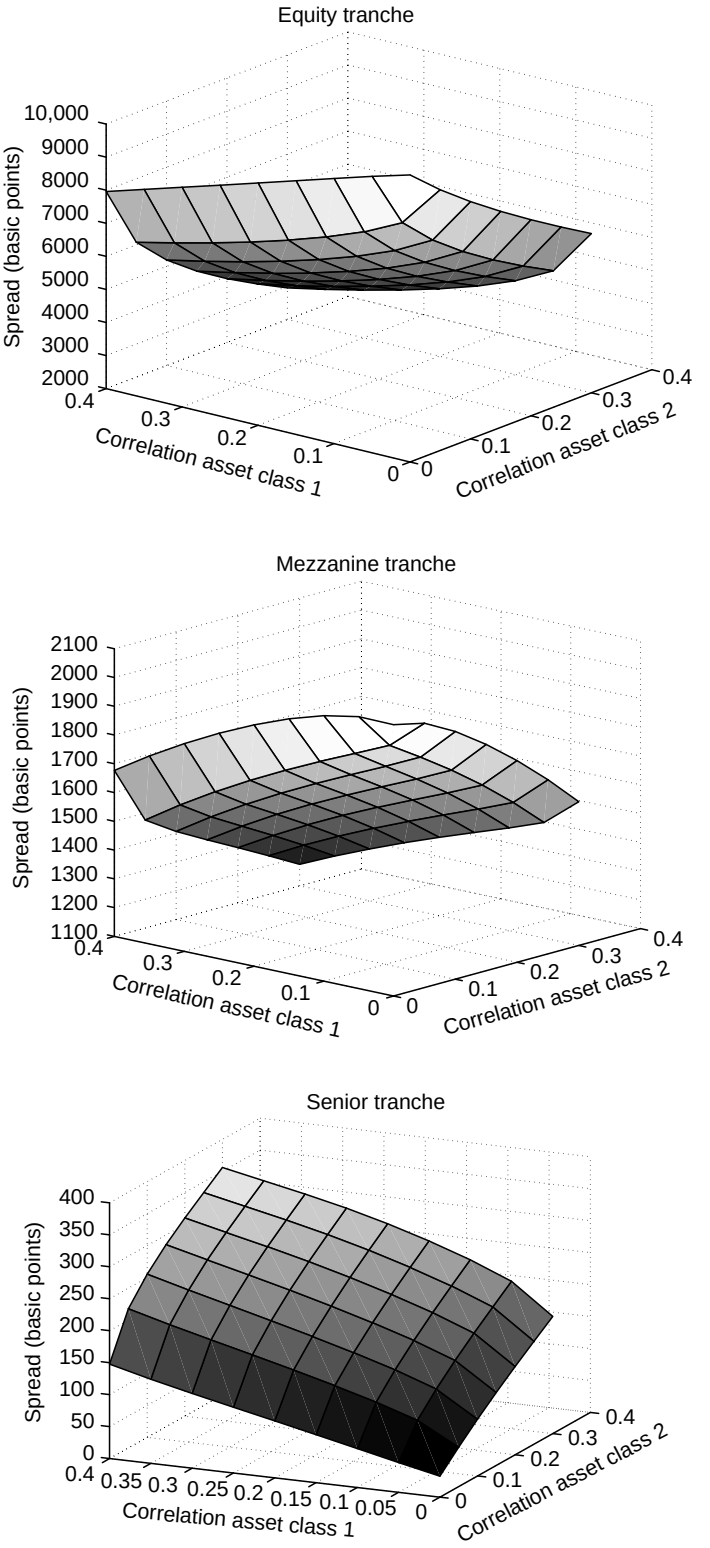


FIGURE 24.5 Spreads under different correlations for a 25%–75% portfolio.

24.5 ECONOMETRIC FRAMEWORK

This section focuses on the parameters estimation of the model given in Equation 24.7. As pointed out in Embretchs et al. (2005), the statistical estimation of parameters in many industrial models are simply assigned by means of economic arguments or proxies variables. We will develop an exercise of formal estimation using some well-known econometric tools as logit–probit regressions.* Because of the features of our data, some cautions must be taken to understand our results. This is due to the shortage of relevant data (for instance, rates of default of high-rated companies) or the sample size, as was also noticed by Embretchs et al. (2005). These authors also provide a more general discussion on the statistical estimation of portfolio credit risk models.

24.5.1 Estimation Techniques

As suggested by Schönbucher (2003) or Embretchs et al. (2005), the estimation of parameters in the expression (Equation 24.7) will be carried out using the models for discrete choice of proportions data. Basically, the idea consists in explaining the sample rates of default p_i (where i refers to the asset class or group) as an approximation to the population rates of default P_i plus an error term, ε_i. The idea behind is to link the population probability with some function $F(\cdot)$ over a set of explanatory factors $\mathbf{X}_i$ and their coefficients β, as follows:

$$p_i = P_i + \varepsilon_i = F(\mathbf{x}_i'\beta) + \varepsilon_i \tag{24.10}$$

To be interpreted as a probability, the function $F(\cdot)$ must be bounded and monotonically increasing in the interval [0,1]. Some widely used functions for F are the standard Normal distribution, which corresponds to the probit model, or the uniform distribution, which results in the linear probability model.

As suggested by Greene (2003), we could use regression methods as well as maximum likelihood procedures to estimate the set of coefficients β of the expression (Equation 24.10). For example, in the case of the probit regression, the relationship between the sample rates of default p_i and their population counterparts are

$$p_i = P_i + \varepsilon_i \rightarrow \Phi^{-1}(p_i) = \Phi^{-1}(P_i + \varepsilon_i)$$

which could be aproximated by (Novales, 1993)

$$\Phi^{-1}(p_i) \simeq \mathbf{x}_i'\beta + \frac{\varepsilon_i}{f(\mathbf{x}_i'\beta)}$$

where $\Phi(\cdot)$ denotes the distribution function of a standard Normal variable. As mentioned in Novales (1993), the last expression suggests that we can estimate the parameter vector β by regressing the sample probits $\Phi^{-1}(p_i)$ on the variables x. Considerations about the heteroscedasticity of the residual can be found in the cited reference.

* Standard references on this type of regressions using grouped data can be found in Novales (1993) or Greene (2003).

To check the model's goodness of fit, Novales (1993) also provides a comparison of different regressions (probit, logit, or lineal) in terms of the mean square error (MSE). The statistic s is defined as

$$s = \sum_{1}^{T} \frac{n_i(p_i - \hat{P}_i)^2}{\hat{P}_i(1 - \hat{P}_i)} \sim \chi^2_{T-k} \tag{24.11}$$

where n_i represents the sample size of the data (subscript i refers to asset class or group) and p_i, $\hat{P}_i$ are the observed and estimated frequencies, respectively. The statistic s follows a chi-square distribution with T–k degrees of freedom, sample length T and k restrictions.

24.5.2 Variables and Estimation

With the intention of illustrating the estimation of the model given by Equation 24.7, we have chosen a set of six explanatory variables for the rates of default: the real growth domestic product (GDP), the consumers price index (CPI), the annual return on the S&P500 index (SP_ret), its annualized standard deviation (SP_std), the 10 year treasury constant maturity rate (10_rate), and the industrial production index (IPI).* As dependent variables we have the annual rates of default for two investment grades: noninvestment grade (SG) and investment grade (IG), both collected from Hamilton et al. (2005). Because of the availability of default rate data, the sample period has been taken from 1970 to 2004, which results in 35 observations. A summary of the main statistics and the correlation coefficients are presented in Tables 24.5 and 24.6, respectively.

To visualize the influence of the proposed explanatory variables in the default rates, Figures 24.6 and 24.7 represent the scatter plots of different investment classes versus

TABLE 24.5 Descriptive Statistics

Descriptive Statistics	Mean	Median	Std	Skewness	Kurtosis	Max	Min
IG	0.0006	0	0.0011	2.2481	8.2051	0.0049	0
SG	0.0389	0.0345	0.0288	1.0213	3.0976	0.1058	0.0042
SP_ret	0.0297	0.0439	0.0675	−0.6110	2.5129	0.1310	−0.1208
SP_std	89.4702	39.3947	106.3397	1.4176	3.7198	390.1884	7.5537
GDP	0.0130	0.0148	0.0087	−0.6077	2.8766	0.0301	−0.0085
CPI	0.0205	0.0159	0.0128	1.2338	3.7824	0.0564	0.0052
10_rate	7.7577	7.4176	2.4231	0.8413	3.2283	13.9214	4.0139
IPI	0.0111	0.0124	0.0185	−0.4614	3.2866	13.9214	−0.0409

* All data are available from the Federal Reserve Bank of St. Louis webpage (www.stlouisfed.org) except the S&P 500 index level, which has been taken from Bloomberg.

TABLE 24.6 Correlation Matrix

	IG	SG	SP_ret	SP_std	GDP	CPI	10_rate	IPI
IG	1.0000							
SG	0.4106	1.0000						
SP_ret	−0.2388	−0.2153	1.0000					
SP_std	0.1760	0.3472	−0.0424	1.0000				
GDP	−0.1431	−0.3541	0.4988	0.0029	1.0000			
CPI	−0.1847	−0.3209	−0.2806	−0.4959	−0.4937	1.0000		
10_rate	−0.1043	−0.2344	0.0611	−0.6113	−0.1070	0.6060	1.0000	
IPI	−0.1499	−0.2930	0.2687	−0.0385	0.6671	−0.3252	−0.1344	1.0000

different explanatory variables. Figure 24.6 seems to corroborate what we could guess departing from the correlation parameters included in Table 24.6: the standard deviation of the S&P 500 returns, the GDP, and the CPI can be good candidates for explaining the default rate in the case of the noninvestment firms. Additionally, at a certain degree, the

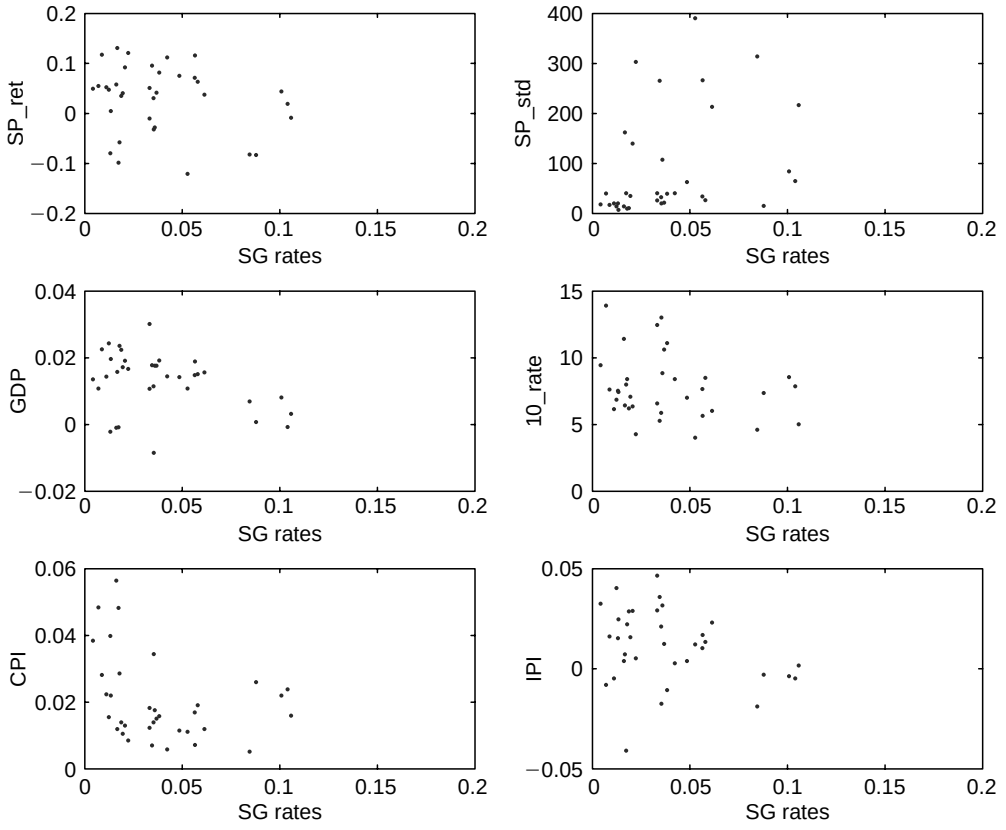


FIGURE 24.6 Noninvestment grade rates of default (SG) versus different explanatory variables.

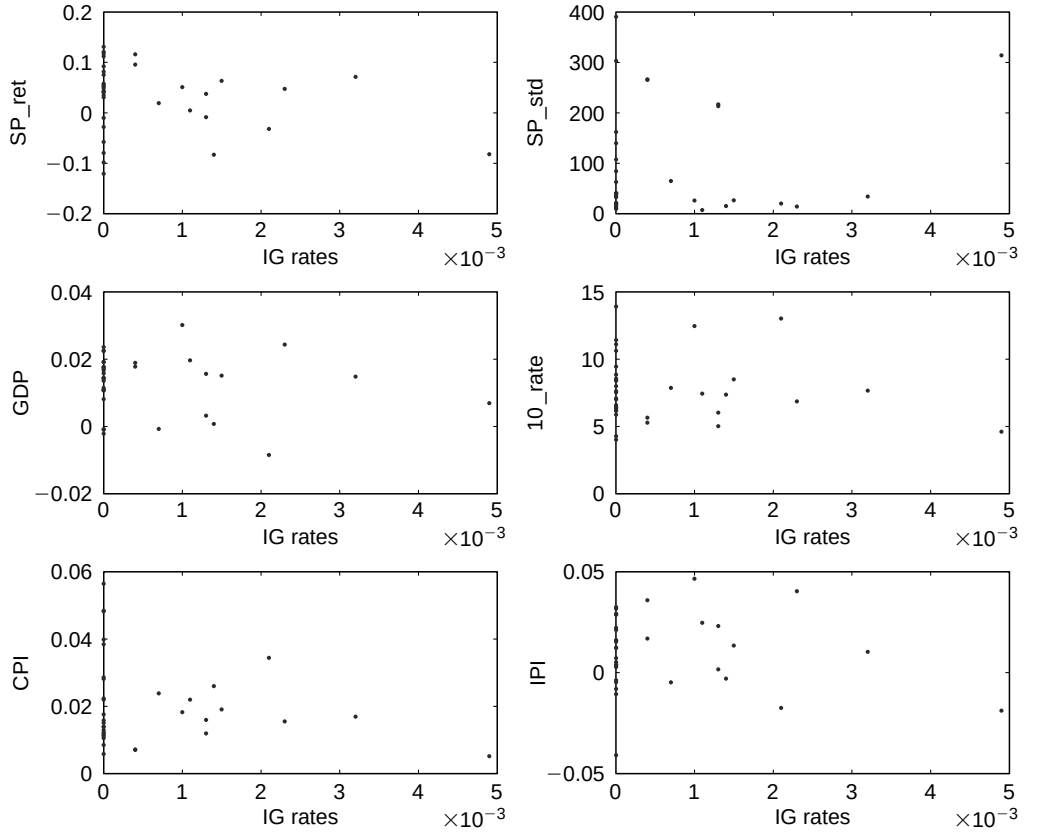


FIGURE 24.7 Investment grade rates of default (IG) versus different explanatory variables.

S&P 500 return can be added to this list as a possible explanatory variable in the case of the investment firms.*

One step ahead is to compute how much of the sample can be explained by the set of variables under study. To answer this question, we regress the explanatory variables on the noninvestment and investment rates. Table 24.7 shows the results. The first row corresponds to the different independent variables under study. The first column contains the model under study—linear, probit, logit—and the different regressed variables (SG and IG default rates). Second to eighth columns display different β's obtained under different models. Finally, the last column shows the s statistic defined in Equation 24.11, which will be used as a naive benchmark: if the whole set of independent variables explains some quantity of the sample, two variables would explain less: the pair of variables whose s value are closest to the benchmark could be good candidates as common factors in the model (Equation 24.7).

* As Figure 24.7 reveals, due to the high number of null observations in the IG sample, conclusions about the factors affecting IG rates should be taken carefully.

TABLE 24.7 Regression of Noninvestment (SG) and Investment (IG) Default Rates with Respect to Different Explanatory Variables

Model	β_0	SP_ret	SP_std	GDP	10_rate	CPI	IPI	s
Linear SG	0.0792	−0.0511	0.0000	−1.9518	0.0025	−1.6900	−0.1222	11.5543
	[0.0344 0.1241]	[−0.1904 0.0881]	[−0.0001 0.0001]	[−3.4391 −0.4646]	[−0.0022 0.0074]	[−2.6699 −0.7102]	[−0.7063 0.4617]	
IG	0.0011	−0.0045	0.0000	−0.0179	0.0001	−0.0391	−0.0060	1.7799
	[−0.0010 0.0033]	[−0.0113 0.0022]	[−0.0000 0.0000]	[−0.0898 0.0541]	[−0.0001 0.0003]	[−0.0865 0.0082]	[−0.0343 0.0222]	
Probit SG	−1.3794	−0.8812	0.0003	−23.7382	0.0489	−26.7956	−1.3676	13.9414
	[−1.8624 −0.8963]	[−2.3799 0.6176]	[−0.0007 0.0014]	[−39.7437 −7.7326]	[−0.0028 0.1005]	[−37.3405 −16.2507]	[−7.6526 4.9174]	
IG	−3.3965	−1.2129	0.0003	−17.1498	0.0482	−16.6621	3.6291	6.0474
	[−4.3212 −2.4717]	[−4.0822 1.6565]	[−0.0018 0.0023]	[−47.7920 13.4924]	[−0.0506 0.1471]	[−36.850 13.5258]	[−8.4033 15.6616]	
Logit SG	−2.3439	−2.1634	0.0007	−54.8455	0.1191	−64.7566	−3.2499	15.0393
	[−3.4585 −1.2293]	[−5.6218 1.2951]	[−0.0017 0.0032]	[−91.7787 −17.9123]	[−0.0001 0.2383]	[−89.0892 −40.4240]	[−17.7527 11.2528]	
IG	−8.1171	−4.1999	0.0011	−64.2390	0.1772	−60.2912	14.5665	6.7882
	[−11.4907 −4.7434]	[−14.6678 6.2680]	[−0.0064 0.0086]	[−176.0283 47.5503]	[−0.1835 0.5380]	[−133.9410 13.3587]	[−29.3303 58.4634]	

Note: SP_ret and SP_std are the yearly returns and standard deviation of the S&P500 Index. GDP, 10_rate, CPI, and IPI are the growth domestic product, 10 year constant maturity rate, consumer price index, and industrial production index, respectively. Finally, s is the fit statistic defined in Equation 24.11.

We start with regressions on SG rates. One main reason recommends this procedure: their data are more relevant to determine the factors that may cause default. Up to a point, conclusions on the factors will be more robust. Previous regressions suggest choosing the variables GDP, CPI, IPI, and SP_stdas common factors in the model given by Equation 24.7. These variables minimize the statistic (Equation 24.11) with respect to other pairs of alternatives. Finally, we select GDP and CPI as common factors according to two main reasons:

1. Firstly, the industrial production index could be seen as a proxy of the GDP and its information could be redundant. Moreover, regressions of probit–logit models using these two variables support the choice of GDP against the IPI.
2. Secondly, regressions on the parameter SP_std give a β close to the precision imposed to our estimated parameters (10^{-4}).

Table 24.8 presents the ordinary least squares (OLS)* estimates for β of independent term, GDP variable, and CPI variable, in the columns, using the SG rates. Confidence intervals at the 95% level are displayed in brackets. The rows in this table also display regression results for linear, probit, and logit models. The last row contains the value of the statistic given by Equation 24.11 obtained for each case. Attending to the goodness-of-fit criteria using *s*, the ordinary least squares (OLS) probit model regression provides the best fit to the sample.† Overall, all the β estimates corresponding to OLS regressions are negative, except for the independent term of the linear model, which leads to higher *s* statistic. Results concerning to OLS regressions can be interpreted as follows: a negative β implies an increasing on default probabilities. In line with this, as expected, a decrease in GDP rates may produce an increase in SG default rates. Surprisingly, an increase in the CPI rate could diminish the rates of default, which might result counter intuitive.

TABLE 24.8 Results for Regressions of Noninvestment (SG) Default Rates on Real Growth Domestic Product (GDP) and Consumer Price Index (CPI) as Explanatory Variables

SG	Model	β_0	GDP	CPI	*s*
Linear	OLS	0.0979	−2.2380	−1.4690	21.8030
		[0.0733 0.1226]	[−3.2486 −1.2274]	[−2.1548 −0.7832]	
Probit	OLS	−1.0785	−26.7705	−21.6288	14.7653
		[−1.3568 −0.8002]	[−38.1822 −15.3587]	[−29.3730 −13.8846]	
Logit	OLS	−1.6293	−61.8990	−51.7790	15.4018
		[−2.2769 −0.9816]	[−88.4563 −35.3416]	[−69.8014 −33.7567]	

* GLS estimates have not been computed due to the sample size.

† Maximum likelihood estimates (available on request) for models probit and logit are close to those parameters obtained for respective OLS models.

TABLE 24.9 Results for Regressions of Investment (IG) Default Rates on Real Growth Domestic Product (GDP) and Consumer Price Index (CPI) as Explanatory Variables

IG	Model	β_0	GDP	CPI	s
Linear	OLS	0.0017	−0.0391	−0.0290	5.0921
		[0.0005 0.0029]	[−0.0884 0.0102]	[−0.0624 0.0045]	
	OLS	0.0009	−0.0181	—	2.2557
		[0.0002 0.0015]	[−0.0623 0.0261]		
Probit	OLS	−3.1421	−14.1617	−11.2345	6.4347
		[−3.6496 −2.6347]	[−34.9687 6.6453]	[−25.3546 2.8856]	
	OLS	−3.4781	−5.9888	—	8.6336
		[−3.7663 −3.1899]	[−24.5193 12.5417]		
Logit	OLS	−7.1801	−50.7973	−40.7355	7.1658
		[−9.0318 −5.3284]	[−126.7275 25.1328]	[−92.2634 10.7923]	
	OLS	−8.3983	−21.1631	—	9.6372
		[−9.4493 −7.3472]	[−88.7533 46.4271]		

Table 24.9 shows regression results for linear, probit, and logit models using IG rates as dependent variable. It is worth to notice that results are not conclusive as 63% of the sample under study are zeros. GLS estimations do not make sense in this context. To avoid numerical problems in the estimation, we have added the quantity 0.00005 to the sample, as suggested by Greene (2003). The first row displays the independent term and explanatory variables. Each pair of the following rows contains firstly the different values of β's obtained using two variables (GDP and CPI); secondly, their values using only the GDP variable. Maximum likelihood estimates (available on request) for the probit and logit models are close to those parameters obtained for respective OLS models. We have estimated GDP variable alone with the intention of analyzing the explanatory power of the GDP on IG rates. First to second rows show the model and procedure used. The last column displays the value for the statistic s. At a certain degree, results on Table 24.8 could support the inverse relationship between the explanatory variables and the IG rates of default, as previously noted for the SG case.

24.5.3 Interpretation of Coefficients

As was pointed out by Elizalde (2005), due to the difficulty of interpreting what the correlation term represents, estimating the correlation term in factor models is not an evident task. Looking at Equation 24.10, the estimate β describes the effect from the explanatory factor x through a nonlinear transformation of the firm's asset value, which itself is unobserved, as it is also noticed by Elizalde (2005). As this fact complicates understanding the proper correlation term, the author enumerates some measures used ad hoc by practitioners, as equity return correlations, to conclude about the insufficiency (and scarcity) of papers that deals with this problem.

Our interpretation of coefficients in the model given by Equation 24.7 goes in the direction of the econometric explanation for the coefficients of the linear, logit, and probit models, that is, the influence of the exogenous variables on the endogenous variables.

TABLE 24.10 Estimated Coefficients for the Linear Probability Model

Linear Model	β_0	GDP	CPI	s	$\chi^2_{95\%}$ (32)	$\chi^2_{99\%}$ (32)
SG	0.0979	−2.2380	−1.4690	21.8030	No reject	No reject
IG	0.0017	−0.0391	−0.0290	5.0921	No reject	No reject

Note: s refers to the fit statistic.

In other words, the (relative) impact of the explanatory variables on the probability of default. Following Novales (1993), this interpretation of estimates for the linear model must differ to that for the logit and probit models.* This is the main reason why we split our results in two tables, Tables 24.10 and 24.11, that include, respectively, the estimation of the linear and logit–probit models.

Regarding the estimates of the linear probability model, Table 24.10 reflects the contribution of the two explanatory random variables to the probability of default. The main conclusions are obtained from the default probabilities of noninvestment grade assets (SG), but can also be extended to the investment grade (IG) assets. Looking at Table 24.10, it is interesting to observe the sign of the coefficients, which is negative: the more we decrease the GDP or the CPI, the more we increase the rates of default. Given the value of the coefficients, the variables have the same contribution to the default probability. With reference to the fit of the model to the data, under the null hypothesis that the goodness of fit to the sample is good, we cannot reject that the linear probability model could explain the results obtained.

Table 24.11 includes the ratio between estimates for SG and IG series for probit and logit models, respectively. By and large, the conclusions are the same for all the series under study. According to Novales (1993), the ratio between the estimated β's measures the relative contribution of the explanatory variables on the default probability. Results are consistent to the results obtained for the linear probability model: the negative sign of the explanatory variables, which reflects an opposite effect between default ratios and the macroeconomic variables. Moreover, the relative contribution of the explanatory variables remains equal, as was also derived from Table 24.10. Finally, we do not reject the goodness of fit of the model using confidence levels of 95% and 99%.

TABLE 24.11 Estimated Coefficients for Probit and Logit Models

	β_0	$\frac{\text{GDP}}{\text{CPI}}$	s	$\chi^2_{95\%}$ (32)	$\chi^2_{99\%}$ (32)
Probit model					
SG	−1.0785	1.2377	14.7653	No reject	No reject
IG	−3.1421	1.2606	6.4347	No reject	No reject
Logit model					
SG	−1.6293	1.1954	15.4018	No reject	No reject
IG	−7.1801	1.2470	7.1658	No reject	No reject

Note: s refers to the fit statistic.

* For example, the relationship between the explanatory and explained variables in the probit model is nonlinear while the linear probability model implies linearity between dependent and independent variables.

24.6 CONCLUSIONS

The current success of the credit derivatives market for tranched products is one of the biggest success seen within the financial industry. The standard pricing model, widely used by the practitioners, is the Gaussian one-factor model (Vasicek, 1991). However, some assumptions underlying this model are probably too restrictive. These features concern, among others, to those of homogeneity of asset classes involved, or the exposure to one sources of systematic risk.

In a more realistic setting, Schönbucher (2003) or Lando (2004) pointed out that a credit portfolio is composed of different asset classes or buckets, attending to criteria as, for example, investment grade, noninvestment grade assets, or industry. The exposure of a credit portfolio to a set of common risk factors could be significant between groups, but should be homogeneous within them. In line with this, the idea of two groups of assets treated in different ways could become a more realistic assumption than that used previously in the literature.

With the aim of contributing to the current literature, this chapter considers a family of models that takes into account the existence of different asset classes or regions of correlation. Thus, we analyze a model in the line of the two assets–two-Gaussian factor model of Schönbucher (2003). In this chapter we generalize the standard model by means of a two-by-two model (two factors and two asset classes). We assume two driving factors (business cycle and industry) with independent *t*-Student distributions, respectively, and allow the model to distinguish between portfolio assets classes. One of the main implications of considering a *t*-Student distribution is that we assign a higher probability to high default events.

Our work contributes to the existing literature in the analysis of these asset class models. To the best of our knowledge, no similar studies has been reported yet in this direction. Regarding to distributional assumptions, we extend the standard Gaussian model by considering the *t*-Student distribution. In this way, we deal with a more general model with the additional advantage that includes the Gaussian model as a particular case. We also provide the econometric framework for assessing the parameters of the posited model. Finally, an empirical application with Moody's data has been presented as an illustration of the methodology proposed.

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CHAPTER 25

CDO Prices and Risk Management: A Comparative Study of Alternative Approaches for iTraxx Pricing

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25.1 INTRODUCTION

Default risk is now spreading the valuation of derivatives in any area. Whenever a derivative is traded over the counter, the default risk of the counterparty should, in principle, enter the valuation. Recently, regulatory institutions have insisted on the need to include this kind of risk in the pricing paradigm, as for example pointed out in the Basel II framework and also in the IAS 39 system.

Counterparty risk is the risk that the counterparty owing some payments (according to an agreed derivative transaction) defaults. Whenever a payoff is at risk of default, it must be included in the risk neutral valuation paradigm. In a sense, doing so creates a payoff depending on the underlying of the basic derivative and on the default of the counterparty. The latter feature transforms the payoff in a credit (actually hybrid) derivative, and default modeling is needed to carry out the risk neutral pricing.

This principle applies to all derivatives, and in particular, credit derivatives whose simplest form are the credit default swaps (CDSs). In this chapter, we study a more complex class of credit derivatives, namely collateralized debt obligations (CDOs). This class has experienced a major breakthrough since the turn of the century. By 2006, synthetic and tranched CDOs were accounting for 24% of the total credit derivatives market. In particular, tranched index trades amounted to 7.6% of the total, a rise of almost 300% in 2 years. Yet, the pricing of these CDO tranches still bears some obscure zones, like the dependence assumptions between the underlying names. This chapter attempts to identify the major sources of simplifications and to study the impact of the main risk drivers on the pricing and the sensitivities of tranche prices.

Our contribution can be viewed in two dimensions. First, we provide a very simple algorithmic decomposition of the pricing kernel of CDOs for numerical application purposes. Next, thanks to a set of numerical simulations using observed prices of iTraxx CDO tranches and fitting the implicit correlation structure, we can assess the impact of the various pricing assumptions on the price levels and sensitivities.

25.2 PRIMER ON CDS AND STANDARDIZED CDO TRANCHES

Concerning single-name products, involving default of single companies, the main data are defaultable (or corporate) bond prices and CDS rates. Corporate bonds are generally used more as building blocks than as concrete market instruments. The situation is different as far as CDSs are concerned. CDSs are now highly standardized products focusing on default of single names. When available, liquid CDS data are preferred to bond data for this reason. The payoffs for CDS are representing the actual payoffs one finds in the market. This consistency and absence of special features in CDSs is the reason why the market tends

to prefer CDSs as sources of single-name credit data over corporate bonds, especially for delta-hedging purposes.

CDSs are quoted through the rates (or spread) S in their premium legs that render them fair at inception. CDS rates on one name are usually quoted for protection extending up to 1–10 years. Mostly, maturities T_i of 1, 3, 5, 7, and 10 years are quoted. These data may be used to calibrate either intensity or structural models to credit data, by finding the model parameters matching the default probabilities implicit in CDS prices to the default probabilities implied by the models themselves. A few prices of CDS options on single names are quoted in the market as well, but bid-offer spreads are usually large and these quotes are not liquid. For the time being it may be safe not to include single-name CDS option data in model calibration, although this might rapidly change.

As far as multiname credit derivatives are concerned, the quoting mechanisms are similar. The premium rate S_{index} on a standardized pool of CDS on different names (the DJiTraxx (Dow Jones iTraxx) index, involving 125 European names, and the DJCDX index, involving 125 U.S. names are the main examples) is quoted when protecting against the whole loss of the pool of names, although some particular conventions on outstanding notionals and recovery rates are taken, which we partly illustrate with the trade example below.

But also standardized CDO tranche quotes are available in the market. Premium rates $S^{\theta}_{\text{index}}$ in the premium legs rendering the different tranches fair at inception are considered, for CDOs on standardized pools of names with standardized attachment points θ_i and standardized maturities T_i. The protection maturities T_i are 5 and 10 years. The standardized attachment/detachment points of the tranches are as percentages, where for the iTraxx index we have

$$\theta_1 = 0,\ \theta_2 = 3\%,\ \theta_3 = 6\%,\ \theta_4 = 9\%,\ \theta_5 = 12\%,\ \theta_6 = 22\%$$

while for the CDX index we have

$$\theta_1 = 0,\ \theta_2 = 3\%,\ \theta_3 = 7\%,\ \theta_4 = 10\%,\ \theta_5 = 15\%,\ \theta_6 = 30\%$$

Just to illustrate this important family of products, we consider an example concerning an iTRAXX tranche trade.

An investor sells EUR 25 mm protection on the 3%–6% tranche with a 5 year maturity. We assume a fair tranche credit spread (or rate) $S^{3,6}_{\text{index}}$ of 140 bp. Therefore, the market maker pays the investor 140 bp per annum quarterly on a notional amount of EUR 25 mm. We assume that each underlying name has the same recovery $\alpha = 40\%$.

- Each single name in the portfolio has a credit position in the index of $1/125 = 0.8\%$ and participates to the aggregate loss in terms of $0.8\% \times \text{LGD} = 0.8\% \times (1 - 0.4) = 0.48\%$, since each time a default occurs the recovery is saved, leading to a loss of $\text{LGD} = 1 - \alpha = 1 - 0.4 = 0.6$ (where LGD is loss given default).
- This means that each default corresponds to a loss of 0.48% in the global portfolio.

- After six defaults, the total loss in the portfolio is EUR $0.48\% \times 6 = 2.88\%$ and the tranche buyer is still protected since the tranche starts at 3% and we are at $2.88\% < 3\%$.
- When the seventh name in the pool defaults the total loss amounts to $0.48\% \times 7 = 3.36\%$ and the lower attachment point (i.e., 3%) of the tranche is reached.
- To compute the loss of the tranche, we have to normalize the total loss with respect to the tranche size. The net loss in the tranche is then $(3.36\% - 3\%)/3\% \times 25$ mm = EUR 3 mm, which is immediately paid by the protection seller to the protection buyer.
- Notional amount on which the premium is paid reduces to $25 - 3$ mm = EUR 22 mm and the investor receives every month a premium of 140 bp on EUR 22 mm until maturity or until the next default.
- Each following default leads to change in the tranche loss (paid by the protection seller) of $0.48\%/3\% \times 25$ mm = EUR 4 mm and the tranche notional decreases correspondingly.
- After the 13th default the total loss exceeds 6% ($13 \times 0.48\% = 6.24\%$) and the tranche is completely wiped out.
- In this case, one last payment is made of $(6\% - 5.76\%)/3\% \times 25$ mm = EUR 2 mm to the protection buyer, which in turn stops paying the premium since the outstanding notional has reduced to zero.

The different tranches offer different kinds of protection to the investor. An equity tranche (0%–3%) buyer suffers from every default in the portfolio, which leads to a decrease of tranche notional on which the periodic premium is paid and conversely to contingent protection payments. On the other side, the buyer of more senior tranches (e.g., 9%–12%) is better protected against few defaults. Indeed, theses tranches are affected only in the case of large number of defaults. This difference leads to different premium paid to buy protection. It is natural to see that the periodic premium paid to buy protection is inversely proportional to the tranche seniority.

The premium for the equity tranche is usually very large, so it is market practice to pay it as a fixed running premium of 500 bp plus an up-front payment (computed in a way that the total value is zero at inception).

Bearing in mind this discussion, in the next section we review the mechanisms implemented for the pricing of CDO tranches.

25.3 MODELING ASSUMPTIONS

In this section, we discuss the assumptions taken regarding the inputs that permit the pricing of the CDO. As pricing is done with a Monte Carlo simulation framework, a great number of scenarios are simulated for the default times of the referenced names inside the portfolio. Each of these scenarios gives the exact timing of the defaults for every underlying debt, provided that default occurs before maturity. Knowing the times of default, we

compute the loss given default (LGD) for each name with the associated recovery rate (RR). With a complete picture of the losses incurred in each simulation, it is possible to determine what would have been the right spread to pay for a given tranche. The final pricing will be the average of all the observed spreads.

25.3.1 Inputs

There are four key issues for the pricing of CDO tranches with the Monte Carlo simulations procedure described above. The interest rate dynamics has to be known to compute the present values of the losses. The probability of default from the names inside the portfolio represents the major input for the simulations. Next, one has to assess what recovery rate is to be considered in the event of a default. Finally, the correlation structure between these different parameters is crucial.

Practically, the yield curve which is used in this chapter is constructed by bootstrapping from the Euribor (Euro interbank offered rate) swap rates. The majority of transactions on the iTraxx are made by financial institutions, hence the Euribor is the closest proxy for their real funding rate.

The distribution of the probability of default for the different names is structured in an intensity-based framework, following Jarrow and Turnbull (1995). The default is therefore modeled as a jump event following a Poisson process. Denoting τ the default time and $\lambda(t)$ the hazard rate* the probability of default is given by

$$S(t) = \Pr(\tau > t) = \mathrm{e}^{-\int_t^T \lambda(s)\,\mathrm{d}s}$$

The hazard rate $\lambda(t)$ is derived for every name from the observed CDS spreads. It is market practice to use a fixed hazard rate for the whole period from a single spread.† Here we have a 5 year CDO, so we could only focus on the underlying 5 year CDS (a reason could also be that the liquidity is particularly high on the 5 year CDS market). Nevertheless, we have 1 year and 3 quotes as well as the 5 year, and the spreads are much different for these maturities,‡ usually implying a much lower hazard rate for the first periods. We suppose that the information given by the 1 and 3 year maturities is relevant, and therefore we choose to construct a bootstrapping methodology to infer the hazard rate function.

The base case recovery rate is taken as a fixed number of 40%, which is a standard approach in the industry (Chen et al. 2005). Secondly, we introduce some dynamics by setting that RR follows a beta distribution parameterized by Renaud and Scaillet (2004). The beta distribution is a very convenient function for RR since it is bounded between 0 and 1, and only requires two parameters that allow a large set of distribution patterns. The probability density function for the standard beta distribution is given as

* The instantaneous default probability for a security that has attained age t conditional on the survival until time t.
† Here we are working with mid-spreads.
‡ On September 1, 2006, the average mid-quotes for the series 6 of the iTraxx were 6.0, 16.8, and 28.2 bp, respectively, for 1, 3, and 5 year CDS maturities.

$$f(x,\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad x \in [0,1]$$

where $B(\alpha,\beta)$ is equal to the beta function.

Finally, the correlation between the probabilities of default can be modeled with a one factor model (Laurent and Gregory 2003, Hull and White 2004). This method uses a vector of correlation to a state of the economy variable which is the same for every name. Another technique is based on copula functions dependent on complete matrices of correlation of default, i.e., a different correlation between every name (Will 2003). In practice, since the default correlation is not readily observable, the same correlation factor is often applied to the whole portfolio.

For the pricing of multiname credit derivatives, it is common to suppose independence between the dynamics of the interest rates and of the arrival rate of default. But some argue that an increasing interest rate can have a negative impact on the probability of survival of a firm, a distressed company will have to support even higher charges. This hypothesis has recently been introduced for pricing single-named credit derivatives by Brigo and Pallavicini (2007).

Another correlation issue arises between the probability of default and the level of the recovery rate. During recession periods, when there are numerous defaults, recovery rates tend to be low. This subject has been studied in credit risk theory by Frye (2000), Hu and Perraudin (2002), and Sibille and Hübner (2007) in the context of CDS. A review of these models has been made by Altman et al. (2005).

In this study, we adopt the copula approach. This assumption enables us to challenge the market practice assumption of an equal correlation factor to the whole portfolio, and to identify what changes in the pricing dynamics induced by a more precise account of the dependence structure between individual default probabilities.

25.3.2 CDS Prices

The price of a CDS is represented by its spread, which is the interest on the notional of the contract that the protection buyer has to pay to the protection seller so that he is fully reimbursed in the case of a default. As shown by Hull and White (2000) among others, the spread can be extracted from the equality of the present value of the expected fixed and floating legs.

$$\text{Fixed Leg} = sN \sum_{j=1}^{T} B(0,t_j)\Delta_{j-1,j}S(t)$$

$$\text{Floating Leg} = N \int_0^T B(0,t)(1-RR)F(\tau = t)\,\mathrm{d}t$$

$$s = \frac{\int_0^T B(0,t)(1-RR)F(\tau = t)\,\mathrm{d}t}{\sum_{j=1}^{T} B(0,t_j)\Delta_{j-1,j}S(t)}$$

where

N is the notional

s the spread

$F(t) = 1 - S(t)$ the probability of default

This equation allows us to extract the hazard rate between time 0 and a 1 year maturity,* because we have no information on the instantaneous probability at time 0 we suppose it to be constant until time 1. The probability is taken from an arbitrage-free setting and is therefore true under the equivalent martingale measure $\mathbb{Q}$. With the hazard rate between 0 and 1, we can extract the hazard rate until time 3. Here we suppose a linear relation between λ_1 and λ_3. When we know λ_3, we can determine λ_5. The survival probability will then be defined as follows:

$$
\begin{aligned}
S(t|0 < t < 1) &= \mathrm{e}^{-\lambda_1 t} \\
S(t|1 < t < 3) &= \exp - \left[\lambda_1 + \int_1^t \frac{\lambda_1(3-s) + \lambda_3(s-1)}{2} \mathrm{d}s \right] \\
S(t|3 < t < 5) &= \exp - \left[\lambda_1 + \int_1^3 \frac{\lambda_1(3-s) + \lambda_3(s-1)}{2} \mathrm{d}s + \int_3^t \frac{\lambda_3(5-s) + \lambda_5(s-3)}{2} \mathrm{d}s \right]
\end{aligned}
\tag{25.1}
$$

As an example, here is a graph which plots the hazard rate as a function of time following the Barclays CDS structure (Figure 25.1).

With the survival probabilities for each names, an RR assumption and a present value function, we can build a complete model for the pricing of the tranches.

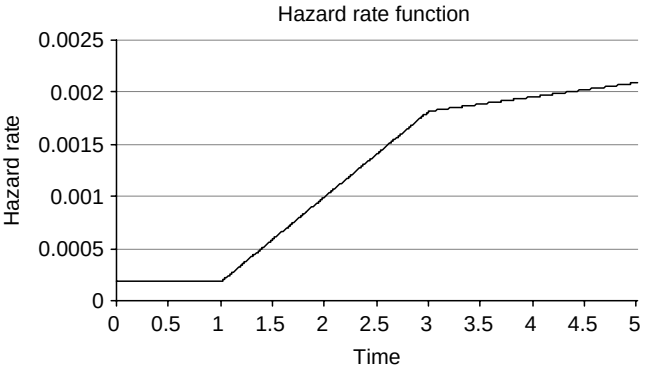


FIGURE 25.1 Graph of the hazard rate as a function of time following the Barclays CDS (credit default swap) structure.

* The recovery rate is supposed to be fixed at 40%.

25.4 PRICING METHODOLOGY

The Gaussian copula model has been introduced by Li (2000) and has become an industry standard. The assumption is to model the default correlation between the names of the referenced portfolio as in a multinormal distribution, with a rather small correlation in the tails (for the extreme values).

Definition 25.1. *Let $X_1, \ldots, X_n$ be normally distributed random variables with means $\mu_1, \ldots, \mu_n$, standard deviations $\sigma_1, \ldots, \sigma_n$, and correlation matrix Σ. Then the distribution function $C_\Sigma(u_1, \ldots, u_n)$ of the random variables*

$$U_i := \Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right), \quad i \in \{1, \ldots, n\}$$

(where $\Phi(\cdot)$ denotes the cumulative univariate standard normal distribution function) is called the Gaussian copula with correlation matrix Σ and can be written as

$$C_\Sigma(u_1, \ldots, u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} k_\Sigma \mathrm{e}^{-\frac{1}{2}(\vec{v}-\vec{\mu})^T \Sigma^{-1}(\vec{v}-\vec{\mu})} dv_1 \cdots dv_n$$

where

$$k_\Sigma = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}}$$

$$v_i = \Phi^{-1}(u_i), \quad i \in \{1, \ldots, n\}$$

This distribution is a very convenient one, but in practice it is usually impossible to parameterize the correlation structure such that the model is price consistent for all the tranches. This has led to the well-known correlation smile. A different correlation structure is implied for every tranche, and the structure obtained looks like a smile with high correlations on the two extreme tranches.*

To imagine a different correlation for every tranche means to suppose a completely different distribution. This proves the distribution to be wrong. Therefore, other copulas have been proposed in the literature as in Burtschell et al. (2005) where a comparison is made between the Gaussian, the Student-*t*, and the Clayton copulas. Here we present the Student-*t* copula that is tested on different degrees of freedom compared to the Gaussian copula.

Definition 25.2. *Let $X_1, \ldots, X_n$ be normally distributed random variables with mean 0, standard deviations 1, and correlation matrix Σ. Let Y be an χ^2-distributed random variable with ν degrees of freedom which is independent of $X_1, \ldots, X_n$. Then, the distribution function $C_{\nu,\Sigma}(u_1, \ldots, u_n)$ of the random variables*

* For further explanations see Hager and Schöbel (2006).

$$U_i := t_\upsilon\left(\frac{\sqrt{\upsilon}}{\sqrt{Y}}X_i\right), \quad i \in \{1,\ldots,n\}$$

*(where $t_\upsilon(\cdot)$ denotes the cumulative univariate Student-*t *distribution function with υ degrees of freedom) is called the Student-*t *copula with correlation matrix Σ and can be written as*

$$C_{\upsilon,\Sigma}(u_1,\ldots,u_n) = \int_{-\infty}^{t_\upsilon^{-1}(u_1)} \cdots \int_{-\infty}^{t_\upsilon^{-1}(u_n)} k_{\nu,\Sigma}\left[1+\frac{1}{\upsilon}(\vec{v}-\vec{\mu})^T\Sigma^{-1}(\vec{v}-\vec{\mu})\right]^{-\frac{\upsilon+n}{2}} d\upsilon_1\cdots d\upsilon_n$$

where

$$k_{\upsilon,\Sigma} = \frac{\Gamma\left(\frac{\upsilon+n}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)\sqrt{(\pi\upsilon)^n \det(\Sigma)}}$$

It is important at this point to keep in mind that the copula functions only introduce correlation in the probabilities of default. The univariate probability functions are the $S_i(t)$ presented before.

With multivariate distribution functions, we can simulate the times of default and realize the pricing. We present here the details of the valuation model for the Gaussian copula, but the methodology is the same for the Student-*t*.

First, correlated random numbers u_i are generated for each firm i from the copula.* Then, these numbers are transformed into probabilities using the normal univariate cumulative density function, $p_i = \Phi(u_i)$. The probabilities obtained are compared with Equation 25.1. If for name i the probability is above the corresponding $S_i(T)$, the firm has survived until maturity T. If it is not the case, the exact time of default τ_i is the time such that $p_i = S_i(\tau_i)$. In our specific example on the iTraxx, this gives

$$\tau_i = \begin{cases} -\dfrac{\ln p_i}{\lambda_1} & \text{if } p_i \le S_i(1) \\ \dfrac{3\lambda_1 - \lambda_3 - \sqrt{2\lambda_1^2 + \lambda_1 k - \lambda_3 k}}{\lambda_1 - \lambda_3} & \text{if } S_i(1) < p_i \le S_i(3) \\ \dfrac{5\lambda_3 - 3\lambda_5 - 2\sqrt{\lambda_3^2 + \lambda_3 k - \lambda_5 k}}{\lambda_3 - \lambda_5} & \text{if } S_i(3) < p_i \le S_i(5) \\ \text{No default} & \text{if } p_i > S_i(5) \end{cases}$$

where

$$k = \ln(1 - p_i)$$

* One way to realize this is to generate independent normal numbers and multiply them with the Cholesky-decomposed correlation matrix.

The advantage of such an analytical formula is a much faster computation which permits a high number of simulations in the Monte Carlo framework. We can also generate an antithetical case with a second set of random numbers u_i which is equal to $-u_i$. This is a consistent approach since the Gaussian copula is symmetric and has a mean equal to 0. It is also good for convergence purposes since the global mean of all generated scenarios will be exactly equal to 0.

The second part of the model performs the pricing given the times of default from each firm for every scenario. First, defaults τ_i are sorted in an ascendant order (from the first default to the last). Because each name weights the same proportion in the index and because the RR assumption is the same for all names, it does not matter to identify exactly which name has defaulted. Given the defaulted times, the following kernel is computed for every default.

$$S_0, L_0 = 0 \tag{25.2}$$

$$N_{1,0}, N_{2,0} = 1 \tag{25.3}$$

$$\text{if } \tau_i \leq T$$

$$S_{\tau_i} = S_{\tau_{i-1}} + \left[\sum_{k=\tau_{i-1}}^{\tau_i} B(0,t_k)(t_k - t_{k-1})s\right] N_{2,\tau_{i-1}} \tag{25.4}$$

$$L_{\tau_i} = L_{\tau_{i-1}} + \frac{1_{N_{1,\tau_{i-1}}<0}(1-R)B(0,\tau_i)}{n(\theta_{\text{end}} - \theta_{\text{beg}})} \tag{25.5}$$

$$N_{1,\tau_i} = N_{1,\tau_{i-1}} - \frac{1-R}{n\theta_{\text{beg}}} \tag{25.6}$$

$$L_{\tau_i} = L_{\tau_i} + 1_{N_{1,\tau_i}<0, N_{2,\tau_{i-1}}} = 1\left[-B(0,\tau_i)N_{1,\tau_i}\frac{\theta_{\text{beg}}}{\theta_{\text{end}} - \theta_{\text{beg}}}\right] \tag{25.7}$$

$$N_{2,\tau_i} = \max\left(0, 1 + 1_{N_{1,\tau_i}<0} N_{1,\tau_i}\frac{\theta_{\text{beg}}}{\theta_{\text{end}} - \theta_{\text{beg}}}\right) \tag{25.8}$$

where S_t and L_t represent, respectively, the sum of the spreads s being received and the losses being incurred,* until time t. t_k are exact payment dates† of spreads. $N_{1,t}$ and $N_{2,t}$ are the proportions that have survived from 0% of notional default to the beginning of the tranche θ_{beg} and from θ_{beg} to the end of the tranche θ_{end}. n is the number of underlying debts (125 for the iTraxx), R is the recovery rate, and $B(0,t)$ the present value at time 0 of one monetary unit received for certain at time t.

When the kernel is finished, which means all defaults have been treated, we obtain the total spread received and the loss incurred for every simulation by solving

* We take this example in the point of view of the buyer of the tranche.

† The spreads on the iTraxx are paid quarterly.

$$S_T = S_{\tau_T} + \left[\sum_{k=\tau_T}^{T} B(0,t_k)(t_k - t_{k-1})s\right] N_{2,\tau_T} \tag{25.9}$$

$$L_T = \min(L_{\tau_T}, 1) \tag{25.10}$$

The spread s that permits to equal the floating leg with the fixed leg is then given by

$$s = \frac{L_T}{S_T}$$

The final pricing of the tranche will be the average of all the spreads from the simulated scenarios.

The pricing kernel can be explained as follows. First, in Equation 25.2 the global losses and received spreads are fixed to 0 at time 0. The proportion of survival before the beginning of the tranche and the proportion of survival inside the tranche are fixed to 100% in Equation 25.3 at time 0. Equation 25.4 increases the fixed leg by the present value of the spread. These payments are proportional to the amount of the tranche that has survived until the new default. This means that the accrued interests until the date of default are taken into account. Equation 25.5 increases the amount of the floating leg by the present value of the LGD, this increase takes place only if the tranche has already been hit. The loss is also proportional to the size n of the portfolio and the size $\theta_{\text{end}} - \theta_{\text{beg}}$ of the tranche. Equation 25.6 then computes the loss on the notional before the beginning of the tranche, θ_{beg}. If this number becomes negative, the tranche has been hit. Equation 25.7 increases the amount of the losses in a particular case: the first loss in the tranche (the condition is written $N_{1,\tau_i} < 0$ and $N_{2,\tau_{i-1}} = 1$). This loss is divided between the tranche and the proportion of defaults before the tranche. The amount of the default that has reached the tranche is equal to the value of $-N_{1,\tau_i}$, but has to be proportional to the size of the tranche by multiplying by $\frac{\theta_{\text{beg}}}{\theta_{\text{end}}-\theta_{\text{beg}}}$ and to the present value at the time of default $B(0,\tau_i)$. Equation 25.8 defines the proportion of survivors inside the tranche. This proportion is also constrained to stay positive. To complete the model, a last payment of the spreads is introduced in Equation 25.9 and a constraint is put on L_T so that the maximum of the losses cannot be higher than 100%.

25.5 RESULTS

This section applies the methodology described above for the pricing of the series 6 of the iTraxx. The first pricing is done on September 21, 2006 with a Gaussian copula and a fixed RR assumption. Subsequently, the date will be changed to see if the same implied correlation structure can lead to qualitatively similar results. Next, the influence of a random RR is tested. Finally, an implied correlation for different degree of freedom of the Student-t copula is tested to see how the correlation smile can be altered.

The second part of this section tests the sensitivity of the model to the following inputs: interest rates, correlation, and CDS spread.

TABLE 25.1 Implied Correlation

Tranche	Price	Implied Correlation (%)
S1	19.94%	16.15
S2	74.00	4.53
S3	22.00	11.16
S4	10.25	15.36
S5	4.00	21.04
S6	1.38	50.32

Note: The table shows the implied correlation structure for every tranche of the Series 6 of the iTraxx on September 21, 2006.

25.5.1 Pricing

It is very difficult to find good proxies for the correlation of default. Correlation on the stock prices is not related to the correlation of default and historical data are low and mostly irrelevant from nature (i.e., it is difficult to explain the correlation of default from current firms with historical defaults of other firms). For this reason, the CDO market works with implied correlation as a common practice. A different correlation for each tranche is implied from the current prices. This is the approach that we take here.

For different correlation levels, we perform 400,000 simulations.* When two prices are found closer than 0.5% to the actual price (one below and one above), an interpolation is made to find the exact implied correlation of the tranche. The results are presented in Table 25.1.

This table clearly shows a smile structure in the correlation. The equity tranche has a bigger correlation than even the S4, and the correlation of S6 is huge. This result implies that the Gaussian copula is not an exact distribution for the arrival of defaults, with a too low correlation in the tails.

25.5.2 Date

To see if the same correlation structure can explain the price on different dates, we run the simulations and compare the prices we obtain with the prices that have been observed.

As we can see in Table 25.2, a constant correlation structure cannot explain very well the price on a future date. Some estimated prices (as for S5) are close to the market price, but on average the error is 14.6%. Positive and negative errors occur, so it is not a global shift upward or downward from the correlation structure.

25.5.3 Student-*t* Copula

The test realized in this section is about the Student-*t* copula. Trying two different levels of degrees of freedom—given that an increase in the number of degrees of freedom would result in a convergence to the Gaussian one—we determine the implied correlation structure and compare the results with the standard Gaussian model (Table 25.3).

* This is the standard number of simulations that are made given the high level of convergence that the model requires.

TABLE 25.2 Price for Different Trading Dates

	20/10/2006		21/11/2006		21/12/2006	
Tranche	P	P^{est}	P	P^{est}	P	P^{est}
S1 (%)	16.13	17.6	11.81	13.62	11.5	14.92
S2	65.00	58.6	53.00	38.07	50.00	42.26
S3	16.00	18.46	14.75	11.66	15.00	13.07
S4	7.5	8.4	5.75	5.56	5.75	6.10
S5	3.25	3.35	2.5	2.22	2.5	2.52
S6	0.938	1.234	0.7	0.941	1.1	1.06

Note: This table illustrates the error in the price we would estimate if we keep the same correlation structure on different trading dates.

The cells marked "NA" represent cases where no dependence structure could fit the observed prices within the considered tranche. Looking at the results, it appears the Student-t copula does not do better than the Gaussian copula to introduce correlation in the probability of default for the pricing of CDO tranches. One of the issue being that no correlation can be implied for certain tranches.

25.5.4 Random Recovery Rate

The RR in the case of a default cannot be predicted. For this reason, it is better to suppose a random RR. However, the probability of default implied by the CDS market is dependent on the hypothesis we take on the RR (if we suppose it is high, for example, the probability of default will be high). This is one of the reason why the RR is often taken as a constant, another being the facility in the computation.

In this section, we realize a pricing of the tranche taking a random RR and following a beta distribution as presented earlier (Table 25.4). The parameters will follow the article by Renaud and Scaillet (2004). An average of 44.96% (the whole hazard rate structure is adapted for this average), an alpha = 1.088 and a beta = 1.3318.

The randomness of RR does not seem to have an influence on the equity tranche. Numerous defaults appearing in this tranche seem to offset the effect. For the other tranches, the new distribution of RR has a material effect. The resulting impact is an

TABLE 25.3 Implied Correlation for Student-t Copula

Tranche	$\nu = 10$ **(%)**	$\nu = 20$ **(%)**	**Gaussian (%)**
S1	0.542	8.55	16.15
S2	NA	NA	4.53
S3	NA	3.04	11.16
S4	0.027	10.70	15.36
S5	4.28	17.83	21.04
S6	43.8	47.91	50.32

Note: The table gives the implied correlation structure for different loss distributions.

TABLE 25.4 Random Recovery Rate

Tranche	P	P_{RR}	$P_{RR} - P$	$\frac{P_{RR}-P}{P}$ (%)
S1 (%)	19.94	19.94	0.00	−0.01
S2	74.00	80.3	6.30	8.51
S3	22.00	21.66	−0.340	−1.55
S4	10.25	9.42	−0.830	−8.10
S5	4.00	3.07	−0.930	−23.25
S6	1.38	1.165	−0.215	−15.58

Note: This table shows the differences we obtain when using a random recovery rate.

increase in the price of S2, compensated by lower prices for the other tranches. Quite logically, the effects balance each other because the underlying risk structure is dependent on the prices of the CDS, not on the assumption about RR.

25.5.4.1 CDS Spreads

We change in this section the level of the spreads S_{CDS} to test the influence it can have on the price.

From Table 25.5, we can conclude that the prices of the tranches are highly sensitive to CDS market price. Because the CDS market provides information on the probability of default, it is not surprising to witness a marked, and even inflated effect. Furthermore, the effect seems to differ in magnitude to a large extent from one tranche to another (the span goes from 16.90% for the equity tranche to 36.77% for the first mezzanine tranche S2).

The convexity of the tranche price is negative for the extreme tranches, while it is positive in the middle range. This finding shows that there is no progressive increase in the gamma as one moves from the senior to the equity tranche. The option-like behavior of the equity tranche is partially offset by the fact that defaults do not eat up the tranche in a linear fashion, therefore putting more probability on low recoveries.

TABLE 25.5 CDS Spread Sensitivity

Tranche	$\frac{\Delta S_{CDS}}{S_{CDS}}$ (%)	P_{CDS}	$P_{CDS} - P$	$\frac{P_{CDS}-P}{P}$ (%)	$C_v = \frac{P^+_{CDS} + P^-_{CDS} - 2P}{P(\Delta S_{CDS})^2}$
S1	−10.00	16.42%	−3.52%	−17.64	−0.741
	10.00	23.31%	3.37%	16.90	
S2	−10.00	52.42	−21.58	−29.16	7.62
	10.00	101.21	27.21	36.77	
S3	−10.00	15.61	−6.39	−29.03	5.19
	10.00	29.53	7.53	34.22	
S4	−10.00	7.57	−2.68	−26.16	2.08
	10.00	13.15	2.90	28.24	
S5	−10.00	3.07	−0.93	−23.32	1.58
	10.00	5.00	1.00	24.90	
S6	−10.00	1.08	−0.30	−21.89	−4.86
	10.00	1.61	0.23	17.03	

Note: This table shows the changes in the prices from a change in the spreads of the CDS. The last column represents the effective convexity of the tranche price taken at its initial level.

TABLE 25.6 Correlation Sensitivity

Tranche	ρ (%)	P	$P_\rho - P$	$\varepsilon_{P,\rho} = \frac{\Delta P/P}{\Delta \rho/\rho}$
S1	15.15	20.53%	0.59%	0.48
	17.15	19.42%	−0.51%	−0.42
S2	3.53	61.42	−12.58	−0.77
	5.53	86.36	12.36	0.76
S3	10.16	18.54	−3.46	−1.76
	12.16	26.02	4.02	2.04
S4	14.36	7.82	−2.43	−3.64
	16.36	11.94	1.69	2.53
S5	20.04	3.53	−0.47	−2.45
	22.04	4.48	0.48	2.51
S6	49.32	1.28	−0.10	−3.47
	51.32	1.46	0.08	2.81

Note: The table gives the price of the tranches for different levels of correlation.

25.5.4.2 Correlation

The last series of sensitivity test focuses on the pricing of the tranche for different levels of correlation (Table 25.6).

The correlation has a major impact on the price of CDOs. Also, the effects are opposite between the equity tranche and the rest of the tranches. A lower correlation desynchronizes the occurrences of defaults, which tends to favor a diversification effect present in the equity tranche. For all other tranches, a lower correlation seems to make a cut into the upper tranches more likely as it is a rarer event.

25.6 CONCLUSION

This research presents an application of standard models for the pricing of the iTraxx. On the theoretical side, it has shown how a simple recursive kernel could be used to provide a numerically effective procedure to price the CDO tranches. Based on market data, we have analyzed various sensitivity aspects of the CDO index, and performed a series of checks of the consequences of the adoption of a Gaussian copula model.

With such a practical methodology, it is possible to forecast changes in the price relative to changes in the correlation and in spread of the underlying CDS. Furthermore, tests on the Student-*t* distribution have shown its poor job in describing the correlation structure in a particular case. Finally, we emphasize that the introduction of a random distribution for the RR should be seriously considered by market participants given the important influence it can have on the price.

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CHAPTER 26

Numerical Pricing of Collateral Debt Obligations: A Monte Carlo Approach

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26.1 INTRODUCTION

The pricing of derivatives based on portfolios composed by assets with default risk has received an increasing attention during the past years. Up to a point, this may be due to the recent appearance of the highly successful credit default swap (CDS) market, where the irruption of indexes like the DJ iTraxx or DJ CDX, and their tradable tranches series have contributed to their fast growing over the past years. For example, the International Swaps and Derivatives Association (ISDA) reports an increase in the notional principal outstanding volume* of CDS from $26.0 trillion at June 30, 2006 to $34.5 trillion at December 31, 2006, which represents a 33% increase during the second half of 2006.† In view of this fast development of standardized credit markets, the recent financial literature has focused its interests on the pricing of these new credit-linked products, with attention to the modeling of some interesting features presented in these assets.

Probably, the main characteristic in basked credit derivatives products is their exposure to default correlation, intended as the different relationships of dependence between defaults that could be given among firms: the higher the correlation between the assets in a credit portfolio, the higher is the probability of suffering a big loss in the portfolio. Then, the role of the correlation is crucial to price assets as, for instance, first-to-default swaps (FtD) or collateral debt obligations (CDO).

The recent literature on credit risk refers, at least, four main approaches to achieve correlation among different obligors. As pointed out by textbooks of Schönbucher (2003), Lando (2004), and Duffie and Singleton (2004), the academic literature provides different frameworks to introduce correlation. The approaches more cited in the literature are the following:

- Structural models
- Intensity models
- Copulas
- Mixture models

* ISDA news release, April 18, 2007. Available at http://www.isda.org. This report refers to credit default swaps on single names, baskets, and portfolios of credits, baskets, and index trades.

† CDS notional volume growth rates were 52% during the first half of 2006. Those reported for the whole years 2006 and 2005 were 102% and 103%, respectively. Source: www.isda.org.

Roughly speaking, the main difference between these different pricing models for CDOs consists on the form that they use to achieve correlation among different obligors. We focus here on the two main approaches to introduce default dependence between assets, in line with the two main paradigms for analyzing credit risk:*

1. On one hand, the structural-based models take into account the capital structure of a firm by relating the value of the issuing company to its liabilities (Schmidt and Stute, 2004). This family of models includes the models of Merton (1974), Black and Cox (1976), and Leland and Toft (1996), among others.
2. On the other hand, the reduced-form approach is based on the relative valuation of credit derivatives by means of market data, with no care about any firm features. Jarrow and Turnbull (1995) or Duffie and Singleton (1999), among others, can be included within this second group.

To put it briefly, the main basic difference between structural and reduced-form models consists on how they consider the causes of default:

1. Structural models explain the event of default as the impossibility of the firm to respond to its liabilities, that is, in an endogenous form.
2. By contrast, the intensity-based model analyzes the dynamics of default from an exogenous point of view, extracting directly the probabilities of default from the observation of market prices (Giesecke, 2004).†

Owing to its direct connection with the literature of bond pricing, the intensity approach has become a standard in the valuation of corporate bonds and their credit derivatives counterparts (CDS, for example). Although, the intensity-based literature has not been capable to extend its influence to the pricing of credit basket derivatives, and specially to tranched products like the collateral debt obligations (CDOs). Some remarkable contributions are those of Duffie and Garlenau (2001), which basically considers correlations between the default times of individual firms. Nevertheless, as pointed out by Hull and White (2004) or Mortensen (2006), this procedure is computationally burdensome.‡

In addition to the lack of tractability of intensity models, another plausible reason that may explain the wide-range usage of factors models by the practitioners may be given by the popularity of some models like the Credit + of Moody's or the KMV (see Schönbucher, 2003 or Schmidt and Stute, 2004 for a digression of these models). Directly inspired from Merton (1974), both models formalize the CDO pricing under an individual level firm

* See Schönbucher (2003) or Duffie and Singleton (2004) for an overview of additional alternatives for modeling default dependencies.

† Recently, Duffie and Lando (2001) have demonstrated that both approaches are connected by information asymmetries.

‡ Recently, Mortensen (2006) has implemented a tractable extension of the Duffie and Garlenau (2001) framework by considering the algorithm of Andersen et al. (2003) for factor models.

framework by modeling the default of individual firms that trigger some threshold previously determined, and combining their different possibilities. Furthermore, the development of techniques that improve the computation times of former models (usually highly time consuming) has contributed to extend the scope of these models.*

In contrast to this individual approach, where the total number of defaulted firms is conditioned by the level of correlation among the obligors, Longstaff and Rajan (2008) posited a joint default approach for the pricing of CDOs: basically, they assume that defaults could be clustered, in the sense that joint defaults could happen simultaneously, an approach closely linked to the contagious scheme of Davis and Lo (2001). Contrary to Duffie and Garlenau (2001) approach, where defaults are considered at an individual level, the assumption of joint default simplifies the computation of CDOs prices dramatically, as is shown later.

The aim of this chapter is to study the pricing of CDO tranches using the intensity-based scheme of clustered defaults developed by Longstaff and Rajan (2008). Our intention is to explore the capability of the model to generate reasonable values for CDOs tranches under different sets of circumtances, particularly those referred to the number of factors used and their impact in the loss distribution. We also explore some natural extensions of the original model, extending its range mainly in two directions: firstly, by analyzing the consequences of adding jumps to the default process; then, by randomizing the impact of defaults in the loss distribution. To the best of our knowledge, no similar studies have yet been reported.

This chapter is organized as follows. Section 26.2 reviews the mechanics of the CDS and their related indexed products like iTraxx or CDX. Section 26.3 describes the model posited in Longstaff and Rajan (2008) and introduces some modifications in that paper. Section 26.4 performs a Monte Carlo study to illustrate numerically some extensions we propose in the original model by Longstaff and Rajan (2008). Finally, the main conclusions are summarized in Section 26.5.

26.2 CREDIT DEFAULT SWAP INDEXES

This section is devoted to the CDS indexes. We outline here their main features, beginning with their most basic component, the credit default swap; then, we review their pricing mechanics, and provide some standard pricing formulas. This part is mainly based on Felsenheimer et al. (2004), Jakola (2006), and references therein.

26.2.1 Credit Default Swap

Before showing the mechanism of a CDX index, it is necessary to describe briefly the main component of this index, the CDS. A CDS is a financial product used to hedge mainly fixed income assets (for instance, bonds) against certain types of credit events.† Then, a CDS is based on an agreement between two parties where one of them—the protection buyer—pays a defined, periodical amount to the other—the protection seller—contingent to the

* We can mention the algorithm of Andersen et al. (2003) as an example of this type of methods.

† For instance, Moody's refers bankruptcy, delayed disbursement of principal, or some legal distresses, among others.

occurrence of a given credit event (usually a default). If there is no default, the protection buyer must pay a defined premium until maturity. By contrast, if default happens, the protection seller must pay to the protection buyer an amount equal to the difference between the face value of the asset (for instance, a bond) and its market value after default. As pointed out by Duffie and Singleton (2004), a CDS can be seen as a default insurance on loans or bonds.*

Up to a point, the success of the CDS market could be explained as these assets provide to investors with a simple, liquid tool to remove the credit risk exposure of their portfolios in an easy way. Blanco et al. (2006) underline that the cost of shorting credit risk in the corporate bond market is high, mainly due to illiquidity costs. By using CDS, investors can short credit risk over a longer period at a known cost by buying protection.

26.2.2 Credit Default Swap Indexes

26.2.2.1 Product

A direct consequence of the fast growing of the credit market has been the appearance of standardized products that diminish the hedging cost of portfolios composed by a large number of firms, by means of transferring their credit risk without a necessity of buying CDS contracts on each individual firm. An example of these kind of products is the CDS index.

A CDS index is a basket of equally weighted CDS single names that serves as underlying for some tranched products, swaps, etc. Felsenheimer et al. (2004) refer that a CDS index cannot be characterized as a classical index, in the sense that no index level is computed. They also noticed that quoted prices for these products are a result of supply and demand within credit markets.

Jakola (2006) refers two main tradable index families: the Dow Jones CDX and the International Index Company iTraxx. Basically, both indexes are portfolios that present similar features as, for instance, the following:†

- Both indexes are composed by the 125 most liquid firms in the CDS market.
- These firms are investment grade rated.
- Index composition is revised every 6 months.

With respect to differences between these indexes, we can mention that the Dow Jones CDX only includes U.S.-names while the iTraxx takes into account only European names.

26.2.2.2 Mechanism

To explain the payment mechanism in a CDS index, we illustrate briefly here its two main situations, depending whether there is default or not. We address the reader to Felsenheimer et al. (2004) for a more detailed exposition.

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* As noted by Blanco, Brennan, and Marsh (2006), the difference between a CDS and an insurance contract is of legal type: under a CDS, holding an insured asset to claim compensation is not necessary.

† See Jakola (2006) for references to the Dow Jones CDX and Felsenheimer et al. (2004) for the iTraxx index.

Investors on CDS index can find any of the following alternative situations:

- No credit event happens. Here, the protection buyer pays a constant premium to the protection seller. This premium is computed as a percentage (named the spread) on a definite notional amount and is paid on a quarterly basis until maturity.
- A credit event happens. Now, the magnitude of the event and the recovery value determine the cash payment between the parties involved. As an example, we will assume that 1.6% of the firms included in the total portfolio default (2 over 125) and a recovery rate of 50%. Here, the protection seller pays the amount of 1.6% $\times$ notional $\times$ 50% to the protection buyer. As could be intended, the protection seller is exposed to recovery risk. With respect to the protection buyer, he continues paying the constant premium on the remaining notional amount, which is now 98.4% of the original. This amount should be paid until maturity or until other credit event.

The existence of a tranched products market besides each CDS index has provided a liquid, simple way of removing the credit exposure associated to fixed, standardized fractions—tranches—of the underlying portfolio losses. In other words, without necessity of creating the full capital structure, an investor can hold individual index tranches according to its needs. Longstaff and Rajan (2008) argue that this huge flexibility has contributed to the fast increase in the trading volume of the index tranches.

26.2.3 Pricing Formulas

To complete this overview of tranched products, this section outlines the basic formulas for pricing a CDO. For a more detailed exposition, we refer the reader to Gibson (2004) or Elizalde (2005) and references therein.

Roughly speaking, pricing a CDO means to compute the fair value of the percentage on a definite notional that must be paid by the CDO issuer, protection buyer, to her counterpart, the CDO investor or protection seller. As previously mentioned, this percentage is called the spread. With small differences with respect to a swap, a CDO is a contract between two parties where the CDO issuer pays periodically a fixed coupon, based on the spread, to the counterparty. This payoff scheme is close to that of a CDS in the sense that the payments are contingent to the occurrence of a credit event on the underlying. However, its difference comes from the following facts:

1. We operate over a pool of firms in the case of a CDO, instead of a single name as in a CDS.
2. We can define the loss amount of the underlying portfolio in percentage terms (tranches) that the protection seller agrees to offer protection, in contrast to a CDS, where the protection seller must face to the total amount of the (single) loss.

Usually, a CDO is issued over a portfolio of bonds, loans, CDS, or other defaultable instruments (Elizalde, 2005). The exposure to the losses in the portfolio is sold in tranches, that is, percentages on the total portfolio losses. Their bounds are usually standardized and

are called attachment points. For instance, usual attachment points are 0%–3% for the equity tranche—riskier—and 15%–30% for the senior tranche—smaller risk.* Intermediate tranches are called mezzanine and their attachment points vary from 3% to 7%, 7% to 10%, and 10% to 15%.

An additional feature of CDOs is the hierarchy in the payoffs: cash flows generated by the pool of firms are assigned by seniority to the different tranches of the CDO; first to the owners of senior tranches, then mezzanine, and to equity tranches subsequently. It is worth saying that equity investors receive the last payments of the credit portfolio. As a result, these investors are highly exposed to the first losses that may occur in the underlying portfolio.

As pointed out by Elizalde (2005), the pricing of a CDO is similar to that of a swap contract in the sense that the spread is fixed as there is no up-front payment between the parties. In line with this argument, we must compute the cash flows received for each part and equate, in present value terms, both amounts.

Let s be the spread of the CDO and let U and L be the upper and lower attachment points that define the tranche, respectively. Also, let R be the recovery rate and X_t be a random variable that defines the total distribution of losses in the portfolio at time t. For the ease of the exposition, we will define later the function $f(\cdot)$ that defines the losses of the tranche holders over the total losses in the portfolio.†

On one hand, the cash flows received by the protection seller will be the (unknown) spread s over their associate tranche $(U - L)$ computed on a principal amount. If any firm of the portfolio defaults, the protection seller will receive the same spread but over the new (reduced) notional amount, $U - L - E[f(X_t)]$.‡ The accrual factor for the payment t is given by Δ_t. Computing the corresponding present value for each date of payment, this results

$$P_{\text{seller}} = s \sum_{t=1}^{T} \mathrm{e}^{-rt} \Delta_t \{ U - L - E[f(X_t)] \} \tag{26.1}$$

On the other hand, the protection buyer expects to receive the discounted amount of the total losses produced on the tranche during the different periods, which are

$$P_{\text{buyer}} = \sum_{t=1}^{T} \mathrm{e}^{-rj} \{ E[f(X_t)] - E[f(X_{t-1})] \} \tag{26.2}$$

By equating expressions given by Equations 26.1 and 26.2, we get the spread

$$s = \frac{\sum_{t=1}^{T} \mathrm{e}^{-rj} \{ E[f(X_t)] - E[f(X_{t-1})] \}}{\sum_{t=1}^{T} \mathrm{e}^{-rt} \Delta_t \{ U - L - E[f(X_t)] \}} \tag{26.3}$$

* Bespoke tranches are also available in over-the-counter markets.

† To be consistent with our notation, recovery rates, attachment points, and the losses in the portfolio are given in percentage terms.

‡ Of course, until the total losses of the portfolio overcome the tranche.

Finally, the exposure to the total portfolio losses faced by the tranche investors is given by

$$\begin{aligned} f(X_t) &= \min[(1-R)X_t, U] - \min[(1-R)X_t, L] \\ &= \max[\min[(1-R)X_t, U] - L, 0] \end{aligned} \tag{26.4}$$

where, of course, the recovery rate affects directly the total portfolio losses.

26.3 MODEL

This section is devoted to the model. First of all, we justify the idea of correlation among firms, and cite the main approaches presented in the literature to model it. Second, we analyze some interesting aspects of the Longstaff and Rajan (2008) model, also incorporating some extensions.

26.3.1 Previous Models

The distribution of losses in basket products like CDS indexes or tranches plays a crucial role because the amount of cash flows delivered between the protection seller and the protection buyer depends dramatically on the number of default events occurred during the contract: the higher the number of firm defaults in the portfolio the higher are the losses. In some cases where the prioritization schemes are embedded in some products like the standard tranches, the losses on a definite percentage of the portfolio can be viewed as a call spread on the total losses of the portfolio (Duffie and Singleton, 2004; Longstaff and Rajan, 2008).

One of the stylized facts generally assumed in the credit risk literature is that firms usually default together in the sense that a default in an individual firm seems to increase the risk of default in some others. A simple empirical exercise can illustrate this idea.

With the intention of checking this effect in real data, Figure 26.1 shows (with an histogram) the distribution of default rates for U.S. firms from 1970 to 2004.*

Figure 26.1 also displays the default rate distribution obtained in a simulated portfolio (dashed line) composed by one hundred independent obligors. For the sake of simplicity, all firms within the portfolio have the same characteristics—even the same probability of default. This is usually called the homogeneous portfolio assumption. It is clear that, in the absence of correlation among firms, the total distribution losses will be described by a binomial distribution with parameters $N = 100$ and p equal to the individual default probability of each obligors. To put it another way, it reduces to an experiment repeated (independently) N times with probability of success equal to p. Finally, this figure also exhibits the simulated rates of default for a portfolio taking into account correlation among firms (solid line).

* The data has been taking from Hamilton et al. (2005).

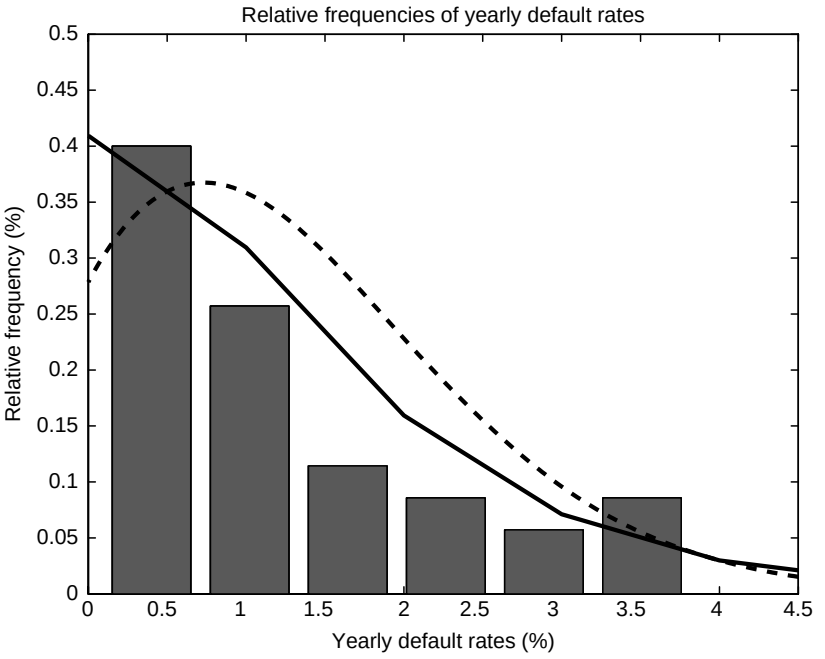


FIGURE 26.1 Relative frequencies of yearly default rates. The empirical yearly default rates are represented in bars. The dashed line corresponds to simulated default rates for a portfolio of 100 independent obligors. The solid line represents the default rates obtained with the Vasicek model for the same portfolio but considering a correlation parameter of 10.00%. Individual default probabilities are fixed at 1.20%. Simulation parameters have been previously estimated from real data. (From Vasicek, O. 1991. Limiting loan loss probability distribution, Working Paper, KMV Corporation.)

Simulations of default rates have been carried on using the Vasicek (1991) model, a standard model used by the industry for pricing CDOs. Simulation parameters for the independent ($\rho = 0$) and dependent cases correspond to the parameters previously obtained from the yearly default rates sample of Hamilton et al. (2005). Estimated parameters of correlation and individual default probability for the Vasicek (1991) model are $\rho = 10.73\%$ and $p = 1.27\%$, respectively. Finally, parameters have been estimated using the large homogeneous portfolio (LHP) hypothesis.*

As Figure 26.1 reveals, a model that incorporates correlation among firms seems to offer a better fit to the sample data than a model without correlation. Additionally, Schönbucher (2003) also presents evidence of default clustering through time. The intuition behind this fact is that default correlation among entities results in a higher risk of joint default. That is to say, the correlation between firms affects crucially the total number of defaults. As a

* See Schönbucher (2003) for a detailed discussion of the former points.

result, models used for pricing portfolios of firms should take into account this empirical feature of data.

Our work is focused on the recent proposal of Longstaff and Rajan (2008), which is inserted within the intensity models family. The next section analyzes some of its main features.

26.3.2 Simultaneous Default Model

Longstaff and Rajan (2008) assume the occurrence of simultaneous defaults within the obligors of the credit portfolio. The main idea behind this approach consists on modeling the loss distribution directly, by using a monotonically decreasing process. The intuition underlying this approach is that the arrival of a credit event (usually triggered by an underlying intensity process) produces a fractional loss in the portfolio. Longstaff and Rajan (2008) posit that the impact of credit events on portfolio losses are constant through time, with a range that implies defaults from one to various tens of firms.

In view of the former, three main components could be noticed in the Longstaff and Rajan (2008) model: firstly, the loss process; secondly, the process that activates the losses; and finally, the magnitude of the losses themselves. What follows is a more detailed exposition of this point, where we also include some extensions to the original model.

26.3.2.1 Loss Process

Using the notation given in Longstaff and Rajan (2008), let L_t denote the total portfolio losses per \$1 notional amount, with $L_0 = 0$. The dynamics of the loss distribution is given by the expression

$$\frac{dL_t}{1 - L_t} = \sum_{i=1}^{n} \bar{\gamma}_i \, dN_{it} \tag{26.5}$$

where

N_{it} is a Poisson process with time-varying intensity λ_t
γ_i is the magnitude of the jump size
n represents the number of factors that affect the losses

As shown in Longstaff and Rajan (2008), the solution to Equation (26.5) is given as

$$L_t = 1 - \exp\left(- \sum_{i=1}^{n} \gamma_i \, dN_{it} \right) \tag{26.6}$$

The mechanism of the model is simple: an underlying intensity process, which captures the instantaneous default probabilities given by the market, causes a jump in the Poisson process, which takes a value 1. This activates a (percentage) loss in the portfolio of magnitude $e^{-\gamma}$.

Figure 26.2 illustrates the evolution through the time of the losses based on an extended version of Longstaff and Rajan (2008) model for different parameters.* We have simulated

* Particularly, we have considered jumps in the intensity process. Moreover, we have taken the impact of losses as a random variable.

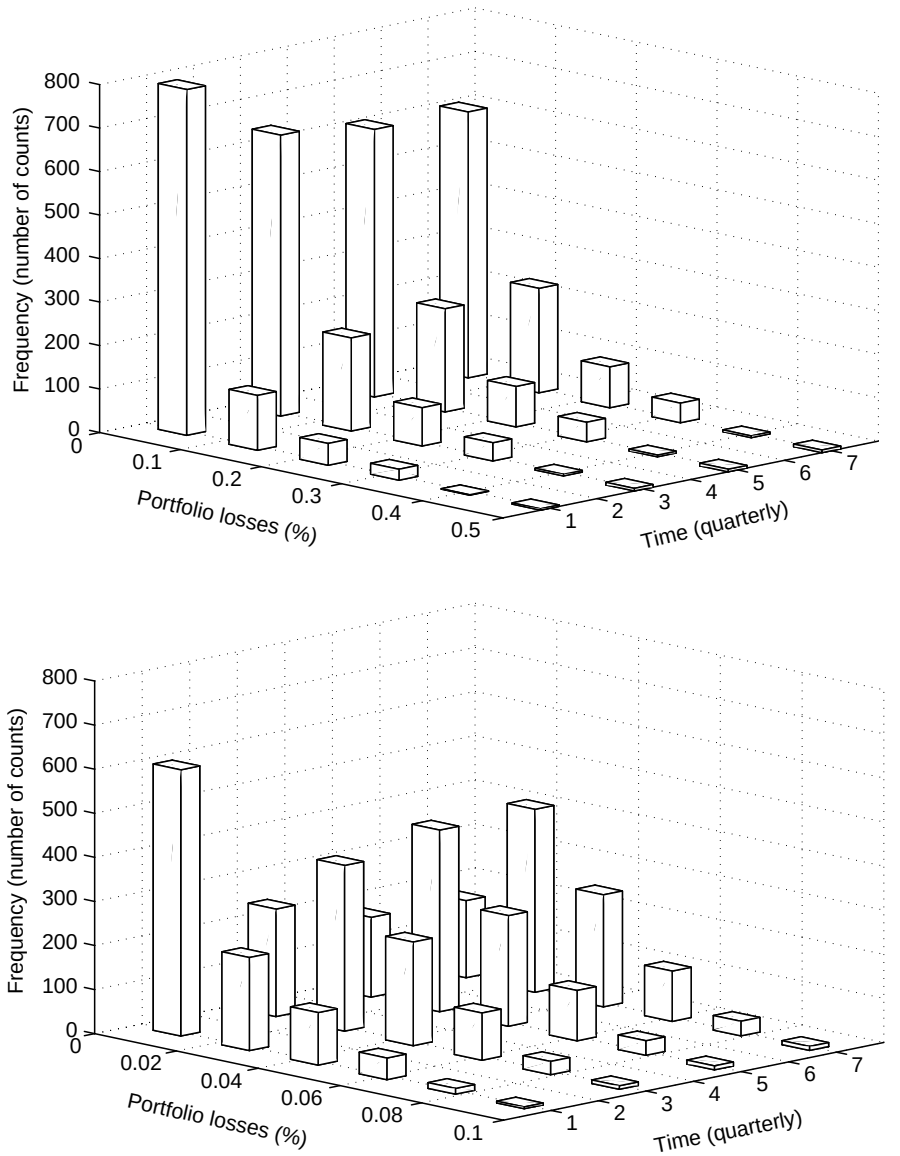


FIGURE 26.2 Evolution of portfolio losses in the Longstaff and Rajan (2008) framework for two different parameter sets. Notice that portfolio losses axes have different scales. (From Longstaff, F.A. and Rajan, A. 2008. To appear in *Journal of Finance*.)

1000 different scenarios of portfolio losses and have collected their evolution in different histograms. As Figure 26.2 exhibits, the number of bigger losses in the portfolio increases when times goes by, with a high number of zero counts in the early stages that decreases with time.

On the basis of principal component analysis over the CDX index and its associated tranches, Longstaff and Rajan (2008) consider that a three-factor model is enough to capture the whole variability of prices in the CDX index tranches. This point is also tested

by Longstaff and Rajan (2008) analyzing simplest versions of the model. Their results suggest that the three-factor model version presents a close fit to the data.

26.3.2.2 Intensity Process

To capture the instantaneous default probability observed in the market, we assume a stochastic process for the intensity of the Poisson process, λ_t. By definition, the process must guarantee positive values. Then, it is common to assume a squared-root type process as that proposed by Cox et al. (1985) for interest rates. We also consider in our study the potential appearance of jumps in the intensity process, as previously used by Duffie and Garlenau (2001) in the context of individual firms:

$$d\lambda_{it} = (\alpha_i - \beta_i \lambda_{it})\, dt + \sigma_i \sqrt{\lambda_{it}}\, dZ_{it} + J_i\, d\pi \tag{26.7}$$

where

β is the mean-reversion speed of the process
α/β determines the long-run mean value at which the intensity converges
σ is the volatility of the intensity process

Finally, we assume that jumps are driven by a Poisson process $d\pi$ with constant intensity δ_i. The size of the jump is given by a random variable J exponentially distributed with mean μ_i.

The objective that we pursue by adding jumps is to increase the volatility of the intensity process. The intuition is that, in line with that posited by Merton (1976) for stock markets, the arrival of abnormal information—modeled by the Poisson process $d\pi$—could produce an increasing, nonmarginal change of magnitude J in the instantaneous default probability.

26.3.2.3 Impact of Losses

The impact of a credit event in the total portfolio losses is given by the parameter γ_i in Equation 26.5. The estimations given for a three-factor model in Longstaff and Rajan (2008) suggest an economic interpretation of the different parameters γ in terms of percentage of losses in the portfolio: the market seems to be pricing three main credit events:

1. Default of a single obligor
2. Industry-specific event that affects one sector of the economy (about 10 firms)
3. Wide crisis that affects the whole economy (about 75 firms)

As the authors pointed out, this interpretation is consistent with that mentioned by Duffie and Garlenau (2001) that considers default events in three categories: firm-specific events, industry events, and recessions of the entire economy.

Again, we adopt here a different approach by randomizing the impact of losses after a credit event. Thus, we consider that losses are distributed as nonnegative random variables. In more detail, we assume exponentially distributed variables with mean γ. Our purpose is to relax the assumption of fixed impact to allow for more flexibility to the total losses portfolio distribution.

26.4 MONTE CARLO STUDY

This section analyzes the different models posited in the previous section by means of simulations.* We start exploring the original model of Longstaff and Rajan (2008) and extend the study to consider two possible alternatives: by introducing jumps in the intensity process, as previously specified; and by randomizing the losses or the jumps.

Before starting this simulation study, some general considerations must be taken into account. The simplest form of the model proposed by Longstaff and Rajan (2008) (one factor and constant loss size) needs a four-parameter vector $\Theta = (\alpha, \beta, \sigma, \gamma)$ including three parameters for the intensity process and one for the loss impact. This means that any extension of the model implies adding more parameters; what is more, the different combinations of parameters that we must introduce to analyze the responses of the model increase the difficulties of interpretation.

On account of this, some restrictions on the study must be imposed, with the intention of extracting some reasonable conclusions about the parameters and their cross interrelationship. For this reason, we proceed in a sequential form by detecting (in the early stages of the model) the set of parameters that could be of interest in subsequent extensions of the model.

As a general rule, our Monte Carlo study tries to identify the changes on the spread of tranches in two main directions:

1. Changes on the parameters of the intensity process
2. Changes on the losses size

In line with this idea, all the following tables mainly address variations on one of these alternative ways.

26.4.1 Longstaff and Rajan Model

As a first step, we analyze the model presented in Longstaff and Rajan (2008) with a constant impact losses γ and intensity process (Equation 26.7) without jumps ($J = 0$). We start by using the parameters obtained therein due to the inexistence of similar studies, a point that is discussed later when necessary.

We begin by simulating the one-factor model. Table 26.1 displays the results we have obtained. The first column indicates the set of parameters for the intensity process under analysis. The second column displays the different impacts on the portfolio losses. Finally, the last column exhibits the spreads obtained for each of the different tranches.

Firstly, it can be seen that variations in the losses size, given by the parameter γ, seem to produce a high variability on the tranche spreads obtained. For instance, considering the lower impact on losses ($\gamma = 0.005$), we obtain the same values independent of the set of parameters used.† The influence of the volatility parameter σ seems to be irrelevant;

* Our study comprises 5000 paths simulations with 1250 steps in each one. Initial parameters for the intensity process of the expression given by Equation (26.7) have been fixed to those of their long-run means. All computations have been carried on a 2.8 Mhz Pentium IV computer with 500 Mb of RAM.

† Differences in basis points of some spreads are explained by the variance of the simulation procedure used.

TABLE 26.1 Simulations for the One-Factor Model of Longstaff and Rajan

Parameters			Size Losses	Tranches (Basis Points)				
α	β	σ	γ	0–3	3–7	7–10	10–15	15–30
0.50	0.60	0.05	0.005	94	0	0	0	0
			0.050	1839	355	0	0	0
			0.100	1852	1852	1079	0	0
			0.350	1828	1828	1828	1828	1634
0.50	0.60	0.15	0.005	94	0	0	0	0
			0.050	1843	355	0	0	0
			0.100	1862	1862	1084	0	0
			0.350	1842	1842	1842	1842	1634
1.50	0.60	0.15	0.005	102	0	0	0	0
			0.050	4577	421	0	0	0
			0.100	4550	4550	1704	0	0
			0.350	4549	4549	4549	4549	3553
8.33	10.00	0.35	0.005	94	0	0	0	0
			0.050	1860	357	0	0	0
			0.100	1815	1815	1067	0	0
			0.350	1828	1828	1828	1828	1627

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

likewise, those differences could be explained by the numerical implementation of the model.

The increase of the parameter α related with the actual mean arrival probability of credit events (α/β) produces large variations on tranche credit spreads, as is also displayed in Table 26.1: proportional increases in α, β parameters (from $\alpha = 1.50$, $\beta = 0.60$ to $\alpha = 8.33$, $\beta = 10.00$) keep the long-run mean of the intensity process constant. As a result, no changes are observed with respect to previous cases.

Table 26.2 displays the simulations for the two-factor version of Longstaff and Rajan (2008). The first main column refers to parameters of the intensity process (Equation 26.7) for $i = 1, 2$. Values under the Parameters column must be intended as follows: values under the α column correspond to $\alpha_1 = 0.50$, $\alpha_2 = 0.02$; values under the β column correspond to $\beta_1 = 0.60$, $\beta_2 = 0.60$, and so on. Second and third columns are those of Size Losses and Tranches similar to those in Table 26.1.

As expected from observing the expression in Equation 26.3, the magnitude of losses is determinant when it comes to price the tranches: to a certain degree, lower γ results on lower prices for tranches. This last begins to be conditioned to the combination of losses of the two processes. For instance, considering low losses ($\gamma_1 = 0.005$, $\gamma_2 = 0.100$ or $\gamma_1 = 0.005$, $\gamma_2 = 0.035$) seems not to generate as a richiest variety of spreads as higher ones (for example, $\gamma_1 = 0.050$, $\gamma_2 = 0.050$).

Again, an increase in the α parameter (that controls the long-run mean of the process given by Equation 26.7) results on higher tranches spreads.

Finally, Table 26.3 shows the simulations for the three-factor model. Basically, this table reveals that the addition of a third factor enhances the spreads obtained for the one- and

TABLE 26.2 Simulations for the Two-Factor Model of Longstaff and Rajan

Parameters						Size Losses		Tranches (Basis Points)				
α		β		σ		γ_1	γ_2	0–3	3–7	7–10	10–15	15–30
0.50	0.02	0.60	0.60	0.15	0.20	0.005	0.100	110	0	0	0	0
						0.005	0.350	109	0	0	0	0
						0.050	0.050	1890	436	61	0	0
						0.100	0.350	1905	1905	1162	75	15
1.50	0.02	0.60	0.60	0.15	0.20	0.005	0.100	117	0	0	0	0
						0.005	0.350	117	0	0	0	0
						0.050	0.050	4641	519	62	0	0
						0.100	0.350	4637	4637	1849	77	15
0.50	0.20	0.60	0.60	0.15	0.20	0.005	0.100	185	0	0	0	0
						0.005	0.350	185	0	0	0	0
						0.050	0.050	2507	1099	493	0	0
						0.100	0.350	2475	2475	1825	636	85

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

two-factor versions of the model. Particularly, in addition to bespoke intensity processes, the heterogeneity in the losses impact results in a diversity of spreads for all tranches.

To put it briefly, results of this section seem to address that changes in the σ (volatility) parameter seem not to produce changes in tranches values. By contrast, variations in (1) the long-run mean and (2) the impact of the portfolio sizes must be taken into account. This is the purpose of the subsequent sections.

26.4.2 Longstaff and Rajan Model with Jumps

Here, we extend the model of Longstaff and Rajan (2008) by adding jumps in the background processes that control the arrival of credit events. As previously cited, the underlying idea is to introduce discontinuities—jumps—in the paths of the intensity process that could be explained as nonmarginal changes in the instantaneous probability of default. The arrival of unexpected information could cause these kind of effects.

To keep the number of parameters under study manageable, the parameters that correspond to jumps in the intensity processes (the arrival intensity δ and the mean μ of

TABLE 26.3 Simulations for the Three-Factor Model of Longstaff and Rajan

	Size Losses			Tranches (Basis Points)				
Set	γ_1	γ_2	γ_3	0–3	3–7	7–10	10–15	15–30
1	0.005	0.050	0.350	143	50	7	7	7
2	0.150	0.050	0.350	1911	1897	1865	975	19
3	0.050	0.150	0.350	1891	788	51	51	16
4	0.350	0.150	0.050	1871	1870	1868	1860	1422

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

Note: Parameters are fixed at $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\alpha_3 = 0.001$, $\beta_1 = \beta_2 = \beta_3 = 0.60$, $\sigma_1 = 0.15$, $\sigma_2 = 0.20$, and $\sigma_3 = 0.15$.

TABLE 26.4 Simulations for the One-Factor Model with Jumps of Longstaff and Rajan

Diffusion Parameters			Jump Parameters		Size Losses	Tranches (Basis Points)				
α	β	σ	δ	μ	γ	0–3	3–7	7–10	10–15	15–30
0.50	0.60	0.15	0.02	0.10	0.005	98	0	0	0	0
					0.050	2342	378	0	0	0
					0.100	2336	2336	1243	0	0
					0.350	2336	2336	2336	2336	2035
0.50	0.60	0.15	0.02	0.20	0.005	99	0	0	0	0
					0.050	2653	388	0	0	0
					0.100	2673	2673	1337	0	0
					0.350	2633	2633	2633	2633	2260
0.50	0.60	0.15	0.10	0.10	0.005	100	0	0	0	0
					0.050	3430	405	0	0	0
					0.100	3421	3421	1509	0	0
					0.350	3397	3397	3397	3397	2805

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

the jump amplitude) will be considered equal for all the intensity processes. All the tables in this and subsequent sections that contain these parameters must be intended in this form.

Tables 26.4 through 26.6 display the simulations for the one-, two- and three-factor versions of the Longstaff and Rajan (2008) model with jumps. By and large, same qualitative conclusions than aforementioned in the previous section arise: the addition of jumps results in a high value of tranche spreads. Basically, jumps introduce more volatility in the default process, which results in higher probability of default. As a result, the more we increase the probability of a credit event, the more credit events we have. Another fact

TABLE 26.5 Simulations for the Two-Factor Model with Jumps of Longstaff and Rajan

Jump Parameters		Size Losses		Tranches (Basis Points)				
δ	μ	γ_1	γ_2	0–3	3–7	7–10	10–15	15–30
0.02	0.10	0.005	0.100	1068	937	907	0	0
		0.005	0.350	1075	946	946	946	929
		0.050	0.050	2737	1330	631	0	0
		0.100	0.350	2720	2720	2131	928	926
0.02	0.20	0.005	0.100	1553	1402	1340	0	0
		0.005	0.350	1556	1407	1407	1407	1373
		0.050	0.050	3184	1791	845	0	0
		0.100	0.350	3145	3145	2623	1370	1367
0.10	0.10	0.005	0.100	2575	2387	2227	0	0
		0.005	0.350	2574	2388	2388	2388	2296
		0.050	0.050	4218	2749	1166	0	0
		0.100	0.350	4159	4159	3705	2345	2335

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

Note: Parameters are fixed at $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\beta_1 = \beta_2 = 0.60$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.20$. For ease of exposition, jump parameters are equal for both processes.

TABLE 26.6 Simulations for the Three-Factor Model with Jumps of Longstaff and Rajan

	Size Losses			Tranches (Basis Points)				
Set	γ_1	γ_2	γ_3	**0–3**	**3–7**	**7–10**	**10–15**	**15–30**
1	0.005	0.050	0.350	1595	1155	881	881	878
2	0.150	0.050	0.350	3050	2871	2728	2275	975
3	0.050	0.150	0.350	3001	1948	1471	1449	954
4	0.350	0.150	0.050	3047	2900	2782	2714	2322

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.
Note: Parameters are fixed at $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\alpha_3 = 0.001$, $\beta_1 = \beta_2 = \beta_3 = 0.60$, $\sigma_1 = 0.15$, $\sigma_2 = 0.20$, and $\sigma_3 = 0.15$. For ease of exposition, jump parameters are equal for the processes, $\delta = 0.02$, and $\mu = 0.10$.

arises from these tables: considering the parameters under study, it seems that higher frequencies of relatively small jumps produce bigger impacts on the tranche spreads than lower frequencies with large leaps.

26.4.3 Longstaff and Rajan Model with Jumps and Random Losses

In view of the important role of the parameter γ in the total portfolio losses, we analyze now the consequences of randomizing the impact of losses in the Longstaff and Rajan (2008) model. We also include jumps in the intensity process with the intention of studying simultaneously the joint effect on the spreads.

Tables 26.7 through 26.9 show the simulations for the one-, two- and three-factor versions of the Longstaff and Rajan (2008) model with jumps in the background default process and random losses. The variables that model the size losses are exponentially distributed with mean γ.

TABLE 26.7 Simulations for the One-Factor Model with Jumps and Exponential Random Losses of Longstaff and Rajan

Diffusion Parameters			Jump Parameters		Size Losses	Tranches (Basis Points)				
α	β	σ	δ	μ	γ	**0–3**	**3–7**	**7–10**	**10–15**	**15–30**
0.50	0.60	0.15	0.02	0.10	0.005	98	0	0	0	0
					0.050	934	269	104	39	7
					0.100	1323	570	309	168	46
					0.350	1938	1328	981	732	392
0.50	0.60	0.15	0.02	0.20	0.005	96	0	0	0	0
					0.050	943	253	96	34	4
					0.100	1434	595	311	169	51
					0.350	2152	1467	1068	774	415
0.50	0.60	0.15	0.10	0.10	0.005	98	0	0	0	0
					0.050	1079	279	107	43	6
					0.100	1640	637	333	181	49
					0.350	2645	1671	1188	847	431

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

TABLE 26.8 Simulations for the Two-Factor Model with Jumps of Longstaff and Rajan

Jump Parameters		Size Losses (Mean)		Tranches (Basis Points)				
δ	μ	γ_1	γ_2	0–3	3–7	7–10	10–15	15–30
0.02	0.10	0.005	0.100	863	427	258	148	46
		0.005	0.350	1006	731	600	492	304
		0.050	0.050	1625	662	308	144	25
		0.100	0.350	2190	1485	1110	823	451
0.02	0.20	0.005	0.100	1157	529	302	169	51
		0.005	0.350	1412	997	782	618	360
		0.050	0.050	1968	774	347	157	27
		0.100	0.350	2677	1862	1388	1056	555
0.10	0.10	0.005	0.100	1647	631	331	182	53
		0.005	0.350	2201	1448	1087	803	419
		0.050	0.050	2622	926	386	165	28
		0.100	0.350	3716	2662	1978	1416	675

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

Note: Parameters are fixed at $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\beta_1 = \beta_2 = 0.60$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.20$. Jump parameters are equal for both processes. Parameter in Size Losses correspond to the mean of an exponential distributed random variable.

By and large, we can observe that considering a random impact of credit events results in a richest diversity of spreads for all tranches. When it comes to comparing against the previous results for the Longstaff and Rajan (2008) model with constant γ, this is more evident. A possible explanation can be related to the fact that taking γ as random provides a wide range of different losses without a necessity of considering combinations of fixed values for this parameter.

As revealed by the simulations in Table 26.1 for the mean value of size losses $\gamma = 0.005$, it is also important to notice that the magnitude of the impact in losses continues being important. Moreover, the effect of introducing γ as random produces a smoothness of tranches values with respect to previous cases, a point that will be discussed in the next section.

TABLE 26.9 Simulations for the Three-Factor Model of Longstaff and Rajan with Jumps and Exponential Random Losses

	Size Losses			Tranches (Basis Points)				
Set	γ_1	γ_2	γ_3	0–3	3–7	7–10	10–15	15–30
1	0.005	0.050	0.350	1404	939	733	580	347
2	0.150	0.050	0.350	2610	1929	1471	1141	645
3	0.050	0.150	0.350	2345	1590	1232	982	598
4	0.350	0.150	0.050	2772	2272	1863	1468	830

Source: From Longstaff, F.A. and Rajan, A., 2008. To appear in *Journal of Finance*.

Note: Parameters are fixed at $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\alpha_3 = 0.001$, $\beta_1 = \beta_2 = \beta_3 = 0.60$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.20$, and $\sigma_3 = 0.15$. For ease of exposition, jump parameters are equal for the three processes, $\delta = 0.02$, and $\mu = 0.10$. Finally, parameters in Size Losses correspond to the mean of an exponential distributed random variable.

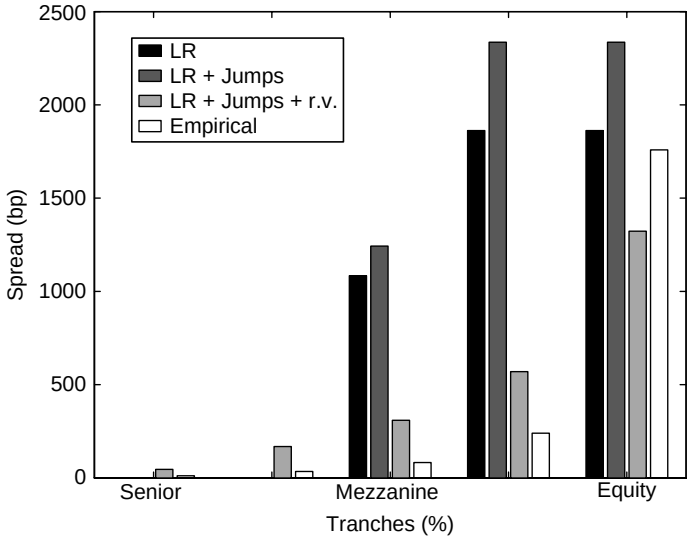


FIGURE 26.3 Comparison of spreads for different tranches and different versions of the Longstaff and Rajan model with one factor. Parameters are $\alpha_1 = 0.50$, $\beta_1 = 0.60$, $\sigma_1 = 0.15$, $\gamma_1 = 0.10$, $\delta_1 = 0.20$, and $\mu_1 = 0.10$. Finally, the γ_1 parameter corresponds to the mean of an exponential distributed random variable for the "random losses" version of the model. (From Longstaff, F.A. and Rajan, A. 2008. To appear in *Journal of Finance*.)

26.4.4 Sectional Comparison

Finally, we study in detail the individual effects of adding jumps and randomizing the losses in the Longstaff and Rajan (2008) for one, two, and three factors.

Figures 26.3 through 26.5 show the different versions of the model under study. LR represents the Longstaff and Rajan (2008) model and it refers to its simplest version. Then, we display the "jumps" and "jumps plus random variable" versions of the model. Finally, we include the mean spreads cited in their article for comparing the performance of the models. It is important to notice that we are interested in comparing the different models (one, two, or three factors) to their extended versions, taking the empirical spreads just as an order reference.*

We begin the study for the one-factor version of Longstaff and Rajan (2008). Figure 26.3 displays the spreads obtained for various tranches under different models. As expected, the one-factor versions of Longstaff and Rajan (2008) with constant loss size produce spreads so far from mean values. Naturally, the version with jumps shows higher spreads, due to the fact that jumps in the default process generate higher probability of arrivals of default events. Finally, it is important to note that a single factor version with random losses can capture the spreads given in the market, attending at bars on Figure 26.1.†

* To put it in other words, there is no intention here of doing an empirical analysis of the model. A study on this direction overcomes the objectives of this chapter.

† How to intend the default process in this context can be a subject of further research.

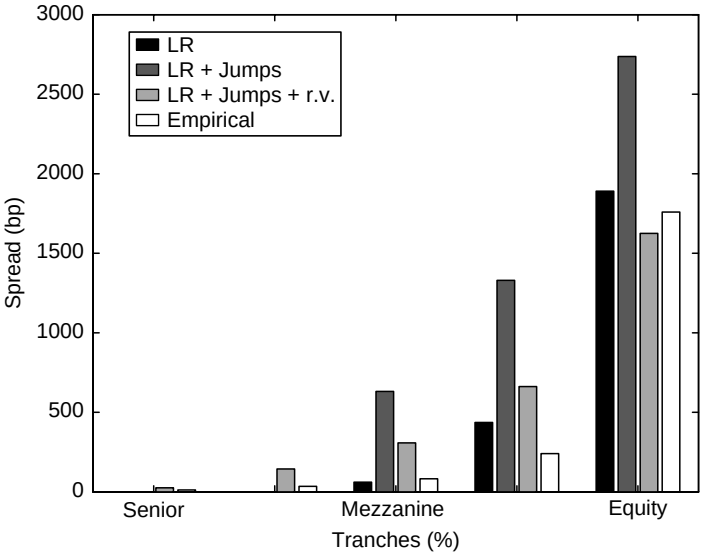


FIGURE 26.4 Comparison of spreads for different tranches and different versions of the Longstaff and Rajan model with two factors. Parameters are $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\beta_1 = \beta_2 = 0.60$, $\sigma_1 = 0.15$, $\sigma_2 = 0.20$, $\gamma_1 = \gamma_2 = 0.05$, $\delta_1 = \delta_2 = 0.20$, and $\mu_1 = \mu_2 = 0.10$. Finally, the parameters γ_i, $i = 1, 2$ correspond to the mean of an exponential distributed random variable for the "random losses" version of the model. (From Longstaff, F.A. and Rajan, A. 2008. To appear in *Journal of Finance*.)

Figure 26.4 compares the spreads for alternative versions of a two-factor model in Equation (26.6). Again, the model with jumps and constant γ generates the highest spreads in equity–mezzanine tranches. We observe here that the addition of a second factor results in spreads obtained by the constant size loss model closer to those obtained with random size losses for every tranches.

Finally, Figure 26.5 shows the spreads for the three-factor version of Longstaff and Rajan (2008) under study. As previously deduced in Figure 26.2, the addition of a third factor serves to provide spreads close to those observed empirically. The extensions that consider "jumps" and "jumps with random losses" seem to produce spreads far away from those observed in the market.

26.5.5 Conclusions

The development of markets where investors are able to trade their exposure to credit events has contributed to the appareance of a huge diversity of standardized financial assets. A wide range of these new products such as FtD or CDS indexes are examples of how the investors can reduce the hedging cost of portfolios including a large number of firms without necessity of buying insurance contracts on each individual firm.

This chapter has focused on these families of basket credit derivatives with special emphasis on CDS indexes, where the role of the default correlation (intended as the

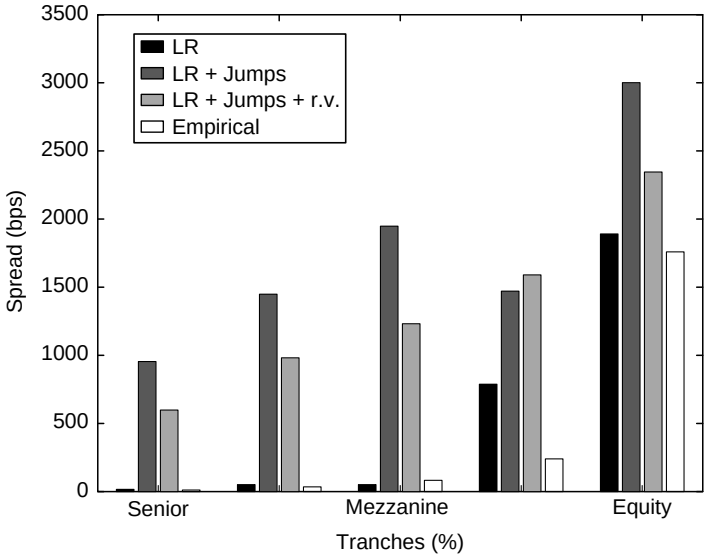


FIGURE 26.5 Comparison of spreads for different tranches and different versions of the Longstaff and Rajan model with three factors. Parameters are $\alpha_1 = 0.50$, $\alpha_2 = 0.02$, $\alpha_3 = 0.001$, $\beta_1 = \beta_2 = \beta_3 = 0.60$, $\sigma_1 = 0.15$, $\sigma_2 = 0.20$, $\sigma_3 = 0.15$, $\gamma_1 = 0.050$, $\gamma_2 = 0.15$, $\gamma_2 = 0.35$, $\delta_1 = \delta_2 = \delta_3 = 0.20$, and $\mu_1 = \mu_2 = \mu_3 = 0.10$. Finally, the parameters γ_i, $i = 1$, 2, 3 correspond to the mean of an exponential distributed random variable for the "random losses" version of the model. (From Longstaff, F.A. and Rajan, A. 2008. To appear in *Journal of Finance*.)

different relationship of dependence between defaults that can be given among firms) is crucial to price these assets.

Then, the aim of this chapter has been to study the pricing of standardized CDO tranches using the intensity-based framework of clustered defaults developed by Longstaff and Rajan (2008). These authors assume that defaults can be bursted, that is, that joint defaults can happen simultaneously, an approach linked to the contagious scheme of Davis and Lo (2001). Then, this model contrasts to the individual approach of Duffie and Garlenau (2001), where the total number of defaulted firms is conditioned by the level of correlation among the obligors.

We have performed a Monte Carlo study to explore numerically the capability of this model to generate values for standardized CDOs tranches under two different circumstances, particularly those referred to the number of factors used and the impact of credit events on the loss distribution.

We have also extended the original model in two directions: by adding jumps to the default process; and by randomizing the impact of defaults on the loss distribution. To the best of our knowledge, no similar study has yet been reported.

Our results seem to suggest that a three-factor version of Longstaff and Rajan (2008) with constant losses impact is flexible enough to reproduce the spreads given by the market. In addition to this, the inclusion of jumps to the default process results in a

high arrival of credit events, as corroborated by the high values spreads for equity tranches. Finally, the alternative of random losses can be helpful when dealing with one- and two-factor models, but it seems to be irrelevant in the case of three-factor models. Anyway, this last point can be developed as a subject for further research.

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Finance/Mathematics

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